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Some New Results on Equivalent Cauchy Sequences and Their Applications to Meir-Keeler Contraction in Partial Rectangular Metric Space

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Abstract The study of fixed points in the metric spaces plays a crucial role in the development of Functional Analysis. It is evolved by generalizing the metric space or improving the contractive conditions. Recently, the partial rectangular metric space and its topology have been the center of study for many researchers. They have defined open and closed balls the equivalent Cauchy sequences and Cauchy sequences, convergent sequences which are used as tools in many achieved results. In this paper, two facts for equivalent Cauchy sequences in a partial rectangular metric space are provided by using an ultra - altering distance function. Furthermore, some results of Cauchy sequences in a partial rectangular metric space are highlighted. There is proved that under some conditions the equivalent Cauchy sequences are Cauchy sequences in a partial rectangular metric space. Some fixed point results have been taken as applications of our new conditions of Cauchy sequences and equivalent Cauchy sequences in a partial rectangular metric space (X, p) for orbitally continuous functions $f : X \to X$. To illustrate the obtained results some examples are given.

Keywords Partial Rectangular Metric Space, Equivalent Cauchy Sequences, Cauchy Sequences, Fixed Point

1 Introduction

The theory of the fixed point depends on either generalizing the metric type space or the contractive type mapping [1]. Firstly, the generalization of a metric space is based on reducing or modifying the metric axioms, for example, quasi-metrics, b-metrics, partial metrics, rectangular metrics, etc., [2], [3], [4], [5].

Secondly, many researchers have generalized Banach's contraction principle in metric spaces and so in the generalized metric spaces [6], [7], [8].

In [9] Matthew introduced a new abstract space called partial metric space. The partial metric space is a generalization of the usual metric spaces in which the distance of a point from itself may not be zero. In these spaces, related to the theory of fixed point many authors have given their contribution such as Došenović and Radenović [10], Radenović [11], Kirk and Shahzad [1].

Another researcher Branciari [12] introduced a generalized metric, which is called a rectangular metric by replacing triangular inequality with similar one which involves four or more points instead of three points.

In addition, interesting results in rectangular metric space are obtained by Hiu Sheng Ding et.al [13], George et.al [14]. Recently Shulka in [15] generalized the concept of rectangular metric space and extended the concept of partial metric space by introducing the partial rectangular metric space.

The concept of Cauchy sequences and equivalent Cauchy sequences are very important in functional analysis and especially in fixed point theory.

In 1983 Leader [16] obtained a sufficient and necessary condition as a characterization of equivalent Cauchy sequences.

In [17] and [18], are proved some conditions for equivalent Cauchy sequences and Cauchy sequences in partial metric spaces.

This research shows some new conditions for equivalent Cauchy sequences, some new results for Cauchy sequences in partial rectangular metric spaces. Furthermore, as applications of these results, we prove a fixed point theorem in partial rectangular metric spaces.

2 Preliminaries

For convenience we commence with the following definitions, lemmas and theorems.

Definition 2.1. [13] Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a function. The pair (X, d) is called a rectangular metric space if the following conditions are held for all $x, y \in X$.

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all distinct points $u, v \in X \setminus \{x, y\}$.

Definition 2.2. [19] Let (X, d) be a rectangular metric space.

1. A sequence $\{x_n\}$ is called convergent to a point $x \in X$ if $\lim_{n\to+\infty} d(x_n, x) = 0$, written $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$;
2. A sequence $\{x_n\}$ is called Cauchy sequence if $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$;
3. (X, d) is called complete if each Cauchy sequence $\{x_n\}$ converges in X.

Definition 2.3. [15] Let X be a non-empty set and $p : X \times X \to \mathbb{R}$ be a function. The function p is called a partial metric on X if for all $x, y \in X$, the following conditions hold:

1. $p(x, y) \geq 0$;
2. $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
3. $p(x, x) \leq p(x, y)$;
4. $p(x, y) = p(y, x)$;
5. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X.

Definition 2.4. [19] Let X be a non-empty set and $p : X \times X \to \mathbb{R}$ be a function. Then p is called a partial rectangular metric on X if the following condition hold for all $x, y \in X$.

1. $p(x, y) \geq 0$;
2. $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
3. $p(x, x) \leq p(x, y)$;
4. $p(x, y) = p(y, x)$;
5. $p(x, y) \leq p(x, u) + p(u, v) + p(v, y) - p(u, u) - p(v, v)$ for all distinct points $u, v \in X \setminus \{x, y\}$.

The pair (X, p) is called a partial rectangular metric space.

Definition 2.5. Let (X, p) be a partial rectangular metric space.

1. A subset $A \subset X$ in a $p-$rectang partial rectangular metric space is bounded if there exists $M > 0$ such that $p(x, y) \leq M$ for every $x, y \in A$.
2. If A is a bounded subset of X, then the diameter of A is denote by $\delta(A)$, where $\delta(A) = \sup\{p(x, y) | x, y \in A\}$.

Definition 2.6. [19] Let (X, p) be a partial rectangular metric space.

1. A sequence $\{x_n\}$ is called convergent to a point $x \in X$ if $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$, written as $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$.
2. A sequence is called Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m)$ exists and it is finite.
3. (X, p) is called complete if for each Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\lim_{n,m\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x) = p(x, x)$.

