

Trigonometric Ratios Using Algebraic Methods

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Received August 6, 2021; Revised September 17, 2021; Accepted October 17, 2021

Cite This Paper in the following Citation Styles

(a): [1] Sameen Ahmed Khan, "Trigonometric Ratios Using Algebraic Methods," *Mathematics and Statistics*, Vol.9, No.6, pp. 899-907, 2021. DOI: 10.13189/ms.2021.090605

(b): Sameen Ahmed Khan, (2021). *Trigonometric Ratios Using Algebraic Methods*. *Mathematics and Statistics*, 9(6), 899-907. DOI: 10.13189/ms.2021.090605

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Abstract The main aim of this article is to start with an expository introduction to the trigonometric ratios and then proceed to the latest results in the field. Historically, the exact ratios were obtained using geometric constructions. The geometric methods have their own limitations arising from certain theorems. In view of the certain limitations of the geometric methods, we shall focus on the powerful techniques of equations in deriving the exact trigonometric ratios using surds. The cubic and higher-order equations naturally arise while deriving the exact trigonometric ratios. These equations are best expressed using the expansions of the cosines and sine of multiple angles using the Chebyshev polynomials of the first and second kind respectively. So, we briefly present the essential properties of the Chebyshev polynomials. The equations lead to the question of reduced polynomials. This question of the reduced polynomials is addressed using the Euler's totient function. So, we describe the techniques from theory of equations and reduced polynomials. The trigonometric ratios of certain rational angles (when measured in degrees) give rise to rational trigonometric ratios. We shall discuss these along with the related theorems. This is a frontline area of research connecting trigonometry and number theory. Results from number theory and theory of equations are presented wherever required.

Keywords Trigonometric Ratios, Algebraic Methods, Higher-order Equations, Chebyshev Polynomials, Minimal Polynomials, Gauss-Wantzel Theorem, Galois Theory

ods to obtain the trigonometric ratios. The same tradition was followed by the Medieval Arab scientists and scientists from India and China [8]. Some of these historic aspects were covered during the *International Year of Light and Light-based Technologies* [9]-[15]. Geometric methods are constrained by the constructible regular polygons [16]. This necessitates the use of equations, which enables us to obtain the trigonometric ratios of angles not possible from the geometric constructions. "A *trigonometric number* is an *irrational number* obtained by taking the sine or cosine of a rational number of degrees (if expressed in radians, the angles is a rational multiple of π)." The only exceptions are $\cos \alpha, \sin \alpha \in \{0, \pm \frac{1}{2}, \pm 1\}$. We shall look at the theorems related to the rational and irrational trigonometric numbers.

For completeness, we briefly cover the procedures for obtaining the exact values of the trigonometric ratios of the routine angles ($15^\circ, 18^\circ, 30^\circ, 36^\circ, 45^\circ, 54^\circ, 60^\circ, 72^\circ$ and 75°). In Section 2, it will be seen that the basic trigonometric identities along with quadratic equations are sufficient for such derivations. In Section 3, we shall focus on the rational angles in degrees. In Section 4, using cubic equations, we obtain some additional results. The same Section has some general results from the *theory of equations* and *number theory* pointing to the limitations inherent in the geometric techniques. Consequently, there is a dependence on higher-order equations in deriving the exact ratios. Section 5, our final Section has the concluding remarks. The study of the trigonometric ratios is an ongoing topic of active research, which is poined by the continued journal articles.

1 Introduction

The exact values of the trigonometric ratios have been a subject of keen interest since the beginning trigonometry and continues to this day [1]-[7]. The Greeks relied on geometric meth-

2 Trigonometric Ratios from Identities and Equations

The very basic identity, $\sin^2 A + \cos^2 A = 1$ is a consequence of the Pythagorean theorem. The identities for the sums

of two angles are

$$\begin{aligned} \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \cos(A + B) &= \cos A \cos B - \sin A \sin B. \end{aligned} \quad (1)$$

Matrices enable us to write them in a compact form

$$\begin{bmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{bmatrix} \begin{bmatrix} \cos B & \sin B \\ -\sin B & \cos B \end{bmatrix} = \begin{bmatrix} \cos(A + B) & \sin(A + B) \\ -\sin(A + B) & \cos(A + B) \end{bmatrix}. \quad (2)$$

