Universal Journal of Accounting and Finance 9(5): 927-934, 2021 http://www.hrpub.org
DOI: 10.13189/ujaf.2021.090504

Simulation for Ruin Probabilities in Insurance with Sequence Markov Dependence Random Variables

Quang Phung Duy[1,*], Thinh Nguyen Huu[1], Chien Doan Quyet[2]

[1]Mathematics Department, Foreign Trade University, Hanoi, Viet Nam
[2]Statistics & Actuarial Science, Soongsil University, Seoul, South Korea

Received June 30, 2021; Revised August 1, 2021; Accepted September 21, 2021

Cite This Paper in the following Citation Styles

(a): *[1] Quang Phung Duy, Thinh Nguyen Huu, Chien Doan Quyet , "Simulation for Ruin Probabilities in Insurance with Sequence Markov Dependence Random Variables," Universal Journal of Accounting and Finance, Vol. 9, No. 5, pp. 927 - 934, 2021. DOI: 10.13189/ujaf.2021.090504.*

(b): *Quang Phung Duy, Thinh Nguyen Huu, Chien Doan Quyet (2021). Simulation for Ruin Probabilities in Insurance with Sequence Markov Dependence Random Variables. Universal Journal of Accounting and Finance, 9(5), 927 - 934. DOI: 10.13189/ujaf.2021.090504.*

Abstract The aim of this paper is to calculate ruin probabilities using Monte Carlo method for two models: i) classical risk model with claim amounts are homogeneous Markov chains; ii) generalized risk models with premiums amounts, claim amounts are homogeneous Markov chains. The sequence of random variables in the article is considered as a series of Markov dependent random variables. The main results of this paper are Lemma 3.1, Lemma 3.2 and Lemma 3.3, which have built mathematical formulas for the simulation of the probability of insurance models considered in this paper. From those lemmas, we build algorithms to simulate ruin probability for insurance models considered in this paper. From these algorithms, we build numerical results illustrating the problems posed in the paper. These results all show that when the initial capital increases, the ruin probability will decrease, and when the time increases, the ruin probability will increase. This result is consistent with the theory of the risk problem in insurance.

Keywords Ruin Probability, Homogeneous Markov Chain, Monte Carlo Method

JEL Codes: 62P05, 60G40, 12E05.

1. Introduction

In risk theory, the premiums amount U(t) at time t: $U(t) = u + rt - \sum_{i=1}^{N_t} X_i$, where u > 0 is the initial capital of that company, and r is the premium rate per a unit of time. The number of claim amounts to time t, N_t is the pure Poisson process with intensity μ and claim amount series $\{X_i\}$ is a series of independent random variables having the same distribution as the probability distribution function F, which have finite mean μ . The ruin probability with finite time t, denoted $\psi(u,t)$, is defined by:

$$\psi(u,t) = P\{\exists \tau \le t : U(\tau) < 0\} \tag{1.1}$$

Ruin probability with infinite time, denoted $\psi(u)$, is defined by:

$$\psi(u) = \psi(u,+\infty) = \lim_{t\to+\infty} \psi(u,t) \tag{1.2}$$

If there exists a number R > 0 satisfying

$$\int_0^{+\infty} e^{Rx}(1-F(x))dx = \frac{r}{\mu} \tag{1.3}$$

then with every $u \ge 0$ we have $\psi(u) \le e^{-Ru}$. If $\int_0^{+\infty} e^{Rx}(1-F(x))dx < +\infty$ then

$$\lim_{u\to+\infty} e^{Ru}\psi(u) = C \tag{1.4}$$

where C is a constant. Equation (1.3) is called aproximate

Cramer – Lundberg and R is called exponential constant Lundberg. (see H. U. Gerber [3] and Grandell [4]). For these dependency structure models, it would often be very hard to calculate the approximation of exponential constant R (see Phung Duy Quang [5], [6]). In [1], authors considered the numerical solution to one type of integro-differential equation by a probability method based on the fundamental martingale of mixed Gaussian processes. As an application, we try to simulate the estimation of ruin probability with an unknown parameter driven not by the classical Lévy process, but by the mixed fractional Brownian motion. In [2], authors studied based on a discrete version of the Pollaczeck–Khinchine formula, a general method of calculating the ultimate ruin probability in the Gerber–Dickson risk model is provided when claims follow a negative binomial mixture distribution. The result is then extended for claims with a mixed Poisson distribution. The formula obtained allows for some approximation procedures.

Analytical results and numerical results are often unknown. Simulation Monte Carlo methods can provide tools for calculating approximately probabilities $\psi(u,t)$.

The aim of this paper is to approximately calculate ruin probability $\psi(u,t)$ in two cases using Monte Carlo simulation method: i) the claim amounts is an homogeneous Markov chain in classical model; ii) premiums amounts, claim amounts are homogeneous Markov chains in the general model does not have effects of interests.

In the second section of this paper, authors will introduce the classical model, the general model that has no effect of interest rates with a series of Markov dependent random variables. In the third section of this paper, authors will introduce simulation algorithms to calculate ruin probability in the models introduced in the second section of this paper. In the fourth section of this paper, authors will introduce simulation results with different homogeneous Markov chain dependent models. Finally, the fifth section concludes the paper.