Definition 2.7. [15] Let (X, p) be a partial rectangular metric space.

1. The sequence $\{x_n\}$ is called 0-Cauchy sequence in (X, p) if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$.
2. (X, p) is called 0-complete if for each 0-Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that $\lim_{n,m\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x) = p(x, x) = 0$.

Lemma 2.8. *[15] Let (X, p) be a partial rectangular metric space.*

1. *If (X, p) is complete, then it is 0-complete.*
2. *If $x_n \neq x, y_n \neq y, x_n \neq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = x$, $\lim_{n\to+\infty} y_n = y$, $p(x, x) = p(y, y) = 0$, then $\lim_{n\to+\infty} p(x_n, y_n) = p(x, y)$.*

Note that every rectangular metric space is a partial rectangular metric space. The converse of this fact does not hold. However, in [15] was shown that every partial rectangular metric space induces a rectangular metric space.

Theorem 2.9. *[15] Let (X, p) be a partial rectangular metric space and*

$$p_r(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$. Then, it yields:

1. *(X, p_r) is a rectangular metric space.*
2. *The sequence $\{x_n\}$ converges in x in (X, p) if and only if $\{x_n\}$ converges in x in (X, p_r).*
3. *The sequence $\{x_n\}$ is Cauchy in (X, p) if and only if $\{x_n\}$ is Cauchy in (X, p_r).*

Definition 2.10. [19] The sequences $\{x_n\}$ and $\{y_n\}$ in a rectangular metric space (X, d) are called equivalent if $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

Definition 2.11. [19] The sequences $\{x_n\}$ and $\{y_n\}$ in a partial rectangular metric space (X, p) are called equivalent if $\lim_{n\to+\infty} p(x_n, y_n)$ exists and it is finite.

Definition 2.12. [19] The sequences $\{x_n\}$ and $\{y_n\}$ in a partial rectangular metric space (X, p) (rectangular metric space (X, d)) are equivalent Cauchy if they are Cauchy and equivalent in (X, p) ((X, d)).

In [20] is given the following Lemma.

Lemma 2.13. *Let (X, p) be a partial metric space. A sequence $\{x_n\}$ is a Cauchy sequence in partial metric space (X, p) if and only if it accomplishes the following condition:*
For each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) - p(x_n, x_n) < \varepsilon$ whenever $n_0 < n < m$.

In [21] are defined the altering distance between the points as the following definition.

Definition 2.14. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if:

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.
3. $\varphi(t) > Mt^\mu$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constants.

Definition 2.15. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an ultra-altering distance function if:

1. φ is non-decreasing;
2. $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) = 0$;
3. $\varphi(t) < t$, for every $t \in \mathbb{R}^+$.

The set of ultra-altering distance functions is denoted by Φ.

3 Main Results

Let (X, p) be a partial rectangular metric space and (X, p_r) the rectangular metric space induced by (X, p).

Lemma 3.1. *Let (X, p) be a partial rectangular metric space. A sequence $\{x_n\}$ is a Cauchy sequence in partial rectangular metric space (X, p), if and only if it satisfies the following condition:*
For each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$p(x_n, x_m) - p(x_n, x_n) < \varepsilon, \quad \text{whenever } n_0 < n < m \quad (*)$$

The proof of the Theorem is similarly with Lemma 1.9 in [20].

Theorem 3.2. *If the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy in (X, p_r), then they are equivalent Cauchy in (X, p).*

Proof: Since (x_n) and (y_n) are equivalent Cauchy in (X, p_r), there is that $\lim_{n\to\infty} p_r(x_n, y_n) = 0$.

From the above result, it is

$$\lim_{n\to+\infty} p_r(x_n, y_n) = \\ = \lim_{n\to+\infty} (2p(x_n, y_n) - p(x_n, x_n) - p(y_n, y_n)) = 0 \quad (3.1)$$

Since $p(x_n, x_n) \le p(x_n, y_n)$ and $p(y_n, y_n) \le p(x_n, y_n)$, it results

$$\lim_{n\to+\infty} (p(x_n, y_n) - p(x_n, x_n)) = 0$$

and

$$\lim_{n\to+\infty} (p(x_n, y_n) - p(y_n, y_n)) = 0 \quad (3.2)$$

By theorem 2.9, since the sequences (x_n) and (y_n) are Cauchy in (X, p_r), they are Cauchy and in (X, p).

Let prove that they are and equivalent in (X, p).

By the lemma 3.1, since (x_n) and (y_n) are Cauchy in (X, p), they satisfy the condition (*). So, in the same way as in the proof of lemma 1.9 in [2], we can proof that sequences $\{p(x_n, x_n)\}$ and $\{p(y_n, y_n)\}$ converge according to Euclidean metric on $\mathbb{R}^+$.

Define $\lim_{n\to+\infty} p(x_n, x_n) = a$ and $\lim_{n\to+\infty} p(y_n, y_n) = b$.