A special case is

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}. \quad (3)$$

These formulae can be derived using various techniques such as the de Moivre’s formula. Alternately, we can use Euler’s formula with complex arguments

$$\begin{aligned} \cos(A + B) + i \sin(A + B) &= e^{i(A+B)} \\ &= e^{iA} e^{iB} \\ &= (\cos A + i \sin A)(\cos B + i \sin B) \\ &= (\cos A \cos B - \sin A \sin B) \\ &\quad + i(\sin A \cos B + \cos A \sin B). \end{aligned} \quad (4)$$

The identities in (1) are readily obtained by the substitution $-B$ for B and then equating the real and imaginary parts. The trigonometric ratios of 30° and 45° are well known. Choosing $A = 45^\circ$ and $B = 30^\circ$ gives

$$\begin{aligned} \sin 75^\circ &= \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}} \\ \cos 75^\circ &= \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}. \end{aligned} \quad (5)$$

The identities involving the difference of two angles are

$$\begin{aligned} \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B. \end{aligned} \quad (6)$$

For the multiple angles, we have

$$\sin 2A = 2 \sin A \cos A \quad (7)$$

and

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A. \end{aligned} \quad (8)$$

The identities in (8) can be written as

$$\begin{aligned} \sin\left(\frac{A}{2}\right) &= \frac{1}{2} \sqrt{2(1 - \cos A)}, \\ \cos\left(\frac{A}{2}\right) &= \frac{1}{2} \sqrt{2(1 + \cos A)}. \end{aligned} \quad (9)$$

If we choose $A = 45^\circ$, then

$$\begin{aligned} \sin\left(\frac{45^\circ}{2}\right) &= \sqrt{\frac{1 - \cos 45^\circ}{2}} = \frac{1}{2} \sqrt{2 - \sqrt{2}} \\ \cos\left(\frac{45^\circ}{2}\right) &= \sqrt{\frac{1 + \cos 45^\circ}{2}} = \frac{1}{2} \sqrt{2 + \sqrt{2}}. \end{aligned} \quad (10)$$

Starting with the exact value of $\cos A$ and the repeated use of the identities in (9) enables us to obtain the exact ratios of the sub-multiples of A using only square-roots [17]. On taking k steps, that is k square-roots, we reach the exact value of $\sin(A/2^k)$ expressed as the k -th root

$$\begin{aligned} 2 \sin A &= \sqrt{2 - 2 \cos(2A)} \\ &= \sqrt{2 - \sqrt{2 + 2 \cos(2^2 A)}} \\ &= \sqrt{2 - \sqrt{2 + \sqrt{2 + 2 \cos(2^3 A)}}} \\ &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos(2^4 A)}}}} \\ &= \dots \end{aligned} \quad (11)$$

The same procedure leads to an exact expression for $\cos(A/2^k)$

$$\begin{aligned} 2 \cos A &= \sqrt{2 + 2 \cos(2A)} \\ &= \sqrt{2 + \sqrt{2 + 2 \cos(2^2 A)}} \\ &= \sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos(2^3 A)}}} \\ &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos(2^4 A)}}}} \\ &= \dots \end{aligned} \quad (12)$$

Many exact values can be obtained by a suitable combination of the various identities. Based on the combinations of the identities, the same trigonometric ratio can have several equivalent forms as seen in the following example

$$\begin{aligned} \sin 9^\circ &= \frac{1}{4} \left(\sqrt{8 - 2\sqrt{10 + 2\sqrt{5}}} \right), \\ &= \frac{1}{4} \left(\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}} \right), \\ &= \frac{1}{4} \left(\frac{1}{2} \left(\sqrt{10} + \sqrt{2} \right) - \sqrt{5 - \sqrt{5}} \right), \\ &= \frac{1}{2} \sqrt{2 - 2\sqrt{2} + \phi}, \end{aligned} \quad (13)$$

where $\phi = (\sqrt{5} + 1)/2$ is the celebrated golden ratio [18].

