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Seidel Laplacian and Seidel Signless Laplacian Spectrum of the Zero-divisor Graph on the Ring of Integers Modulo n

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Abstract Let G be a simple graph of order n and let $S(G)$ be the Seidel matrix of G, defined as $S(G) = [s_{ij}]$ where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent and $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent and $s_{ij} = 0$ if $i = j$. Let $D_S(G) = diag(n - 2d_1 - 1, n - 2d_2 - 1, ..., n - 2d_n - 1)$ be the diagonal matrix where d_i denotes the degree of the i^{th} vertex of G. The Seidel Laplacian matrix of a graph G is defined as $SL(G) = D_S(G) - S(G)$ and the Seidel signless Laplacian matrix of a graph G is defined as $SL^+(G) = D_S(G) + S(G)$. The zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices x, y are adjacent if and only if $xy = 0$. In this paper, we find the Seidel polynomial and Seidel Laplacian polynomial of the join of two regular graphs using the concept of schur complement and coronal of a square matrix. Also we describe the computation of the Seidel Laplacian and Seidel signless Laplacian eigenvalues of the join of more than two regular graphs, using the well known Fiedler's lemma and apply these results to describe these eigenvalues for the zero-divisor graph on $\mathbb{Z}_n$. Further we find the Seidel Laplacian and Seidel signless Laplacian spectrum of the zero-divisor graph of $\mathbb{Z}_n$ for some values of n, say $n = p^3, p^4, pq, p^2q$, where p, q are distinct primes. We also prove that 0 is a simple Seidel Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$, for any n.

Keywords Seidel Laplacian Matrix, Zero-divisor Graph

1 Introduction

Throughout this paper, G denotes a simple, finite, undirected and connected graph. If G has n vertices, then the adjacency matrix, $A(G) = [a_{ij}]_{n\times n}$ where, $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. Van Lint and Seidel [1] introduced the Seidel matrix of G, defined as $S(G) = [s_{ij}]$ where, $s_{ij} = \begin{cases} -1; & if v_i \sim v_j \\ 1; & if v_i \nsim v_j \\ 0; & otherwise \end{cases}$. Clearly, $S(G) = J - I - 2A(G)$. Also, if $\overline{G}$ denotes the complement of a graph G, then $S(\overline{G}) = -S(G)$. The Seidel spectrum of G is denoted by $spec^S(G)$. Let $D_S(G) = diag(n - 2d_1 - 1, n - 2d_2 - 1, ..., n - 2d_n - 1)$ be the diagonal matrix where d_i is the degree of the vertex v_i. The Seidel Laplacian matrix [2] of a graph G is defined as

$$SL(G) = D_S(G) - S(G)$$

and the Seidel signless Laplacian matrix of a graph G is defined as

$$SL^{+}(G) = D_S(G) + S(G).$$

For a complete graph K_n,

$$SL(K_n) = J_n - nI_n$$

and

$$SL^{+}(K_n) = (2-n)I_n - J_n.$$

For a null graph, $\overline{K_n}$,

$$SL(\overline{K_n}) = nI_n - J_n$$

and

$$SL^{+}(\overline{K_n}) = (n-2)I_n + J_n.$$

2 Basic definitions and notations

Let $\Phi(G;x)$ and $Spec(G)$ denote the characteristic polynomial and the spectrum of G respectively. $\Phi_{SL}(G;x)$ and $\Phi_{SL^{+}}(G;x)$ denote the Seidel Laplacian polynomial and Seidel signless Laplacian polynomial of a graph G. In this paper, $\mathbf{1}_n$ denotes the all-one column vector of order $n \times 1$, and $\phi(n)$ is the number of positive integer less than n and relatively prime to n.

Definition 2.1. The join of two graphs G_1 and G_2; denoted by $G_1 \bigtriangledown G_2$, is obtained by joining each vertex of G_1 to all the vertices of G_2.

Definition 2.2. [3] Let the vertices of a graph G be labeled as $1, 2, 3, ..., n$ and let $H_1, H_2, ..., H_n$ be a collection of vertex disjoint graphs. The generalized join of $H_1, H_2, ..., H_n$ denoted by $G[H_1, H_2, ..., H_n]$, is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j accordingly if i and j are adjacent in G or not.

Definition 2.3. [4] The sum of the entries of the matrix $(xI - A)^{-1}$ is defined as the coronal of A, and is denoted by $\Gamma_A(x)$. It can be seen that, $\Gamma_A(x) = (\mathbf{1}_n)^T.(xI - A)^{-1}.\mathbf{1}_n$. We note that, if k is the sum of each row, then $\Gamma_A(x) = \dfrac{n}{x-k}$.

I. Beck [5] initiated the idea of zero-divisor graph $G(R)$ of a commutative ring R in connection with some coloring problems and in 1999, Anderson and Livingston [6], modified the definition of zero-divisor graph as a simple graph $\Gamma(R)$ where, only nonzero zero-divisors of R are considered as vertices. The concept of compressed zero-divisor graph; determined by the equivalence classes of the zero-divisors of R, was introduced by Mulay [7], with the purpose of simplifying the representation of $\Gamma(R)$ In [8, 9], the authors describe the adjacency matrix, eigenvalues and some graph parameters of the zero-divisor graphs $\Gamma(\mathbb{Z}_{p^2q})$, $\Gamma(\mathbb{Z}_{p^2q^2})$ and $\Gamma(\mathbb{Z}_{p^k})$. S. Chattopadhyay et.al [10], have explored the combinatorial structure of $\Gamma(\mathbb{Z}_n)$ as the generalized union of its induced subgraphs. The combinatorial properties of the proper divisor graph on n have been investigated in [11]. It is very interesting and challenging that the combinatorial and spectral properties of zero-divisor graphs can be studied in terms of its compressed graph.