Note that $|p(x, x) - p(y, y)| \le p_r(x, y)$ for all $x, y \in X$. Thus, for $x = x_n$ and $y = y_n$

$$|p(x_n, x_n) - p(y_n, y_n)| \le p_r(x_n, y_n)$$

By the equivalence of the sequences (x_n) and (y_n) in (X, p_r), we have $\lim_{n\to+\infty} p_r(x_n, y_n) = 0$ and so

$$\lim_{n\to+\infty} p(x_n, x_n) = \lim_{n\to+\infty} p(y_n, y_n) = a = b. \quad (3.3)$$

From (3.2) and (3.3), we have

$$\lim_{n\to+\infty} p(x_n, y_n) = \lim_{n\to+\infty} p(x_n, x_n) = \lim_{n\to+\infty} p(y_n, y_n) = a$$

So, the sequences (x_n) and (y_n) are equivalent in (X, p). Since they are Cauchy sequences in (X, p), we conclude that sequences (x_n) and (y_n) are equivalent Cauchy in (X, p). ■

The following Example shows that the converse of Theorem 3.2 does not hold.

Example 3.3. Let $X = [0, a]$ and $b \ge a \ge 3$ and

$$p(x, y) = \begin{cases} x, & x = y; \\ \frac{3b+x+y}{2}, & x, y \in \{1, 2\}, x \ne y; \\ \frac{b+x+y}{2}, & \text{otherwise} \end{cases}$$

Then p is a partial rectangular metric but not a partial metric by [19].

Take the sequences $x_n = 1$ for $n > 1$ and $y_n = 1 + \frac{1}{n}$ for $n > 2$.

These sequences are Cauchy, because $p(x_n, x_m) = p(1, 1) = 1$ and

$$\begin{aligned} p(y_n, y_m) &= \frac{b + y_n + y_m}{2} = \\ &= \frac{b}{2} + \frac{1}{2} + \frac{1}{2n} + \frac{1}{2} + \frac{1}{2m} \to \frac{b}{2} + 1, \\ &\quad (n, m \to +\infty) \end{aligned}$$

In addition, they are equivalent Cauchy in (X, p) because

$$\lim_{n\to+\infty} p(x_n, y_n) = \lim_{n\to+\infty} \left(\frac{b}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2n}\right) = \frac{b}{2} + 1$$

Even though the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in (X, p_r), by definition 2.10, they are not equivalent Cauchy in (X, p_r) because

$$\begin{aligned} \lim_{n\to+\infty} p(x_n, x_n) &= 1, \\ \lim_{n\to+\infty} p(y_n, y_n) &= \lim_{n\to+\infty} \left(\frac{b}{2} + \frac{1}{2} + \frac{1}{2n} + \frac{1}{2} + \frac{1}{2n}\right) \\ &= \frac{b}{2} + 1 \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to+\infty} p_r(x_n, y_n) = \lim_{n\to+\infty} [2p(x_n, y_n) - p(x_n, x_n) - p(y_n, y_n)] \\ &= \lim_{n\to+\infty} \left[2p\left(1, 1 + \frac{1}{n}\right) - p(1, 1) - p\left(1 + \frac{1}{n}, 1 + \frac{1}{n}\right)\right] \\ &= b \neq 0 \end{aligned}$$

Remark 3.4. If the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy in partial rectangular metric space (X, p), and

$$\lim_{n\to+\infty} p(x_n, x_n) = \lim_{n\to+\infty} p(y_n, y_n) = \lim_{n\to+\infty} p(x_n, y_n)$$

then the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy in (X, p_r).

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in the partial rectangular metric space (X, p).

Define $\delta_{i,j} = \sup\{p(x_n, x_n), p(y_m, y_m), p(x_n, y_m) | n \geq i, m \geq j\}$.

Theorem 3.5. *If $\{x_n\}$ and $\{y_n\}$ are two bounded sequences in the partial rectangular metric space (X, p), where*

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \lim_{n\to+\infty} p(y_n, y_{n+1}) = 0$$

and satisfy one of the following conditions:

1. *For each $\varepsilon > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$ and $\varepsilon_0 \in (0, \varepsilon)$ such that*
$$\varphi(\delta_{i,j}) \leq \varepsilon + \delta$$
implies
$$p(x_{i+r}, y_{j+r}) \leq \varepsilon_0,$$
where $\varphi \in \Phi$ and $i, j \in \mathbb{N}$.

2. *For each $\varepsilon > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$ such that*
$$\varepsilon \leq \psi(\delta_{i,j}) - \varphi(\delta_{i,j}) < \varepsilon + \delta$$
implies
$$\psi(\delta_{i+r,j+r}) < \varepsilon$$
where $\varphi, \psi \in \Phi$, ψ is continuous, φ is lower semi continuous function and $i, j \in \mathbb{N}$,

then they are equivalent Cauchy.

Proof. Firstly, let prove theorem when the first condition is hold.

Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in the partial rectangular metric space (X, p). Define

$$a_n = \delta_{n,n} = \sup\{p(x_i, x_i), p(y_i, y_i), p(x_i, y_j) | i \geq n, j \geq n\} < +\infty$$

since the sequence $\{x_n\}$ and $\{y_n\}$ are bounded.