Identities in (1) enable us to obtain

$$\begin{aligned} \sin 3A &= 3 \sin A - 4 \sin^3 A \\ \cos 3A &= 4 \cos^3 A - 3 \cos A. \end{aligned} \quad (14)$$

These identities are sufficient to obtain the trigonometric ratios of 18° , 36° , 54° and 72° . We start with $\theta = 18^\circ = 90^\circ/5$ and

obtain

$$\begin{aligned} 2\theta &= 90^\circ - 3\theta \\ \sin(2\theta) &= \sin(90^\circ - 3\theta) = \cos(3\theta) \\ 2 \sin \theta \cos \theta &= 4 \cos^3 \theta - 3 \cos \theta \\ &= \cos \theta(1 - 4 \sin^2 \theta). \end{aligned} \tag{15}$$

This leads to the equation

$$4 \sin^2 \theta + 2 \sin \theta - 1 = 0, \tag{16}$$

whose solution is

$$\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5} - 1}{4}. \tag{17}$$

The basic identity readily gives the required value of $\cos 18^\circ = \sin 72^\circ$

$$\cos 18^\circ = \sin 72^\circ = \sqrt{1 - \sin^2 18^\circ} = \frac{\sqrt{5 + \sqrt{5}}}{2\sqrt{2}}. \tag{18}$$

Using the identities in (8), the trigonometric ratios of 36° and 54° are

$$\begin{aligned} \sin 36^\circ &= \cos 54^\circ = \sqrt{1 - \cos^2 36^\circ} = \frac{\sqrt{5 - \sqrt{5}}}{2\sqrt{2}} \\ \cos 36^\circ &= \sin 54^\circ = 1 - 2 \sin^2 18^\circ = \frac{\sqrt{5} + 1}{4}. \end{aligned} \tag{19}$$

In this Section, we relied on trigonometric identities along with quadratic equations. In Section 4, we shall see that we require not only cubic or quartic but even higher order equations.

3 Irrationality of Trigonometric Ratios

In 1956, Ivan Morton Niven published a book [19, 20], which had the following theorem summarizing the results on the irrationality of the trigonometric ratios

Theorem 1 *Niven's Theorem: The only rational values of α in the interval $0^\circ \leq \alpha \leq 90^\circ$ for which the sine of α degrees is also a rational number are*

$$\sin 0^\circ = 0, \sin 30^\circ = \frac{1}{2}, \sin 90^\circ = 1.$$

This theorem restricted the rational values to the sets, $\cos \alpha, \sin \alpha \in \{0, \pm \frac{1}{2}, \pm 1\}$, $\sec \alpha, \csc \alpha \in \{\pm 1, \pm 2\}$ and $\tan \alpha, \cot \alpha \in \{0, \pm 1\}$. The arithmetic properties of trigonometric functions have been a recurring topic in the mathematical literature. Part of the aforementioned results were known as early as 1922 with several independent contributions: Swift (1922 [21]), Underwood (1921 [22]), Lehmer (1933 [23]), Olmsted (1945 [24]), and later by Niven (1956) who stated it in the above form. It came to be known as *Niven's Theorem* [19, 20]. Later on, there appeared several proofs to this important theorem employing a variety of techniques such as induction, de Moivre formulas, Chebyshev polynomials, cyclo-tomic polynomials among others (see [25]-[35] and references therein).

The most recent proofs of Niven's theorem are by Bonaventura Paolillo and Giovanni Vincenzi. They are from the year 2020 [34] and the year 2021 [35] respectively. The first theorem makes use of the tangent function and leads to some additional results summarized below

Theorem 2 *Paolillo-Vincenzi Theorem-I: If α is rational in degrees, say $\alpha = (m/n)180^\circ$ for some rational number m/n , and $\tan^2(\alpha)$ is rational, then $\tan^2(\alpha) \in \{0, 1, \frac{1}{3}, 3\}$.*