3 Fiedler's Lemma and its generalisation

The Abel's impossibility theorem implies that the algebraic determination of all the zeroes of a polynomial of degree greater than or equal to five, is impossible in closed form. So, we need resort to the tools of Linear Algebra to explore the eigenvalues of large matrices by means of its sub matrices. In this direction, the famous Fiedler's lemma is very relevant to find the eigenvalues of large symmetric matrices. In [12], H.S.Ramane et.al find the Seidel Laplacian polynomial of the join of two regular graphs. In this section, we find the same in a fairly shorter method, by applying the following lemma and extend the result to the join of more than two regular graphs.

3.1 Fiedlers's Lemma

Lemma 3.1. [13] Consider two symmetric matrices $A_{m\times m}$ and $B_{n\times n}$ and let $(\alpha_i, \mathbf{u}_i)$,$i = 1, 2, ..., m$ and $(\beta_i, \mathbf{v}_i)$, $i = 1, 2, ..., n$, be the eigen pairs of A and B respectively, where $\|\mathbf{u}_1\| = 1 = \|\mathbf{v}_1\|$. For any arbitrary constant ρ, let the matrix $\hat{C} = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}$ has eigenvalues γ_1, γ_2. Then, $C = \begin{bmatrix} A & \rho\mathbf{u}_1\mathbf{v}_1^T \\ \rho\mathbf{v}_1\mathbf{u}_1^T & B \end{bmatrix}$ has eigenvalues $\alpha_2, ..., \alpha_m, \beta_2, ..., \beta_n, \gamma_1, \gamma_2$.

Lemma 3.2. [14] If G is a k- regular graph of order n, and $k, \lambda_2, ..., \lambda_n$ are the adjacency eigenvalues of G, then $\theta_1 = n - 2k - 1, \theta_2 = -1 - 2\lambda_2, \theta_3 = -1 - 2\lambda_3, ..., \theta_n = -1 - 2\lambda_n$ are the Seidel eigenvalues of G.

Theorem 3.1. Let G_i be r_i regular of order n_i, for $i = 1, 2$. Then the Seidel Laplacian polynomial of $G_1 \bigtriangledown G_2$ is given by

$$\Phi_{SL}(G_1 \bigtriangledown G_2; x) = \frac{x(x+n_1+n_2)}{(x+n_1)(x+n_2)} \cdot \Phi_{SL}(G_1; x+n_2) \cdot \Phi_{SL}(G_2; x+n_1).$$

Proof. Let $S(G_1)$ and $S(G_2)$ denote the Seidel adjacency matrices of G_1 and G_2 respectively. Then, the Seidel Laplacian matrices of G_1 and G_2 are given as follows.

$$SL(G_1) = (n_1 - 2r_1 - 1)I_{n_1} - S(G_1),$$
$$SL(G_2) = (n_2 - 2r_2 - 1)I_{n_2} - S(G_2).$$

Since G_1 is regular, by Lemma 3.2, $n_1 - 2r_1 - 1$ is an eigenvalue of $S(G_1)$ with eigenvector $\mathbf{1}_{n_1}$. Also, 1 is an eigenvalue of I_{n_1} with eigenvector $\mathbf{1}_{n_1}$. Thus, $n_1 - 2r_1 - 1$ is an eigenvalue of both $S(G_1)$ and $(n_1 - 2r_1 - 1)I_{n_1}$ with associated eigenvector $\mathbf{1}_{n_1}$, which means that, 0 is an eigenvalue of $SL(G_1)$ with eigenvector $\mathbf{1}_{n_1}$. Similarly, 0 is an eigenvalue of $SL(G_2)$ with corresponding eigenvector $\mathbf{1}_{n_2}$.

Clearly,

$$SL(G_1 \bigtriangledown G_2) = \begin{bmatrix} (n_1+n_2-2(r_1+n_2)-1)I_{n_1} - S_1 & J_{n_1\times n_2} \\ J^T_{n_1\times n_2} & (n_1+n_2-2(r_2+n_1)-1)I_{n_2} - S_2 \end{bmatrix}$$

$$= \begin{bmatrix} SL(G_1) - n_2 I_{n_1} & J_{n_1\times n_2} \\ J^T_{n_1\times n_2} & SL(G_2) - n_1 I_{n_2} \end{bmatrix}.$$

Taking $A = SL(G_1) - n_2 I_{n_1}$, $B = SL(G_2) - n_1 I_{n_2}$ $\alpha_1 = -n_2$, $\beta_1 = -n_1$, $\mathbf{u}_1 = \frac{1}{\sqrt{n_1}}\mathbf{1}_{n_1}$, $\mathbf{u}_2 = \frac{1}{\sqrt{n_2}}\mathbf{1}_{n_2}$, $\rho = \sqrt{n_1 n_2}$ and applying Fiedler's Lemma, it can be seen that

$$spec(SL(G_1 \bigtriangledown G_2)) = spec(SL(G_1) - n_2 I_{n_1})\backslash\{-n_2\} \cup spec(SL(G_2) - n_1 I_{n_2})\backslash\{-n_1\} \cup spec\hat{C},$$

where the matrix $\hat{C} = \begin{bmatrix} -n_2 & \sqrt{n_1 n_2} \\ \sqrt{n_1 n_2} & -n_1 \end{bmatrix}$. Clearly, $spec\hat{C} = \{0, -n_1 - n_2\}$. Thus,

$$\Phi_{SL}(G_1 \bigtriangledown G_2; x) = \frac{x(x+n_1+n_2)}{(x+n_1)(x+n_2)} \cdot \Phi_{SL}(G_1; x+n_2) \cdot \Phi_{SL}(G_2; x+n_1).$$