The sequence $\{a_n\}$ is decreasing and positive. Hence, it converges and

$$\lim_{n\to+\infty} a_n = \inf\{a_n | n \in \mathbb{N}\} = a \geq 0$$

Now we prove $a = 0$. Suppose that $a > 0$. From the condition 1), for $\varepsilon = a$, there is $\delta > 0$, $r \in \mathbb{N}$, $\varepsilon_0 \in (0, a)$ such that $\varphi(\delta_{i,j}) \leq a + \delta$ implies $p(x_{i+r}, y_{j+r}) \leq \varepsilon_0 < a$, whenever $i, j \in \mathbb{N}$.

For this $\delta > 0$ and also from the fact that $a = \inf\{a_n | n \in \mathbb{N}\}$, there exists $P \in \mathbb{N}$ such that for $n > P$, we have $a_n < a + \delta$.

For $i \geq P, j \geq P$, we have $\delta_{i,j} \leq \delta_{p,p} = a_p < a + \delta$.

For $i \geq P, j \geq P$, by the properties of function φ and condition 1), we have

$$\varphi(\delta_{i,j}) \leq \varphi(a + \delta) < a + \delta,$$

implies

$$p(x_{i+r}, y_{j+r}) \leq \varepsilon_0 < a$$

But, it is obvious that $k = i + r \geq P + r > P$, $l = j + r \geq P + r > P$ and $p(x_k, y_l) \leq \varepsilon_0 < a$. So,

$$a_p = \delta_{P,P} = \sup\{p(x_k, x_k), p(y_l, y_l), p(x_k, y_l) | k \geq P, l \geq P\} \leq \varepsilon_0 < a = \inf\{a_n | n \in \mathbb{N}\}$$

which is a contradiction. Thus $a = 0$.

Since $0 \leq p(x_n, y_n) \leq a_n$ and $\lim_{n\to+\infty} a_n = 0$ hold, then

$$\lim_{n\to+\infty} p(x_n, y_n) = 0 \tag{3.4}$$

which shows that the sequences $\{x_n\}$ and $\{y_n\}$ are equivalent in (X, p).

Now we notice that sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in (X, p).

For every $i, j \in \mathbb{N}$, $0 \leq p(x_i, y_j) \leq a_{min\{i,j\}}$ and consequently it is implied

$$\lim_{i,j\to+\infty} p(x_i, y_j) = 0 \tag{3.5}$$

Since $p(x_n, x_n) \leq p(x_n, y_n)$, $p(y_n, y_n) \leq p(x_n, y_n)$ and $\lim_{n\to+\infty} p(x_n, y_n) = 0$, then

$$\lim_{n\to+\infty} p(x_n, x_n) = 0 \text{ and } \lim_{n\to+\infty} p(y_n, y_n) = 0 \quad (3.6)$$

If for every $n \in \mathbb{N}, x_n = x_{n+1}$ then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is constant. Consequently, $x_n = x$ for each $n \in \mathbb{N}$ and $\lim_{n,m\to+\infty} p(x_n, x_m) = p(x, x)$.

Since $\lim_{n\to+\infty} p(x_n, x_n) = 0 = p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

Using the same method, it can be proved that if for every $n \in \mathbb{N}, y_n = y_{n+1}$, then the sequence $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy.

If the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are not constant, then for $k \in \mathbb{N}$, there exits $n > k$ such that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$.

As a result, for $n > k, m \in \mathbb{N}$

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, y_n) + p(y_n, y_{n+1}) + p(y_{n+1}, x_m) \\ &\quad - p(y_n, y_n) - p(y_{n+1}, y_{n+1}) \\ p(y_n, y_m) &\leq p(y_n, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, y_m) \\ &\quad - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \end{aligned}$$

By (3.4), (3.5), (3.6) and the condition in theorem, we have

$$\lim_{m,n\to+\infty} p(x_n, x_m) = 0 \text{ and } \lim_{m,n\to+\infty} p(y_n, y_m) = 0$$

Consequently, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in (X, p). ■

Secondly, we prove the theorem when the second condition is hold. Let $\{x_n\}, \{y_n\}$ be two bounded sequences ne (X, p). Using the same method as we prove the theorem for condition 1), is proved that the sequence $\{a_n\}$ is convergent and $\lim_{n\to+\infty} a_n = a \geq 0$, where $a_n = \delta_{n,n} = \sup\{p(x_i, x_i), p(y_j, y_j), p(x_i, y_j) | i \geq n, j \geq n\} < +\infty$.

Let us show $a = 0$. Suppose that $a > 0$. From the condition 2), we have for $\varepsilon = a > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$ such that $a < \psi(\delta_{i,j}) - \varphi(\delta_{i,j}) \leq a + \delta \Rightarrow \psi(\delta_{i+r,j+r}) < a$, whenever $i, j \in \mathbb{N}$. Then for $i, j \in \mathbb{N}$,

$$\psi(\delta_{i+r,j+r}) < a < \psi(\delta_{i,j}) - \varphi(\delta_{i,j}) \leq a + \delta \quad (3.7)$$

Since $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \delta_{n,n} = a$, we have that for every $i, j \in \mathbb{N}$, $\delta_{\max\{i,j\}} \leq \delta_{i,j} \leq \delta_{\min\{i,j\}}$. Knowing that $\lim_{n\to+\infty} \delta_{\max\{i,j\}} = \lim_{n\to+\infty} \delta_{\min\{i,j\}} = a$, we have that $\lim_{n\to+\infty} \delta_{i,j} = a$.