Using Theorem (2) and some basic identities (such as $\cos^2 \alpha = 1/(1 + \tan^2 \alpha)$, $\cos(2\alpha) = (1 - \tan^2 \alpha)/(1 + \tan^2 \alpha)$ and $\sin(2\alpha) = 2 \tan \alpha/(1 + \tan^2 \alpha)$), we conclude that $\cos^2(\alpha), \sin^2(\alpha) \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. From this set, we conclude the results stated in the theorem. For certain angles, the squares of their trigonometric ratios are rational [36]. There is one more theorem due to Bonaventura Paolillo and Giovanni Vincenzi

Theorem 3 *Paolillo-Vincenzi Theorem-II: If α is rational in degrees, say $\alpha = r 360^\circ$ for some rational number r , then the only rational values of trigonometric functions of α are as follows: $\cos \alpha, \sin \alpha = 0, \pm \frac{1}{2}, \pm 1$, $\sec \alpha, \csc \alpha = \pm 1, \pm 2$ and $\tan \alpha, \cot \alpha = 0, \pm 1$.*

The proof is based on the periodicity of the cosine function [35]. Irrational numbers can be learned through basic trigonometry [37]. The trigonometric functions can be defined as solutions of differential equations. Hence, the irrationality of the values trigonometric functions can be deduced using techniques from calculus [37]-[39].

The irrationality of the trigonometric ratios has found applications within mathematics and several areas of physical sciences [40]-[41]. As an example, we note the Gregory numbers defined as

$$\begin{aligned} G_x &= \tan^{-1} \left(\frac{1}{x} \right) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)x^{2k+1}}, \end{aligned} \tag{20}$$

where x is any integer or any rational number. For instance, with $x = 1$, $G_1 = \tan^{-1}(1) = 45^\circ$ is a Gregory number. Apart from the two exceptional case, $G_{-1} = -45^\circ$ and $G_1 = 45^\circ$, every Gregory number when expressed in degrees is an irrational number. Historically, the identities involving the arctangent function have been widely used to compute the values of π to a very large number of decimal places [27]. The following theorem by Arno Berger [33] summarises the *rational linear independence* of trigonometric numbers

Theorem 4 *Berger Theorem: Let r_1 and r_2 to be two rational numbers such that either $r_1 - r_2$ and $r_1 + r_2$ is not an integer, then the three numbers $1, \cos(r_1\pi)$ and $\cos(r_2\pi)$ are rationally independent.*

4 Cubic and Higher-Order Equations

In this Section, we shall employ the cubic and higher-order equations to derive the exact values for angles, which were not

covered in the previous sections. The idea is to derive the expression for $\cos(nx)$ and $\sin(nx)$ as polynomials of $\cos x$ and $\sin x$. This is best done using the Chebyshev polynomials of *first kind* and *second kind* respectively [5, 6, 7, 42].

4.1 Expansion of $\cos(n\theta)$

Let us first examine the pattern of $\cos(n\theta)$ for the first few values of n by the repetitive use of the identities in (1)

$$\cos(2\theta) = 2 \cos^2 \theta - 1, \tag{21}$$

$$\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta, \tag{22}$$

$$\cos(4\theta) = 8 \cos^4 \theta - 8 \cos^2 \theta + 1, \tag{23}$$

$$\cos(5\theta) = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta. \tag{24}$$

The identities in (21)-(24) suggest that $\cos(n\theta)$ is a n -th degree polynomial in $\cos \theta$. This is indeed the case. We note the general result

$$\cos(n\theta) = T_n(\cos \theta), \tag{25}$$

where $T_n(x)$ denotes the Chebyshev polynomials of the *first kind* [5, 6, 7, 42, 43]. The first few are

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1. \end{aligned} \tag{26}$$

Chebyshev polynomials of the *first kind* satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{27}$$

The leading coefficient of each $T_n(x)$ is 2^{n-1} . In the context of solving polynomial equations, we have the following observations about $T_n(x)$. The even-order $T_n(x)$ are function of x^2 , leading to equations of degree $n/2$ in terms of polynomials of x^2 . The odd-order $T_n(x)$ are functions of the odd powers of x and do not have the constant term. So, the odd-order $T_n(x)$ can be factorised into x times a polynomial of degree $(n - 1)/2$ in x^2 .