□

3.2 Generalization of Fiedler's Lemma

Let M_j be an $m_j \times m_j$ symmetric matrix, with corresponding eigenpairs $(\alpha_{rj}, \mathbf{u}_{rj})$, $1 \leqslant r \leqslant m_j$, where $j \in \{1, 2, ..., k\}$. Also, for $q \in \{1, 2, ..., k-1\}$ and $l \in \{q+1, ..., k\}$, let $\rho_{q,l}$ be arbitrary real numbers. Let $\widehat{\alpha}$ be the $k-$ tuple

$$\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k}) \tag{1}$$

where each $\alpha_{i_j,j}$ is selected from the collection of $\{\alpha_{1,j}, ..., \alpha_{m_j,j}\}$ where $j \in \{1, 2, ..., k\}$. Taking $\widehat{\rho}$ as a $\frac{k(k-1)}{2}$- tuple of real numbers,

$$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k}), \tag{2}$$

consider the symmetric matrices

$C_{\widehat{\alpha}}(\widehat{\rho}) =$

$$\begin{bmatrix} M_1 & \rho_{1,2}\mathbf{u}_{i_1,1}\mathbf{u}^T_{i_2,2} & \rho_{1,3}\mathbf{u}_{i_1,1}\mathbf{u}^T_{i_3,3} & \cdots & \rho_{1,k}\mathbf{u}_{i_1,1}\mathbf{u}^T_{i_k,k} \\ \rho_{1,2}\mathbf{u}_{i_2,2}\mathbf{u}^T_{i_1,1} & M_2 & \rho_{2,3}\mathbf{u}_{i_2,2}\mathbf{u}^T_{i_3,3} & \cdots & \rho_{2,k}\mathbf{u}_{i_2,2}\mathbf{u}^T_{i_k,k} \\ \rho_{1,3}\mathbf{u}_{i_3,3}\mathbf{u}^T_{i_1,1} & \rho_{2,3}\mathbf{u}_{i_3,3}\mathbf{u}^T_{i_2,2} & M_3 & \cdots & \rho_{3,k}\mathbf{u}_{i_3,3}\mathbf{u}^T_{i_k,k} \\ \vdots & \vdots & & \ddots & \vdots \\ \rho_{1,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}^T_{i_1,1} & \rho_{2,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}^T_{i_2,2} & \cdots & M_{k-1} & \rho_{k-1,k}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}^T_{i_k,k} \\ \rho_{1,k}\mathbf{u}_{i_k,k}\mathbf{u}^T_{i_1,1} & \rho_{2,k}\mathbf{u}_{i_k,k}\mathbf{u}^T_{i_2,2} & \cdots & \cdots & M_k \end{bmatrix}, \tag{3}$$

$$\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho}) = \begin{bmatrix} \alpha_{i_1,1} & \rho_{1,2} & \cdots & \rho_{1,k-1} & \rho_{1,k} \\ \rho_{1,2} & \alpha_{i_2,2} & \cdots & \rho_{2,k-1} & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1,k-1} & \rho_{2,k-1} & \cdots & \alpha_{i_{k-1},k-1} & \rho_{k-1,k} \\ \rho_{1,k} & \rho_{2,k} & \cdots & \rho_{k-1,k} & \alpha_{i_k,k} \end{bmatrix}_{k\times k}. \tag{4}$$

Theorem 3.2. [15] For $j \in \{1, 2, ..., k\}$, let M_j, $\widehat{\rho}$, $\widehat{\alpha}$ be as defined (1) and (2). Suppose that for each j, the system of eigen vectors $\{\mathbf{u}_{rj}, r \in I_j\}$ is orthonormal. Then the matrix $C_{\widehat{\alpha}}(\widehat{\rho})$ in (3) has the multi set of eigenvalues $\left(\bigcup_{j=1}^{k}\{\alpha_{1,j}, ..., \alpha_{m_j,j}\}\backslash\{\alpha_{i_j,j}\}\right) \cup \{\gamma_1, ..., \gamma_k\}$, where $\gamma_1, \gamma_2, ..., \gamma_k$ are eigenvalues of the matrix $\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho})$ in (4).

The authors D.M. Cardoso et al [15] and Magi.P.M. et al [16] have used the above result to find distance related spectrum of the generalized join of graphs.

4 Seidel Laplacian spectrum of the joined union of regular graphs

Consider $G[H_1, H_2, ..., H_k]$, where the vertices of G are labeled as $1, 2, ..., k$, and the Seidel matrix $S(G) = [s_{ij}]$. Let H_j be r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $S(H_j)$ and $SL(H_j)$ denote the Seidel matrix and Seidel Laplacian matrix of $H_j, j = 1, 2, ..., k$. The degree of each vertex of H_j, in the joined graph $G[H_1, H_2, ..., H_k]$ is $r_j + \sum_{i\sim j} n_i$. Hence, if S_j denotes the j^{th} diagonal block in the Seidel Laplacian matrix of $G[H_1, H_2, ..., H_k]$, then,

$$\begin{aligned} S_j &= \left(\sum_{i=1}^{k} n_i - 2(r_j + \sum_{j\sim i} n_i) - 1\right) I_{n_j} - S(H_j) \\ &= (n_j - 2r_j - 1)I_{n_j} - S(H_j) + \left(\sum_{j\not\sim i} n_i - \sum_{j\sim i} n_i\right) I_{n_j} \\ &= SL(H_j) + \left(\sum_{i=1}^{k} s_{ij}n_i\right) I_{n_j} \\ &= SL(H_j) + \tau_j I_{n_j}. \end{aligned}$$

where $\tau_j = \sum_{i=1}^{k} s_{ij}n_i$.

Thus, we see that, the Seidel Laplacian matrix of the G- union of $H_1, H_2, ..., H_k$ is given by

$$SL(G[H_1, H_2, ..., H_k]) = \begin{bmatrix} SL(H_1) + \tau_1 I_{n_1} & -s_{1,2}J_{n_1\times n_2} & \cdots & -s_{1,k}J_{n_1\times n_k} \\ -s_{1,2}J^T_{n_1\times n_2} & SL(H_2) + \tau_2 I_{n_2} & \cdots & -s_{2,k}J_{n_2\times n_k} \\ \vdots & & \ddots & \vdots \\ -s_{1,k}J^T_{n_1\times n_k} & -s_{2,k}J^T_{n_2\times n_k} & \cdots & SL(H_k) + \tau_k I_{n_k} \end{bmatrix} \quad (5)$$

where $\tau_j = \sum_{i=1}^{k} s_{ij}n_i$.