Now it is clear that $\lim_{i,j\to+\infty} \delta_{i,j} = a$. So, for $r \in \mathbb{N}$ even $\lim_{i,j\to+\infty} \delta_{i+r,j+r} = a$.

From the continuity of the function ψ, and from the lower semi continuity of the function φ taking the limit in (3.7) we have $\psi(a) \leq \psi(a) - \varphi(a)$. Thus $\varphi(a) = 0$ and from the properties of the function φ, we have $a = 0$. So, $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \delta_{n,n} = 0$. Hence $\lim_{n\to+\infty} p(x_n, y_n) = 0$. Consequently $\{x_n\}$ and $\{y_n\}$ are equivalent. ■

Using the same method in the proof of theorem for condition 1) $\{x_n\}$ and $\{y_n\}$ are Cauchy in (X, p).

Remark 3.6. Fulfilling the conditions 1) and 2) ensures that $\{x_n\}$ and $\{y_n\}$ are equivalent. Adding the condition that $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$ and $\lim_{n\to+\infty} p(y_n, y_{n+1}) = 0$, there is assured that $\{x_n\}$ and $\{y_n\}$ are Cauchy.

Remark 3.7. The proof of theorem 3.5, emphases that if the sequences $\{x_n\}$ and $\{y_n\}$ fulfill the conditions of the theorem they are $0-$equivalent and $0-$Cauchy.

Remark 3.8. Generally, the converse of the theorem is not true. This means that there exist equivalent Cauchy sequences in (X, p), which cannot fulfill any of the conditions in the theorem. This can be seen in example 3.3. There is shown that the sequences $x_n = 1$ and $y_n = 1 + \frac{1}{n}$, for $n \geq 2$ are equivalent Cauchy in (X, p). On the other hand, these sequences do not satisfy any of conditions 1) and 2), because otherwise $\lim_{n\to+\infty} p(x_n, y_n) = 0$, while $\lim_{n\to+\infty} p(x_n, y_n) = b \neq 0$. Similarly, these sequences do not accomplish even the condition 2). $\lim_{n\to+\infty} p(x_n, x_{n+1}) = \lim_{n\to+\infty} p(y_n, y_{n+1}) = 0$, while $\lim_{n\to+\infty} p(x_n, x_{n+1}) = p(1, 1) = 1 \neq 0$ and $\lim_{n\to+\infty} p(y_n, y_{n+1}) = \frac{b}{2} + 1 \neq 0$.

Remark 3.9. The conditions 1 and 2 of theorem 3.5, do not imply that the sequences $\{x_n,\}_{n\in\mathbb{N}}$ and $\{y_n,\}_{n\in\mathbb{N}}$ are bounded.

Indeed. For every pair of sequences $\{x_n,\}_{n\in\mathbb{N}}$ and $\{y_n,\}_{n\in\mathbb{N}}$ such that $\delta_{i,j} = +\infty$, the condition 1 and 2 of theorem 3.5 are completed because for each $\varepsilon > 0$, for every choice of $\delta > 0, r \in \mathbb{N}, \varepsilon_0 \in]0, \varepsilon[$, there do not exists any pair $i, j \in \mathbb{N}$, such that $\varphi(\delta_{i,j}) < \delta_{i,j} \leq \varepsilon + \delta$ or

$$\varepsilon \leq \psi(\delta_{i,j}) - \varphi(\delta_{i,j}) \leq \psi(\delta_{i,j}) < \delta_{i,j} < \varepsilon + \delta$$

Let $\{x_n\}$ be a sequence in a partial rectangular metric space (X, p). Define

$$\delta_{i,j} = \max\{p(x_n, x_n), p(x_n, x_m) | n \geq i, m \geq j\}$$

for $i, j \in \mathbb{N}$.

Theorem 3.10. *If the sequence $\{x_n\}$ in a partial rectangular metric space (X, p) is bounded, where $\lim_{n\to+\infty} p(x_n, x_{n+1}) = a$ and it satisfies one of the following conditions:*

1. *For each $\varepsilon > 0$, there is a $\delta > 0$, $r \in \mathbb{N}$ and $\varepsilon_0 \in (0, \varepsilon)$ such that*

$$\varphi(\delta_{i,j}) \leq \varepsilon + \delta \quad \textit{implies} \quad p(x_{i+r}, x_{j+r}) \leq \varepsilon_0$$

where $\varphi \in \Phi$ and $i, j \in \mathbb{N}$.

2. *For each $\varepsilon > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$, such that*

$$\varepsilon \leq \psi(\delta_{i,j}) - \varphi(\delta_{i,j}) < \varepsilon + \delta \quad \textit{implies} \quad \psi(\delta_{i+r,j+r}) < \varepsilon$$

where $\varphi, \psi \in \Phi$, ψ is continuous, φ is lower semi continuous function and $i, j \in \mathbb{N}$,

then it is a Cauchy sequence.