The polynomials, $T_n(x)$ can also be obtained from the following $n \times n$ determinant equation

$$T_n(x) = \begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2x \end{vmatrix}. \tag{28}$$

4.2 Expansion of $\sin(n\theta)$

Now, we look for the pattern of $\sin(n\theta)$ for the first few values of n

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ &= \sin \theta \{2 \cos \theta\}, \end{aligned} \tag{29}$$

$$\begin{aligned} \sin(3\theta) &= 3 \sin \theta - 4 \sin^3 \theta, \\ &= \sin \theta \{4 \cos^2 \theta - 1\}, \end{aligned} \tag{30}$$

$$\begin{aligned} \sin(4\theta) &= 4 \sin \theta \cos \theta (2 \cos^2 \theta - 1), \\ &= \sin \theta \{8 \cos^3 \theta - 4 \cos \theta\}, \end{aligned} \tag{31}$$

$$\begin{aligned} \sin(5\theta) &= 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta, \\ &= \sin \theta \{16 \cos^4 \theta - 12 \cos^2 \theta + 1\}. \end{aligned} \tag{32}$$

The identities in (29)-(32) suggest that $\sin(n\theta)$ is $\sin \theta$ times a $(n - 1)$ -th degree polynomial in $\cos \theta$. We note the general result

$$\sin(n\theta) = \sin \theta U_{n-1}(\cos \theta), \tag{33}$$

where $U_n(x)$ denotes the Chebyshev polynomials of the *second kind* [5, 6, 7, 42, 43]. The first few are

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_2(x) &= 4x^2 - 1, \\ U_3(x) &= 8x^3 - 4x, \\ U_4(x) &= 16x^4 - 12x^2 + 1, \\ U_5(x) &= 32x^5 - 32x^3 + 6x, \\ U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1. \end{aligned} \tag{34}$$

Chebyshev polynomials of the *second kind* satisfy the recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{35}$$

The leading coefficient of each $U_n(x)$ is 2^n . In the context of solving polynomial equations, we have the following observations about $U_n(x)$. The even-order $U_n(x)$ are function of x^2 , leading to equations of degree $n/2$ in terms of polynomials of x^2 . The odd-order $U_n(x)$ are functions of the odd powers of x and do not have the constant term. So, the odd-order $U_n(x)$ can be factorised into x times a polynomial of degree $(n - 1)/2$ in x^2 .

The polynomials, $U_n(x)$ can also be obtained from the following $n \times n$ determinant equation

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2x \end{vmatrix}. \tag{36}$$

The Chebyshev polynomials of degree n have n different simple roots, called Chebyshev roots, in the interval $[-1, 1]$. The n roots of $T_n(x)$ are

$$x_k = \cos\left(\frac{\pi(k + 1/2)}{n}\right), \quad k = 0, \dots, n - 1. \quad (37)$$

The n roots of $U_n(x)$ are

$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \quad k = 1, \dots, n. \quad (38)$$

4.3 General Cubic Equation Formula

The most general cubic equation is

$$ax^3 + bx^2 + cx + d = 0. \quad (39)$$

The three solutions are expressed through the following

$$\begin{aligned} \Delta_0 &= b^2 - 3ac, \\ \Delta_1 &= 2b^3 - 9abc + 27a^2d, \\ C &= \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \\ \omega &= \frac{-1 + i\sqrt{3}}{2}, \end{aligned} \quad (40)$$

where ω is the primitive cube root of unity from the equation $x^3 - 1 = 0$. The three roots are

$$x_k = -\frac{1}{3a} \left(b + \omega^k C + \frac{\Delta_0}{\omega^k C} \right), \quad k \in \{0, 1, 2\}. \quad (41)$$

In the present context of trigonometric equations, we are dealing with cubic and higher order equations with real coefficients (mostly integer coefficients). We also know that the relevant trigonometric equations do not have rational solutions. From the Galois theory, we know that *when none of the roots are rational and when all three roots are distinct and real, it is impossible to express the roots using only real radicals.*