Remark 4.1. Since for a k-regular graph G of order n, $SL(G) = (n - 2k - 1)I_n - S(G)$, from Lemma 3.2 it follows that, $SL(G)$ has an eigenvalue 0 with multiplicity at least 1. For example, the Seidel Laplacian spectrum of the cycle C_4, which is 2-regular is $spec^{SL}(C_4) = \left\{ \begin{array}{cc} 0 & -4 \\ 3 & 1 \end{array} \right\}$. However, the Seidel Laplacian matrix of complete graphs and null graphs (complement of complete graphs) has an eigenvalue 0 with multiplicity 1.

Theorem 4.1. Consider $G[H_1, H_2, ..., H_k]$ and $S = [s_{ij}]_{k\times k}$ is the Seidel matrix of G and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $\{\sigma^{SL}_{j1} = 0, \sigma^{SL}_{j2}, ..., \sigma^{SL}_{jn_j}\}$ be the Seidel Laplacian eigenvalues of H_j, for $j = 1, 2, ..., k$. Then,

$$spec^{SL}(G[H_1, H_2, ..., H_n]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\sigma^{SL}_{ji} + \tau_j)\right) \cup spec(T_{SL}(G)),$$

where $\tau_j = \sum_{i=1}^{k} s_{ij}n_i$ and

$$T_{SL}(G) = \begin{bmatrix} \tau_1 & -s_{1,2}n_2 & \cdots & -s_{1,k}n_k \\ -s_{1,2}n_1 & \tau_2 & \cdots & -s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,k}n_1 & -s_{2,k}n_2 & \cdots & \tau_k \end{bmatrix}.$$

Proof. Since H_j is regular, 0 is a Seidel Laplacian eigenvalue of H_j with eigenvector $\mathbf{1}_{n_j}$, for every j. Thus from (5), it is evident that each of the diagonal blocks $SL(H_j) + \tau_j I_{n_j}$ is a symmetric matrix which has τ_j as an eigenvalue with eigenvector $\mathbf{1}_{n_j}$, $j = 1, 2, ..., k$.

As in (3), taking

$$M_j = SL(H_j) + \tau_j I_{n_j}, \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = (\tau_j \quad , \quad \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j})$$

and the real numbers

$$\rho_{l,q} = -s_{lq}\sqrt{n_l n_q}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, and applying Theorem 3.2, we find that,

$$spec^{SL}(G[H_1, H_2, ..., H_n]) = \left(\bigcup_{j=1}^{k} spec\left(SL(H_j) + \tau_j I_{n_j}\right) \backslash \{\tau_j\}\right) \cup spec((\widetilde{S})) \tag{6}$$

where $\tau_j = \sum_{i=1}^{k} s_{ij} n_i$ and $\widetilde{S} = \begin{bmatrix} \tau_1 & \rho_{1,2} & \cdots & \rho_{1,k} \\ \rho_{1,2} & \tau_2 & \cdots & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,k} & \rho_{2,k} & \cdots & \tau_k \end{bmatrix}$.

Obviously, $\rho_{l,q} = -s_{lq}\sqrt{n_l n_q} = \begin{cases} \sqrt{n_l n_q} & \text{if } lq \in \text{E(G)} \\ -\sqrt{n_l n_q} & \text{if } lq \notin \text{E(G)} \end{cases}$ for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$.

Assign the weight $n_j = |V(H_j)|$ to the vertex j of G for $j = 1, 2, ..., k$ and consider the matrix W which is a diagonal matrix of vertex weights,

$$W = \begin{bmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & n_k \end{bmatrix}.$$

Let $T_{SL}(G)$ be the combinatorial (vertex weighted) Seidel Laplacian matrix of G, given by

$$T_{SL}(G) = \begin{bmatrix} \tau_1 & -s_{1,2}n_2 & \dots & -s_{1,k}n_k \\ -s_{1,2}n_1 & \tau_2 & \dots & -s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,k}n_1 & -s_{2,k}n_2 & \dots & \tau_k \end{bmatrix}. \tag{7}$$

The vertex weighted combinatorial Laplacian matrix of graphs can be seen in [17]. It is easy to verify that $T_{SL}(G) = W^{-\frac{1}{2}}\widetilde{S}W^{\frac{1}{2}}$. Thus $\widetilde{S}$ and $T_{SL}(G)$ are similar and hence $spec(\widetilde{S}) = spec(T_{SL}(G))$. Hence from (6),

$$spec^{SL}(G[H_1, H_2, ..., H_n]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\sigma_{ji}^{SL} + \tau_j)\right) \cup spec(T_{SL}(G))). \tag{8}$$

□

5 The zero-divisor graph $\Gamma(\mathbb{Z}_n)$

The commutative ring $\mathbb{Z}_p$ is an integral domain for any prime p and so it is quite trivial to study its zero-divisor graph. Hence in the following sessions, it is assumed that n is not a prime. It is very relevant to note that $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph on $p-1$ vertices. Also $\Gamma(\mathbb{Z}_8)$, $\Gamma(\mathbb{Z}_{pq})$ are complete bipartite graphs. Using elementary number theory, the number of non zero zero-divisors is calculated to be $n - \phi(n) - 1$ [9].

For distinct primes $p_1, p_2, ..., p_r$, let $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$ be the canonical decomposition of n. A proper divisor of n is a positive divisor d such that $d \mid n$, $1 < d < n$. Let $p(n)$ denote the number of proper divisors of n. Then, $p(n) = \sigma_0(n) - 2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1.

Let $\mathbf{S}(d) = \{k \in \mathbb{Z}_n : gcd(k, n) = d\}$. Then $\{\mathbf{S}(d_1), \mathbf{S}(d_2), ..., \mathbf{S}(d_{p(n)})\}$ forms an equitable partition of $V(\Gamma(\mathbb{Z}_n))$ Also, $|\mathbf{S}(d_i)| = \phi(\frac{n}{d_i})$, for every $i = 1, 2, ...p(n)$. The following Lemmas are inevitable in the analysis of $\Gamma(\mathbb{Z}_n)$.