Proof. Substituting $y_j = x_j$ in theorem 3.5 we can prove the theorem and that

$$\lim_{n\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x_n) = 0$$

. ■

Remark 3.11. The converse of theorem 3.10 is not true. This means that there exist Cauchy sequences in (X, p), which cannot fulfill none of the conditions of theorem 3.5. As example, we can see again the sequence $x_n = 1 + \frac{1}{n}$ for $n \geq 2$ in the space (X, p) of the example 3.3 in which is shown that the sequence $\{x_n\}$ is a Cauchy sequence in (X, p).

But, this sequence cannot satisfy any of conditions 1) and 2) because otherwise

$$\lim_{n\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x_n) = 0$$

while

$$\lim_{n\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, x_n) = \frac{b}{2} + 1 \neq 0$$

4 Fixed point theorem in a partial rectangular metric space

Meir and Keeler [22] have given the following theorem on fixed points for a contractive function in metric space.

Theorem 4.1. *[22] Let (X, d) be a metric space and $T : X \to X$ a function which satisfies the contractive condition:*
For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon$$

Many researchers have generalized Meir-Keeler contraction and have proved it in different metric spaces.

The following theorem which is given as an application of the obtained results in the above paragraphs, is a generalization of Meir-Keeler contraction in partial rectangular metric space.

Theorem 4.2. *[22] Let (X, d) be a complete metric space and $T : X \to X$ be a function. If T satisfies the following condition:*
If for a given $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in X$

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

then it has a unique fixed point in X.

Definition 4.3. Let (X, p) be a partial rectangular metric space and $T : X \to X$ a self-mapping on X. The orbit of x of function T is defined by

$$O(x) = \{x, Tx, T^2x, \cdots, T^n x \cdots\}$$

and

$$\delta_{i,j}(O(x) \cup O(y)) = \sup\{p(T^m x, T^n x), p(T^m y, T^k y), p(T^k y, T^q y) | m, n \geq i, k, q \geq j\}$$

for $i, j \in \mathbb{N}$ and $x, y \in X$.

Definition 4.4. [20] Let (X, p) be a partial rectangular metric space. $T : X \to X$ is called orbitally continuous if

$$\lim_{i,j\to+\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to+\infty} p(T^{n_i}x, z) = p(z, z)$$

implies

$$\lim_{i,j\to+\infty} p(TT^{n_i}x, TT^{n_j}x) = \lim_{i\to+\infty} p(TT^{n_i}x, Tz) = p(Tz, Tz), \text{ for each } x \in X$$

Equivalently, T is orbitally continuous if $T^{n_i}x \to z$ for $n_i \to +\infty$ with respect to p_r and $T^{n_i+1}x \to Tz$ with respect to p_r for each $x \in X$ as $n_i \to +\infty$.

Theorem 4.5. *Let (X, p) be a complete partial rectangular metric space and $T : X \to X$ orbitally continuous. If T satisfies one of the following conditions:*

1. *For each $x, y \in X$ such that the sequences $\{T^n x\}$ and $\{T^n y\}$ are bounded, and for every $\varepsilon > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$ and $\varepsilon_0 \in (0, \varepsilon)$ such that*
$$\varphi(\delta_{i,j}(O(x) \cup O(y))) \leq \varepsilon + \delta$$
implies
$$p(T^{i+r}x, T^{j+r}y) \leq \varepsilon_0,$$
where $\varphi \in \Phi$ and $i, j \in \mathbb{N}$

2. *For each $x, y \in X$ such that the sequences $\{T^n x\}$ and $\{T^n y\}$ are bounded, and for every $\varepsilon > 0$, there exist $\delta > 0$, $r \in \mathbb{N}$ such that*
$$\varepsilon \leq \psi(\delta_{i,j}(O(x) \cup O(y))) - \varphi(\delta_{i,j}(O(x) \cup O(y))) < \varepsilon + \delta$$
implies
$$\psi(\delta_{i+r,j+r}(O(x) \cup O(y))) < \varepsilon,$$
where $\varphi, \psi \in \Phi$, ψ is continuous, φ is lower semi continuous function and $i, j \in \mathbb{N}$,

then T has a unique fixed point $u \in X$, and for each $x \in X$, $\lim_{n\to+\infty} T^n x = u$.

Proof. Take $x \in X$. Let $\{x_n\}$ be an iterative sequence defined by $x_{n+1} = Tx_n$, for each $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$, which is $x_{n_0} = x_{n_0+1}$, then x_{n_0} is fixed point of T.

Assume then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Substituting $x = x_n$ and $y = x_{n+1}$ in the definition 4.3, we have

$$\begin{aligned}\delta_{i,j}(O(x_n) \cup O(x_{n+1})) &= \sup\{p(T^m x_n, T^t x_n), \\ &\quad p(T^k x_{n+1}, T^q x_{n+1}), p(T^m x_n, T^k x_{n+1}) | \\ &\quad m, t \geq i, k, q \geq j\} \\ &= \sup\{p(T^{m+n}x, p(T^{t+n}x), \\ &\quad p(T^{k+n+1}x, T^{q+n+1}x), \\ &\quad p(T^{m+n}x, T^{k+n+1}x) | m, t \geq i, k, q \geq j\} \\ &= \delta_{i+n,j+n+1}(O(x))\end{aligned}$$

Also, $p(T^{i+r}x_n, T^{j+r}x_{n+1}) = p(x_{n+i+r}, x_{n+j+r+1})$.