4.4 Trigonometric Ratios of $20^\circ (\pi/9)$

Let us now consider the case of 20° . In (30), we choose $\theta = 20^\circ$, with $x = \sin 20^\circ$ and obtain the cubic equation

$$x^3 - \frac{3}{4}x + \frac{1}{8}\sqrt{3} = 0. \quad (42)$$

The solution for $\sin 20^\circ$ is

$$\sin 20^\circ = \frac{1}{2\sqrt[3]{2}} \left[\sqrt[3]{i - \sqrt{3}} - \sqrt[3]{i + \sqrt{3}} \right]. \quad (43)$$

In (22), we choose $\theta = 20^\circ$, with $x = \cos 20^\circ$ and obtain the cubic equation

$$x^3 - \frac{3}{4}x - \frac{1}{8} = 0. \quad (44)$$

The solution for $\cos 20^\circ$ is

$$\cos 20^\circ = \frac{1}{2\sqrt[3]{2}} \left[\sqrt[3]{1 + i\sqrt{3}} + \sqrt[3]{1 - i\sqrt{3}} \right]. \quad (45)$$

4.5 Trigonometric Ratios of $1^\circ (\pi/180)$

The exact value of $\sin 1^\circ$ can be calculated from $\sin 3^\circ$ through the cubic equation. Let $x = \sin 1^\circ$, then

$$4x^3 - 3x + \sin 3^\circ = 0. \quad (46)$$

The solution of this equation is

$$\begin{aligned} x &= \sin 1^\circ \\ &= \frac{1 + i\sqrt{3}}{4} \left(\sqrt[3]{\sin 3^\circ + i \cos 3^\circ} \right. \\ &\quad \left. - \sqrt[3]{-\sin 3^\circ + i \cos 3^\circ} \right). \end{aligned} \quad (47)$$

Some additional but equivalent expressions for $\sin 1^\circ$ are available in [44].

4.6 Minimal Polynomials

If the coefficients of a polynomial equation are rational, then the denominators of the coefficients can be removed. This results in an equivalent polynomial with only integer coefficients. As noted, the Chebyshev polynomials have only integer coefficients and their solutions leads to the values of the trigonometric functions. In Section 3, we noted that $\cos(2\pi/n)$ is a rational number if $n = 1, 2, 3, 4$, and 6 . This implies that for these values of n , $\cos \theta$ satisfies a linear equation with integer coefficients. We also noted that $\cos(2\pi/n)$ is obtained from quadratic equations with integer coefficients for $n = 5, 8$ and 12 . The *algebraic numbers* are roots of polynomials with integer coefficients. If a real (or complex) number is a root of an irreducible polynomial of degree n with integer coefficients, it is said to be an algebraic number with algebraic degree n . The corresponding irreducible polynomial is its minimal polynomial. Here, irreducible polynomial means that it cannot be factored into lower degree polynomials with integer coefficients.

As an example, we will determine the value of $\cos 30^\circ$ (i.e., $\cos(\pi/6)$), using the Chebyshev polynomials. The corresponding polynomial with $\theta = \pi/6$ and $x = \cos(\pi/6)$ is $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$. Then, the equation is

$$\begin{aligned} T_6(x) &= \cos(\pi) \\ 32x^6 - 48x^4 + 18x^2 - 1 &= -1 \\ 32x^6 - 48x^4 + 18x^2 &= 0 \\ 2x^2(4x^2 - 3)^2 &= 0. \end{aligned} \quad (48)$$

We recover the familiar solution $\cos 30^\circ = \sqrt{3}/2$. It is the second degree polynomial $(4x^2 - 3)$, which leads to the value of $x = \cos(\pi/6)$ and is called as the reduced polynomial. Consequently, $\cos 30^\circ$ is an algebraic number of degree two. In the case of $\cos(20^\circ)$ (i.e., $\cos(2\pi/18)$), the corresponding polynomial with $x = \cos(2\pi/18)$ is $T_{18}(x)$. This is a polynomial of degree eighteen. But we know that the $\cos(20^\circ)$ satisfies a cubic equation in (44) and is an algebraic number of degree three. This leads us to the question of the reduced polynomials and their degrees. The Chebyshev polynomials need not be the reduced polynomials all the time! This is evident from the previous two examples of $\cos(\pi/6)$ and $\cos(2\pi/18)$ which were reduced from 6 to 2 and 18 to 3 respectively.