Lemma 5.1. [10] Let $\Gamma(\mathbf{S}(d_i))$ denote the subgraph of $\Gamma(\mathbb{Z}_n)$, induced by $\mathbf{S}(d_i)$ for $i = 1, 2, ..., p(n)$. Then,
$\Gamma(\mathbf{S}(d_i)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d_i})} & \text{if } n \nmid d_i^2 \\ K_{\phi(\frac{n}{d_i})} & \text{if } n \mid d_i^2 \end{cases}$.

Lemma 5.2. [10] $\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(\mathbf{S}(d_1)), \Gamma(\mathbf{S}(d_2)), ..., \Gamma(\mathbf{S}(d_{p(n)}))]$.

The subgraph $\Gamma(\mathbf{S}(d_i))$ is regular for each $i = 1, 2, ..., p(n)$. For example, in $\Gamma(\mathbb{Z}_{p^3})$, $\mathbf{S}(p)$ induces $\overline{K}_{p(p-1)}$ and $\mathbf{S}(p^2)$ induces K_{p-1}. In $\Gamma(\mathbb{Z}_{p^2q})$, $\mathbf{S}(p), \mathbf{S}(q), \mathbf{S}(p^2)$ induce $\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{q-1}$ respectively whereas $\mathbf{S}(pq)$ induces K_{p-1}.

The proper divisor graph, denoted by Υ_n plays a vital role in the analysis of $\Gamma(\mathbb{Z}_n)$, the vertices of which are labeled as $d_1, d_2, ..., d_{p(n)}$. See[11]. Also $d_i \sim d_j$ in Υ_n if and only if n divides the product $d_i d_j$. [10].

5.1 Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$

Theorem 5.1. If $d_1, d_2, ..., d_{p(n)}$ are the proper divisors of n, then the Seidel Laplacian matrix of $\Gamma(\mathbb{Z}_n)$ is given by

$$SL(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} S_1 & -s_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \cdots & -s_{1,p(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{p(n)}})} \\ -s_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & S_2 & \cdots & -s_{2,p(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{p(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,p(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{p(n)}})} & -s_{2,p(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{p(n)}})} & \cdots & S_{p(n)} \end{bmatrix}$$

where,

$$S_j = \begin{cases} \left(\phi(\frac{n}{d_j}) + \sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})\right) I_{\phi(\frac{n}{d_j})} - J_{\phi(\frac{n}{d_j}} & \text{if } n \nmid d_j^2 \\ J_{\phi(\frac{n}{d_j})} + \left(\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i}) - \phi(\frac{n}{d_j})\right) I_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases}$$

and

$$s_{ij} = \begin{cases} -1 & \text{if } n \mid d_i d_j \\ 1 & \text{if } n \nmid d_i d_j \end{cases}.$$

Proof. In the proper divisor graph Υ_n, the vertices d_i and d_j are adjacent if and only if $n \mid d_i d_j$. Thus if $S = [s_{ij}]_{p(n)\times p(n)}$ denotes the Seidel adjacency matrix of Υ_n, then

$$s_{ij} = \begin{cases} -1 & \text{if } n \mid d_i d_j \\ 1 & \text{if } n \nmid d_i d_j \end{cases}.$$

By Lemma 5.2, $\Gamma(\mathbb{Z}_n)$ is the Υ_n - join of $\Gamma(\mathbf{S}(d_1)), \Gamma(\mathbf{S}(d_2)), ..., \Gamma(\mathbf{S}(d_{p(n)}))$. The induced subgraphs $\Gamma(\mathbf{S}(d_j))$ are either a complete graph or a null graph on $\phi(\frac{n}{d_j})$ vertices.
By Lemma 5.1,

$$\Gamma(\mathbf{S}(d_j)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ K_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases}.$$

The Seidel Laplacian matrix of K_n and $\overline{K_n}$ are given by

$$SL(K_n) = J_n - nI_n,$$
$$SL(\overline{K_n}) = nI_n - J_n.$$

Thus, if S_j denotes the j^{th} diagonal block in the Seidel Laplacian matrix of $\Gamma(\mathbb{Z}_n)$, which corresponds to the vertices of $\Gamma(\mathbf{S}(d_j))$, then, $S_j = SL(\Gamma(\mathbf{S}(d_j))) + \tau_j I_{\phi(\frac{n}{d_j})}$ where, $\tau_j = \sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})$, from (5). Hence,

$$S_j = \begin{cases} \phi(\frac{n}{d_j})I_{\phi(\frac{n}{d_j})} - J_{\phi(\frac{n}{d_j})} + \tau_j I_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ J_{\phi(\frac{n}{d_j})} - \phi(\frac{n}{d_j})I_{\phi(\frac{n}{d_j})} + \tau_j I_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2 \end{cases}.$$

Thus, the result follows from (5), taking $G = \Upsilon_n$ and $H_j = \Gamma(\mathbf{S}(d_j))$. □

Applying Theorem 4.1 and Theorem 5.1, we determine the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ in the following corollary.

Corollary 5.1. Let $d_1, d_2, ..., d_{p(n)}$ be the proper divisors of n. Then, $\Gamma(\mathbb{Z}_n)$ has Seidel Laplacian eigenvalues $\phi(\frac{n}{d_j}) + \sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})$ with multiplicity $\phi(\frac{n}{d_j}) - 1$ corresponding to the divisors d_j such that $n \nmid d_j^2$ and $\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i}) - \phi(\frac{n}{d_j})$

with multiplicity $\phi(\frac{n}{d_j})-1$ corresponding to the divisors d_j such that $n \mid d_j^2$ and the remaining Seidel Laplacian eigenvalues are the eigenvalues of

$$T_{SL}(G)=\begin{bmatrix} \tau_1 & -s_{1,2}\phi(\frac{n}{d_2}) & \dots & -s_{1,p(n)}\phi(\frac{n}{d_{p(n)}}) \\ -s_{1,2}\phi(\frac{n}{d_1}) & \tau_2 & \dots & -s_{2,p(n)}\phi(\frac{n}{d_{p(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,p(n)}\phi(\frac{n}{d_1}) & -s_{2,k}\phi(\frac{n}{d_2}) & \dots & \tau_{p(n)} \end{bmatrix} \quad (9)$$

where $\tau_j=\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i}), j=1,2,...,p(n)$.