If T satisfies the condition 1) in the theorem for $x = x_n$ and $y = x_{n+1}$, we have that there is $\varphi \in \Phi$ such as for each $\varepsilon > 0$, there are $\delta > 0$, $r \in \mathbb{N}$ and $\varepsilon_0 \in (0, \varepsilon)$ which satisfy

$$\varphi(\delta_{i,j}(O(x) \cup O(y))) = \varphi(\delta_{i+n,j+n+1}(O(x))) \leq \varepsilon + \delta$$

implies

$$p(T^{i+r}x, T^{j+r}y) = p(x_{n+i+r}, x_{n+j+r+1}) \leq \varepsilon_0$$

where $i, j \in \mathbb{N}$.

Eventually, it can be seen that the sequence $\{x_n\}$ fulfills the condition 1) in the theorem 3.10, thus $\{x_n\}$ is Cauchy sequence.

Suppose that T satisfies the condition 2). Therefore, for $x = x_n$ and $y = x_{n+1}$ there exist $\varphi, \psi \in \Phi$ in which ψ is continuous, φ is lower semi continuous and for every $\varepsilon > 0$, exist $\delta > 0$, $r \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &\leq \psi(\delta_{i,j}(O(x) \cup O(y))) - \varphi(\delta_{i,j}(O(x) \cup O(y))) \\ &= \psi(\delta_{i+n,j+n+1}(O(x))) - \varphi(\delta_{i+n,j+n+1}(O(x))) \\ &< \varepsilon + \delta \end{aligned}$$

implying

$$\psi(\delta_{i+r,j+r}(O(x) \cup O(y))) = \psi(\delta_{i+n,j+n+1}(O(x))) < \varepsilon$$

Consequently, $\{x_n\}$ satisfies the conditions 2) of theorem 3.10, therefore $\{x_n\}$ is Cauchy sequence and $\lim_{n,m \to +\infty} p(x_n, x_m) = \lim_{n \to +\infty} p(x_n, x_n) = 0$. Since (X, p) is complete we deduce that $\{x_n\}$ converges to some $u \in X$ and

$$\lim_{n,m \to +\infty} p(x_n, x_m) = \lim_{n \to +\infty} p(x_n, x_n) = p(u, u) = 0$$

Due to theorem 2.9 we have that the sequence $\{x_n\}$ converges to u and with respect rectangular metric p_r so $\lim_{n \to +\infty} p_r(x_n, u) = 0$.

Since the function T is orbitally continuous we deduce that

$$\lim_{n \to +\infty} p_r(Tx_n, Tu) = \lim_{n \to +\infty} p_r(x_{n+1}, Tu) = p_r(u, Tu) = 0$$

So $Tu = u$. Therefore, u is a fixed point of T in X.

Furthermore, let's prove that for every $y \neq x$ and $y_n = T^n y$ converges to u.

The sequence $y_n = T^n y$ completes the conditions $1), 2)$ in theorem, so it is Cauchy sequence in (X, p) and $\lim_{n,m \to +\infty} p(y_n, y_m) = \lim_{n \to +\infty} p(y_n, y_n) = 0$. Since (X, p) is complete it deduces that $\{y_n\}$ converges to some $v \in X$ and $\lim_{n,m \to +\infty} p(y_n, y_m) = \lim_{n \to +\infty} p(y_n, y_n) = p(v, v) = 0$. The sequences $\{x_n\}$ and $\{y_n\}$ satisfy the condition of theorem 3.5, therefore they are equivalent Cauchy sequence and $\lim_{n \to +\infty} p(x_n, y_n) = 0$. By lemma 2.8, there is $\lim_{n \to +\infty} p(x_n, y_n) = p(u, v) = 0$, and by theorem 2.9 there is $\lim_{n \to +\infty} p_r(x_n, y_n) = p_r(u, v) = 0$ which implies that $u = v$, and the fixed point of T is unique. ■

Below, there is given an example which verifies that theorem 4.5 is applicable.

Example 4.6. Let $X = \{\frac{1}{n} : n \in \mathbb{N} \cup \{0, 3\}\}$. Define the map $p : X \times X \to \mathbb{R}^+$ such that

$$p(x, y) = \begin{cases} x, & x = y \\ 4, & x, y \in \{1, 3\} \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

The function $p(x, y)$ is not a partial metric because

$$\begin{aligned} p(1, 3) = 4 &> p\left(1, \frac{1}{n}\right) + p\left(\frac{1}{n}, 3\right) - p\left(\frac{1}{n}, \frac{1}{n}\right) \\ &= 1 + 3 - \frac{1}{n} = 4 - \frac{1}{n}. \end{aligned}$$

Furthermore, it is not a rectangular metric since $p(x, x) = x \neq 0$, for every $x \neq 0$.

However, $p(x, y)$ is a partial rectangular metric because it accomplishes the conditions of the definition 4 [19].

Let $T : X \to X$ be a function defined by

$$Tx = \begin{cases} 0, & x = 0 \\ \frac{1}{n+1}, & x = \frac{1}{n}, n \in \mathbb{N} \\ 1, & x = 3 \end{cases}$$

The function T satisfies the condition (1) of the theorem 4.5, where $\varphi(t) = \frac{t}{2}$.