The results on the reduced polynomials and their degrees are done using the Euler’s totient function $\phi(n)$, ubiquitous in the field of number theory [45]-[47]. The Euler’s totient function $\phi(n)$ is defined as “the number of positive integers $\leq n$ that are relatively prime to n (i.e., do not contain any factor in common with n).” These numbers, which are relative prime to n are called as *totatives* of n . As an example, there are 8 totatives of 20 (1, 3, 7, 9, 11, 13, 17 and 19). So, $\phi(20) = 8$. In *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [48], values of $\phi(n)$ are designated by the Sequence A000010 [49]. The values of $\phi(n)$ form the sequence, 1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, The OEIS provides additional information, the related sequences and references. By definition, $\phi(p) = p - 1$, where p is any prime number. The totient function satisfies the inequality $\phi(n) \geq \sqrt{n}$ for all n except for $n = 2$ and $n = 6$. For any composite n , $\phi(n) \leq n - \sqrt{n}$. A curious identity satisfied by the totient function is $\phi(n^k) = n^{k-1}\phi(n)$. A particular result is $\phi(2^k) = 2^{k-1}$, since $\phi(2) = 1$ [50, 51].

For any prime $p > 2$, the minimal polynomial of $\sin(2\pi/p)$ is

$$S_p(x) = \sum_{k=0}^{(p-1)/2} (-1)^k \binom{p}{2k+1} (1-x^2)^{\frac{p-1}{2}-k} x^{2k}, \quad (49)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. The polynomial $S_p(x)$ is of order $(p - 1)$. The derivation for the reduced polynomials is available in [52]-[56]. The $S_p(x)$ for first few values of p are

$$S_3(x) = -4x^2 + 3 \quad (50)$$

$$S_5(x) = 16x^4 - 20x^2 + 5 \quad (51)$$

$$S_7(x) = -64x^6 + 112x^4 - 56x^2 + 7 \quad (52)$$

$$S_{11}(x) = -1024x^{10} + 2816x^8 - 2816x^6 + 1232x^4 - 220x^2 + 11. \quad (53)$$

The degree of $S_p(x)$ is $(p - 1)$. If $n \neq 4$, and $n = 2^r m$, where m is odd, then the degree of the irreducible polynomial of $\sin(2\pi/n)$ is

$$\deg \left[\sin \left(\frac{2\pi}{n} \right) \right] = \begin{cases} \phi(n) & \text{if } r = 0, 1 \\ \frac{1}{2}\phi(n) & \text{if } r = 2 \\ \frac{1}{4}\phi(n) & \text{if } r \geq 3. \end{cases} \quad (54)$$

For $\sin(2\pi/n)$, the degrees of the corresponding irreducible polynomials are 1, 1, 2, 1, 4, 2, 6, 2, 6, 4, 10, 1, 12, 6, 8, 4, . . . , which is the Sequence A093819 in the OEIS [57]. The degree of the irreducible polynomial of $\sin(\pi/n)$ is

$$\deg \left[\sin \left(\frac{\pi}{n} \right) \right] = \begin{cases} 1 & \text{if } n = 2 \\ \phi(n) & \text{if } n = 0, 1, 3(\text{mod}4) \\ \frac{1}{2}\phi(n) & \text{otherwise.} \end{cases} \quad (55)$$

For $\sin(\pi/n)$, the degrees of the corresponding irreducible polynomials are 1, 1, 2, 2, 4, 1, 6, 4, 6, 2, 10, 4, 12, 3, 8, 8, . . . , which is the Sequence A055035 in the OEIS [58].