Definition 5.1. [18] A matrix $[a_{ij}]$ is strictly diagonally dominant if in every row of the matrix, the magnitude of the diagonal entry is strictly greater than the sum of the magnitudes of all other non-diagonal entries, that is if, $|a_{ii}|>\sum_{j\neq i}|a_{ij}|$, for all i.

Theorem 5.2. (Levy- Desplanques, Theorem [18]) A strictly diagonally dominant matrix is non-singular.

Theorem 5.3. 0 ia a simple Seidel Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$, for any n.

Proof. First, we show that 0 is an eigenvalue of the matrix $T_{SL}(G)$ of multiplicity 1. For this, we prove that $T_{SL}(G)$ is singular with nullity 1. Arrange the proper divisors of n in the ascending order, $d_1<d_2<...<d_{p(n)}$. It is obvious that, $\phi(\frac{n}{d_1})>\phi(\frac{n}{d_i}), i=2,3,...,p(n)$. We note that $T_{SL}(G)$ is a square matrix of size $p(n)$. Since, $\tau_j=\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})$,

$$\tau_j+\sum_{i\neq j}-s_{ij}\phi(\frac{n}{d_i})=0.$$

Hence it follows from (9), that the sum of each row of $T_{SL}(G)$ is zero. The first column of $T_{SL}(G)$ can be transformed to the zero column on adding $2^{nd},3^{rd},...,p(n)^{th}$ columns to it. Hence $T_{SL}(G)$ is singular which implies that 0 is an eigenvalue of $T_{SL}(G)$. To prove that the multiplicity is 1, it suffices to prove that the rank of $T_{SL}(G)$ is $p(n)-1$. For this, consider the matrix T obtained from $T_{SL}(G)$ by deleting the first row and first column of $T_{SL}(G)$. Since

$$s_{ij}=\pm 1, \quad \phi(\frac{n}{d_1})>\phi(\frac{n}{d_i}), i=2,3,...,p(n) \quad \Rightarrow \quad |\tau_j|>\sum_{i\neq j, i\neq 1}|s_{i,j}\phi(\frac{n}{d_i})|.$$

Thus, T is strictly diagonally dominant. For example, consider the first row of T, say $[\tau_2,-s_{2,3}\phi(\frac{n}{d_3}),-s_{2,4}\phi(\frac{n}{d_4}),...,-s_{2,p(n)}\phi(\frac{n}{d_{p(n)}})]$. Since, $\tau_2=s_{1,2}\phi(\frac{n}{d_1})+s_{2,3}\phi(\frac{n}{d_3})+....+s_{2,p(n)}\phi(\frac{n}{d_{p(n)}})$, it follows that, $|\tau_2|>|s_{2,3}\phi(\frac{n}{d_3})|+...+|s_{2,p(n)}\phi(\frac{n}{d_{p(n)}})|$.
Hence by Theorem 5.2, T is non-singular, and hence the rank of $T_{SL}(G)$ is $p(n)-1$.

By Corollary 5.1, the remaining Seidel eigenvalues of $\Gamma(\mathbb{Z}_n)$ are $\phi(\frac{n}{d_j})+\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})$ and $\sum_{i=1}^{p(n)} s_{ij}\phi(\frac{n}{d_i})-\phi(\frac{n}{d_j})$, neither of which is zero. This proves the theorem. □

Theorem 5.4. For distinct primes p and q, $p<q$, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by

$$Spec^{SL}(\Gamma(\mathbb{Z}_{pq}))=\left\{\begin{array}{cccc} 0 & -(p+q-2) & p-q & q-p \\ 1 & 1 & p-2 & q-2 \end{array}\right\}.$$

Proof. The proper divisors of pq are p and q. By Lemma 5.1 and Lemma 5.2, the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ is the join of $\Gamma(\mathbf{S}(p))$ and $\Gamma(\mathbf{S}(q))$, where $\Gamma(\mathbf{S}(p))=\overline{K}_{q-1}$ and $\Gamma(\mathbf{S}(q))=\overline{K}_{p-1}$. That is

$$\Gamma(\mathbb{Z}_{pq})=\overline{K}_{q-1}\triangledown\overline{K}_{p-1}.$$

Clearly $spec^{SL}(\overline{K}_{q-1})=\left\{\begin{array}{cc} 0 & q-1 \\ 1 & q-2 \end{array}\right\}$ and $spec^{SL}(\overline{K}_{p-1})=\left\{\begin{array}{cc} 0 & p-1 \\ 1 & p-2 \end{array}\right\}$. And the result follows from Theorem 3.1. □

Theorem 5.5. For any prime p, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is

$$spec^{SL}(\Gamma(\mathbb{Z}_{p^3}))=\left\{\begin{array}{ccc} 0 & 1-p^2 & p^2-2p+1 \\ 1 & p-1 & p^2-p-1 \end{array}\right\}.$$

Proof. The proper divisors of p^3 are p and p^2. By lemma 5.1 it can be seen that the subgraphs of $\Gamma(\mathbb{Z}_{p^3})$, induced by $\mathbf{S}(p)$ and $\mathbf{S}(p^2)$ are $\overline{K}_{p(p-1)}$ and K_{p-1} respectively. Also by Lemma 5.2,

$$\Gamma(\mathbb{Z}_{p^3})=\overline{K}_{p(p-1)}\triangledown K_{p-1}.$$

$$spec^{SL}(\overline{K}_{p(p-1)}) = \left\{ \begin{array}{cc} 0 & p(p-1) \\ 1 & p^2-p-1 \end{array} \right\},$$

$$spec^{SL}(K_{p-1}) = \left\{ \begin{array}{cc} 0 & 1-p \\ 1 & p-2 \end{array} \right\}.$$