Below, it is shown that $\varphi \in \Phi$.

1. For all $t > 0, \varphi(t) = t/2 < t$ and $\varphi(0) = 0$,
2. φ is a non-decreasing function.

In addition, the function T satisfies the contractive condition (1) of the theorem 4.5.

Indeed, for each $x \in X, O(x)$ is bounded because X is bounded.

For $x = 0$, it is yielded $O(0) = \{0\}$.

For $x = \frac{1}{n}, n \in \mathbb{N}, O(x) = \{\frac{1}{n}, \frac{1}{n+1}, \ldots\}$.

For $x = 3, O(3) = \{3, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$.

Let consider the following cases.

Case 1: $x = 3, y = \frac{1}{n}, n \in \mathbb{N}$

$$\begin{aligned} O(3) \cup O(\frac{1}{n}) &= \{3, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\} \\ \delta_{ij}\left(O(3) \cup O(\frac{1}{n})\right) &= \max\left\{\frac{1}{i}, \frac{1}{j}\right\}, 2 < i < j \end{aligned}$$

For $0 < \varepsilon \leq \frac{1}{2}$, there exist $k \in \mathbb{N}, k \geq 2$, $r > 2$, such that $\frac{1}{k+1} < \varepsilon \leq \frac{1}{k}, \delta = \frac{1}{k-1} - \varepsilon, \varepsilon_0 = \frac{1}{k+1}$, $i, j \in \mathbb{N}$ and $\min\{i, j\} > \max\{k-1, 2\}$, where

$$\begin{aligned} \delta_{ij}\left(O(3) \cup O(\frac{1}{n})\right) &= \max\{\frac{1}{i}, \frac{1}{n}\} = \frac{1}{\min\{i, j\}}. \\ \varphi(\delta_{ij}(O(3) \cup O(\frac{1}{n}) &= \varphi(\frac{1}{\min\{i, j\}} = \frac{1}{2\min\{i, j\}} \\ &< \frac{1}{2\max\{k-1, 2\}} < \frac{1}{k-1} \\ &= \frac{1}{k-1} - \varepsilon + \varepsilon = \delta + \varepsilon. \end{aligned}$$

We have that $p(T^{i+r}3, T^{j+r}\frac{1}{n}) = \max\{\frac{1}{i+r}, \frac{1}{j+r}\} = \frac{1}{\min\{i,j\}+r} < \frac{1}{k-1+r} < \frac{1}{k+1} = \varepsilon_0$.

If $\varepsilon > \frac{1}{2}$, then there exist $\delta = \frac{1}{4}, r > 2, \varepsilon_0 = \frac{3}{8} < \varepsilon, i, j \in$ **N**, where $i, j > 2$, such that

$$\varphi(\delta_{ij}(O(3) \cup O(\frac{1}{n})) = \frac{\max\{\frac{1}{i}, \frac{1}{j}\}}{2} < \frac{1}{4} < \varepsilon + \delta,$$

and

$$p\left(T^{i+r}3, T^{j+r}\frac{1}{n}\right) = \max\left\{\frac{1}{i+r}, \frac{1}{j+r}\right\} = \frac{1}{4} < \frac{3}{8} <= \varepsilon_0.$$

Case 2: For $x = \frac{1}{n}, y = \frac{1}{m}, n, m \in$ **N**, we have

$$\delta_{ij}(O(\frac{1}{n}) \cup O(\frac{1}{m})) = \max\{\frac{1}{n+i}, \frac{1}{m+j}\}.$$

For $0 < \varepsilon < \frac{1}{2}$, there exists $k \in \mathbb{N}$, such that $\frac{1}{k+1} < \varepsilon \leq \frac{1}{k}$. Using the same method as in Case 1, for $\delta = \frac{1}{k-1} - \varepsilon, \varepsilon_0 = \frac{1}{k+1}, i, j \in \mathbb{N}$, such that $\min\{i, j\} > \max\{k-1, 2\}$, and $r > 2$, the condition (1) of theorem 4.5, is completed.

Furthermore, for $\varepsilon > \frac{1}{2}$, the function T satisfies the condition (1) of theorem 4.5.

Case 3: For $x = 0, y = 3$ we have

$$\delta_{ij}(O(0) \cup O(3)) = \max\{\frac{1}{i}, 0\} = \frac{1}{i} \text{ for } i > 2, j \in \mathbb{N}.$$

The condition (1) of theorem 4.5 is accomplished in this case, too. Therefore, the function T satisfies the condition (1) of theorem 4.5, for all $x, y \in X$, and it has a unique fixed point $x = 0, T0 = 0$.

5 Conclusions

In this paper, we have obtained some results on equivalent Cauchy sequences and Cauchy sequences in a partial rectangular metric space. The highlight of the paper are theorem 3.5 and theorem 3.10, where there are given some sufficient conditions, under which two sequences are equivalent Cauchy sequences and a sequence is Cauchy sequence in a partial rectangular metric space. As application of these two crucial results, is theorem 4.5, where there is proved the existence and uniqueness of a fixed point for a function that accomplishes one of two contractive conditions which are the generalizations of the Meir-Keeler contraction in partial metric space.

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