The minimal polynomial of $\cos(2\pi/p)$ is

$$C_p(x) = S_p \left(\sqrt{\frac{1-x}{2}} \right). \quad (56)$$

The degree of $C_p(x)$ is $(p - 1)/2$. The $C_p(x)$ for first few values of p are

$$C_3(x) = 2x + 1 \quad (57)$$

$$C_5(x) = 4x^2 + 2x - 1 \quad (58)$$

$$C_7(x) = 8x^3 + 4x^2 - 4x - 1 \quad (59)$$

$$C_{11}(x) = 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1. \quad (60)$$

The degree of the irreducible polynomial of $\cos(2\pi/n)$ is

$$\deg \left[\cos \left(\frac{2\pi}{n} \right) \right] = \begin{cases} 1 & \text{if } n = 1, 2 \\ \frac{1}{2}\phi(n) & \text{if } n \geq 3. \end{cases} \quad (61)$$

For $\cos(2\pi/n)$, the degrees of the corresponding irreducible polynomials are 1, 1, 1, 2, 1, 3, 2, 3, 2, 5, 2, 6, 3, 4, 4, 8, . . . , which is the Sequence A023022 in the OEIS [59]. The degree of the irreducible polynomial of $\cos(\pi/n)$ is

$$\deg \left[\cos \left(\frac{\pi}{n} \right) \right] = \begin{cases} 1 & \text{if } n = 1 \\ \phi(n) & \text{if } n = 0(\text{mod}2) \\ \frac{1}{2}\phi(n) & \text{if } n = 1(\text{mod}2). \end{cases} \quad (62)$$

For $\cos(\pi/n)$, the degrees of the corresponding irreducible polynomials are 1, 1, 1, 2, 2, 2, 3, 4, 3, 4, 5, 4, 6, 6, 4, 8, . . . , which is the Sequence A055034 in the OEIS [60].

4.7 Quartic and Higher-Order Equations

Like the cubic equation, there is a formula for the quartic equation. But for fifth and higher degree equations, there is no such formula. In fact, there is a theorem which states that such formulae do *not* exist!

Theorem 5 Abel-Ruffini Theorem: *In general, polynomial equations higher than fourth degree are incapable of algebraic solution in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.*

This theorem is also known as the *Abel’s impossibility theorem*. The proof of this negative theorem is based on the Galois theory. Galois theory has been able to prove the impossibility of several problems of antiquity including: (a) Trisecting the angle (using only a compass and a straightedge); (b) Squaring the circle (i.e., constructing a square of area equal to the area of a given circle); (c) Doubling the cube, (i.e., constructing a cube with twice the volume of a given cube); and (d) constructing a heptagon.

5 Concluding Remarks

The trigonometric functions occur across mathematics and sciences. Obtaining exact values has attracted lot of attention since the beginning of the subject. Geometric techniques provide exact values in the case of several angles but are not suitable for other angles. There are several theorems which limit the use of geometric techniques in finding the exact values. The derivation of the exact trigonometric values is intimately tied to several areas such as *theory of equations, number theory*

and *algebraic geometry*. The irrational sets of trigonometric ratios of rational angles were covered along with related theorems. The new theorems extending the classical results were also covered. Results from number theory and theory of equations were presented wherever required. Trigonometric functions also arise in the study of the hypersingular integral equations [61] and the superstability solutions of The pexiderized trigonometric functional equations[62]

Some of the numerical results can also be derived using the *Microsoft Excel* [63]-[68]. The other alternatives are the symbolic packages, such as the *Mathematica* [69, 70]. MS Excel is valuable for certain types of numerical analysis [64]-[68]. It has been useful in numerous applications such as the analysis of quadratic surfaces [71]-[74]; networks of equal resistors [75]-[78]; chemical physics [79]; and theory of numbers [80, 81].

Acknowledgements

I am grateful to Professor Ramaswamy Jagannathan (Institute of Mathematical Sciences, Chennai, India) for my training in accelerator physics, in particular, the novel area “Quantum theory of charged-particle beam optics” [82]-[84], which was the topic of my doctoral thesis. So, also in the subject of “Quantum Algebras” [85]. He came up with the brilliant idea to apply the Foldy-Wouthuysen transformation technique to explore Helmholtz optics and Maxwell optics. The resulting formalism of optics based on quantum methodologies was selected for the *Year in Optics 2016* by the *Optical Society* [of America]. The two papers [86, 87] covering the formalism were summarised in a dedicated issue of *Optics & Photonics News* [88].

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