Hence, the result follows from Theorem 3.1. □

Theorem 5.6. For any prime p, the Seidel Laplacian spectrum of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ is

$$spec^{SL}(\Gamma(\mathbb{Z}_{p^4})) = \left\{ \begin{array}{cccc} 0 & 1-p^3 & p^3-2p^2+1 & p^3-2p+1 \\ 1 & p-1 & p^2-p-1 & p^3-p^2 \end{array} \right\}.$$

Proof. The divisors of p^4 are p, p^2, p^3. The proper divisor graph of p^4 is the path P_3, in which $p \sim p^3 \sim p^2$. The subgraph induced by $\mathbf{S}(p)$ is the null graph $\overline{K}_{p^2(p-1)}$, whereas the subgraphs induced by $\mathbf{S}(p^2)$ and $\mathbf{S}(p^3)$ are the complete graphs $K_{p(p-1)}$ and K_{p-1} respectively. It can be seen that,

$$\Gamma(\mathbb{Z}_{p^4}) = P_3[\overline{K}_{p^2(p-1)}, K_{p(p-1)}, K_{p-1}].$$

Applying Corollary 5.1, we see that, $p^3-2p+1, p^3-2p^2+1, 1-p^3$ are Seidel Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$ with multiplicities p^3-p^2-1, p^2-p-1 and $p-2$ respectively. And the remaining three Seidel Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$ are the eigenvalues of the matrix,

$$T_{SL}(\Upsilon_{p^4}) = \left[\begin{array}{ccc} p^2-2p+1 & p-p^2 & p-1 \\ p^2-p^3 & p^3-p^2-p+1 & p-1 \\ p^3-p^2 & p^2-p & p-p^3 \end{array} \right].$$

It can be seen that, the above matrix has three eigen values, $\lambda_1 = 0, \lambda_2 = 1-p^3, \lambda_3 = p^3-2p+1$. □

6 Seidel signless Laplacian spectrum of the join of regular graphs

We note that, each diagonal block in the Seidel signless Laplacian matrix of the join of two regular graphs is a symmetric matrix which bears an eigenvalue with all-one vector as the corresponding eigenvector, which facilitates the use of Fiedler's Lemma in the investigation of its spectrum. Since the main theorems of this section are in the same frame work of Fiedlers Lemma, we avoid repetition in proofs, except Theorem 6.1, where the concept of Schur complement and Coronal of a square matrix are incorporated.

Lemma 6.1. [19] Let M, N, P, Q be matrices and let M be invertible. Let $S = \left[\begin{array}{cc} M & N \\ P & Q \end{array} \right]$, then $detS = detM.\ det(Q - PM^{-1}N)$.

Lemma 6.2. [4] Let A be an $n \times n$ matrix and $J_{n\times n}$ denote an all one matrix. Then,

$$det(xI_n - A - \alpha J_{n\times n}) = (1 - \alpha\Gamma_A(x)).det(xI_n - A),$$

where α is a real number.

Theorem 6.1. Let G_i be r_i-regular graph of order n_i, for $i = 1, 2$. The Seidel signless Laplacian polynomial of $G_1 \bigtriangledown G_2$ is

$$\Phi^+_{SL}(G_1 \bigtriangledown G_2; x) = \frac{(x-\kappa_1)(x-\kappa_2) - n_1n_2}{(x-\kappa_1)(x-\kappa_2)} \cdot \Phi^+_{SL}(G_1; x+n_2) \cdot \Phi^+_{SL}(G_2; x+n_1)$$

where, $\kappa_1 = 2n_1 - n_2 - 4r_1 - 2$ and $\kappa_2 = 2n_2 - n_1 - 4r_2 - 2$.

Proof. Let SL_1^+ and SL_2^+ denote the Seidel signless Laplacian matrices of G_1 and G_2 respectively. Both SL_1^+ and SL_2^+ are symmetric and it can be seen that their row sums are the constants $2(n_1 - 2r_1 - 1)$ and $2(n_2 - 2r_2 - 1)$ respectively. Also

$$\begin{aligned} SL^+(G_1 \bigtriangledown G_2) &= \left[\begin{array}{cc} SL_1^+ - n_2I_{n_1} & -J_{n_1\times n_2} \\ -J^T_{n_1\times n_2} & SL_2^+ - n_1I_{n_2} \end{array} \right] \\ &= \left[\begin{array}{cc} S_1 & -J_{n_1\times n_2} \\ -J^T_{n_1\times n_2} & S_2 \end{array} \right], \end{aligned}$$

where $S_1 = SL_1^+ - n_2 I_{n_1}$ and $S_2 = SL_2^+ - n_1 I_{n_2}$. Both S_1 and S_2 are symmetric and it can be easily seen that their row sums are the constants $\kappa_1 = 2(n_1 - 2r_1 - 1) - n_2$ and $\kappa_2 = 2(n_2 - 2r_2 - 1) - n_1$ respectively. Thus the coronal of S_1 and S_2 is found to be $\Gamma_{S_1}(x) = \dfrac{n_1}{x - \kappa_1}$ and $\Gamma_{S_2}(x) = \dfrac{n_2}{x - \kappa_2}$. Applying Lemma 6.2, and Lemma 6.1, it can be seen that

$$det(xI - SL^+(G_1 \bigtriangledown G_2)) = det(xI - S_1) \cdot det(xI - S_2) \cdot (1 - \Gamma_{S_1}(x)\Gamma_{S_2}(x)). \tag{10}$$

where $\Gamma_{S_1}(x) = \dfrac{n_1}{x - (n_1 - 2r_1 - 1)}$ and $\Gamma_{S_2}(x) = \dfrac{n_2}{x - (n_2 - 2r_2 - 1)}$.
Simplifying (10), we obtain the result. □

As in section:4, Theorem 3.2 can be applied to find the Seidel signless Laplacian spectrum of the join of regular graphs. Consider $G[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ with the Seidel matrix $S(G) = [s_{ij}]$ where $s_{ij} = -1$ if the vertices i and j are adjacent and $s_{ij} = 1$ if the vertices i and j are not adjacent and $s_{ij} = 0$ for the diagonal entries. Let H_j be r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $SL^+(H_j)$ denote the Seidel signless Laplacian matrix of $H_j, = 1, 2, ..., k$.

$$SL^+(G[H_1, H_2, ..., H_k]) = \begin{bmatrix} SL^+(H_1) + \tau_1 I_{n_1} & s_{1,2} J_{n_1 \times n_2} & \cdots & s_{1,k} J_{n_1 \times n_k} \\ s_{1,2} J^T_{n_1 \times n_2} & SL^+(H_2) + \tau_2 I_{n_2} & \cdots & s_{2,k} J_{n_2 \times n_k} \\ \vdots & & \ddots & \vdots \\ s_{1,k} J^T_{n_1 \times n_k} & s_{2,k} J^T_{n_2 \times n_k} & \cdots & SL^+(H_k) + \tau_k I_{n_k} \end{bmatrix} \tag{11}$$

where $\tau_j = \sum_{i=1}^k s_{ij} n_i$.

Theorem 6.2. Consider $G[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ and $S = [s_{ij}]_{k \times k}$ is the Seidel matrix of G and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $\{\sigma_{j1}^{SL^+} = 2(n_j - 2r_j - 1), \sigma_{j2}^{SL^+}, ..., \sigma_{jn_j}^{SL^+}\}$ be the Seidel signless Laplacian eigenvalues of H_j, for $j = 1, 2, ..., k$. Then, the Seidel signless Laplacian spectrum of the G-join of the graphs $H_1, H_2, ..., H_k$ is given by,

$$spec^{SL^+}(G[H_1, H_2, ..., H_n]) = \left(\bigcup_{j=1}^{k} \bigcup_{i=2}^{n_j} (\sigma_{ji}^{SL^+} + \tau_j) \right) \cup spec(T_{SL^+}(G))$$

where $\tau_j = \sum_{i=1}^k s_{ij} n_i$ and

$$T_{SL^+}(G) = \begin{bmatrix} 2(n_1 - 2r_1 - 1) + \tau_1 & s_{1,2} n_2 & \cdots & s_{1,k} n_k \\ s_{1,2} n_1 & 2(n_2 - 2r_2 - 1) + \tau_2 & \cdots & s_{2,k} n_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k} n_1 & s_{2,k} n_2 & \cdots & 2(n_k - 2r_k - 1) + \tau_k \end{bmatrix}$$

Proof. Since H_j is r_j-regular, $SL^+(H_j) = (n_j - 2r_j - 1)I_{n_j} + (S(H_j))$, where $S(H_j)$ is the Seidel matrix of H_j. By lemma 3.2, $n_j - 2r_j - 1$ is a Seidel eigenvalue of H_j with corresponding eigenvector $\mathbf{1}_{n_j}$. Thus $SL^+(H_j)$ has eigenvalue $2(n_j - 2r_j - 1)$ with corresponding eigenvector $\mathbf{1}_{n_j}$, for every $j = 1, 2, ..., k$. Hence, as evident from (11), the j^{th} diagonal block of the Seidel signless Laplacian matrix of $G[H_1, H_2, ..., H_n]$ is the symmetric matrix $SL^+(H_j) + \tau_j I_{n_j}$ which has an eigenvalue $2(n_j - 2r_j - 1) + \tau_j$ with eigenvector $\mathbf{1}_{n_j}$, for $j = 1, 2, ..., k$. Thus, taking

$$M_j = SL^+(H_j) + \tau_j I_{n_j}, \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = (2(n_j - 2r_j - 1) + \tau_j \ \ , \ \ \frac{1}{\sqrt{n_j}} \mathbf{1}_{n_j})$$

and the real numbers

$$\rho_{l,q} = s_{lq} \sqrt{n_l n_q}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, the result follows from Theorem 3.2. □

Example 6.1. Consider the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$, where p and q are distinct primes. The proper divisors of p^2q are $d_1 = p, \quad d_2 = q, \quad d_3 = p^2, \quad d_4 = pq$. The proper divisor graph of p^2q is the path P_4, where $p \sim pq \sim p^2 \sim q$. The subgraphs induced by $\mathbf{S}(p), \mathbf{S}(q), \mathbf{S}(p^2), \mathbf{S}(pq)$ are $\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{(q-1)}$ and K_{p-1} respectively. Hence,

$$\Gamma(\mathbb{Z}_{p^2q}) = P_4[\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{(q-1)}, K_{p-1}].$$

Applying Theorem 6.2, we see that, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^2q})$ is the multi set
$$\left\{ \begin{matrix} p^2+pq-3p-1 & p^2+pq-p-2q-1 & pq-p^2-p-1 & p^2-pq-p+3 \\ pq-p-q & p^2-p-1 & q-2 & 3-p \end{matrix} \right\}$$
together with the spectrum of the matrix,
$$T_{SL^+}(\Upsilon_{p^2q}) =$$
$$\begin{bmatrix} p^2+2pq-4p-q & p^2-p & q-1 & 1-p \\ (p-1)(q-1) & 2p^2+pq-2p-2q-1 & 1-q & p-1 \\ (p-1)(q-1) & p-p^2 & pq-p^2-p+q-2 & 1-p \\ -(p-1)(q-1) & p^2-p & 1-q & p^2-pq-2p+4 \end{bmatrix}.$$

Conclusion

The induced subgraphs which are either cliques or null graphs, shape the combinatorial structure of the zero divisor graph on the ring $\mathbb{Z}_n$. The all-one matrix blocks constituting the matrix described in this paper, facilitates the use of Fiedler's lemma in the computation of the spectrum.

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