2. Insurance Model with Homogeneous Markov Chain Dependent Random Variables

2.1. Classical Risk Model

In the classical risk model, we assume that the capital of the insurance company at time t is:

$$U(t) = u + rt - S_t = u + rt - \sum_{k=1}^{N_t} X_k \tag{2.1}$$

Where u is the initial capital, r is the cost of credit, X_t is the claim amount at time t; N_t is the number of claims up to time t (N_t is the pure Poisson process with intensity μ, the interval between two claims, which is independent and co-distributed, following an exponential distribution with parameter μ, expectation $\frac{1}{\mu}$); X_t is a homogeneous Markov chain independent of N_t; the total claim amounts up to time t is $S_t = \sum_{k=1}^{N_t} X_k$.

Ruin probability to time t is determined by:

$$\psi(u,t) = P(\exists \tau \le t : U(\tau) < 0) \tag{2.2}$$

2.2. The General Risk Model where There Is No Interest Rate Effect

In the general risk model where there is no interest rate effect, we assume that the capital of the insurance company at time t is:

$$U(t) = u + \sum_{i=1}^{N_t^1} X_i - \sum_{j=1}^{N_t^2} Y_j \tag{2.3}$$

Where u is the initial capital; the series of premium amounts $X_1, X_2, \ldots, X_n$ depends on homogeneous Markov chain; series of claim amounts $Y_1, Y_2, \ldots, Y_n$ depends on homogeneous Markov chain (X_t is independent on Y_t); N_t^1 is the number of premium amounts up to time t with N_t^1 is the pure Poisson process with intensity $\mu_1 > 0$ (the time interval between two premium is independent and co-distributed, following an exponential distribution with parameter μ_1, the expectation is $\frac{1}{\mu_1}$), X_t is independent on N_t^1; N_t^2 is the number of claim amounts to time t with N_t^2 is the pure Poisson process with intensity $\mu_2 > 0$ (the time interval between two claims is independent and co-distributed, following an exponential distribution with parameter μ_2, the expectation is $\frac{1}{\mu_2}$), Y_t is independent on N_t^2; N_t^1 is independent on N_t^2.

The ruin probability to time t is determined by:

$$\psi(u,t) = P(\exists \tau \le t : U(\tau) < 0) \tag{2.4}$$

3. The Monte Carlo Simulation Method Approximates the Ruin Probabilities in the Insurance Model

3.1. The Algorithm to Simulate A Homogeneous Markov Chain

We assume that X_n are defined on the probability space (Ω, A, P). $\{X_n\}_{n\ge 0}$ is an homogeneous Markov chain, such that for any n the values of X_n are taken from a set of non – negative numbers $G = \{x_1, x_2, ..., x_n, ...\}$ with $x_o = x_i$ and

$p_{ij} = P\left[\omega \in \Omega : X_{n+1}(\omega) = x_j \middle| X_n(\omega) = x_i\right](n \in N, x_i \in G, x_j \in G)$,

where $0 \le p_{ij} \le 1, \sum_{j=1}^{+\infty} p_{ij} = 1.$

Let Y_i denote a generic distributed as the i^{th} row of the matrix, that is,

$$P(Y_i = y_j) = p_{ij}; y_j \in G$$

Let us assume we use inversion to generate such a Y_i. Generate U with U is uniformly distributed. Let $Y_i = x_1$, if $U \le p_{i0}$; $Y_i = x_2$, if $p_{io} < U \le p_{i0} + p_{i,1}$; and in general $Y_i = x_j$, if $\sum_{k=0}^{j-1} p_{iik} < U \le \sum_{k=0}^{j} p_{ik}$. In the following algorithm, whenever we say "generate a Y_i", we mean doing so using this inverse transform method using uniform distribution.

We will build a homogeneous Markov chain simulation algorithm.

Algorithm 3.1.

Input: $G = \{x_1, x_2, ..., x_n, ...\}$; $P = [p_{ij}]_{n \times n}$ is transition matrix.

Output: X_n : X_n is a homogeneous Markov chain, X_n are taken from G

Steps of the algorithm:

Step 1. Choose an initial value, $X_o = x_{i_o}$. Set n = 1

Step 2. Generate Y_{i_o}, and set $X_1 = Y_{i_o}$.

Step 3. If n < N, then set $i = X_n$, generate Y_i, set n = n + 1 and set $X_n = Y_i$; otherwise stop.

Step 4. Go back to Step3.

3.2. The Algorithm to Simulate Ruin Probability for the Model (2.1)

We see model (2.1) with a series of random variables $\{X_k\}_{k=1}^n$ depends on homogeneous Markov chain. If we call $\{\tau_i\}_{i\ge1}$ as a series of independent random variables, with the same distribution $E\{\mu\}$ (indicates the time between claims $\{T_i\}_{i=1}^{N_t}$), then we have:

$$N_t := \max\left\{k : \sum_{i=1}^{k} \tau_i := T_k \le t\right\};$$
$$\tau_o = T_o = 0, \tau_i = -\frac{\ln v_i}{\mu}; v_i \sim U(0;1)(i \ge 1) \quad (3.1)$$

In which, random numbers v_i ($i \ge 1$) is independent.

We, now, consider event A(t) (up to time t) of the problem (2.1):

$$\psi(u,t) = P\{A(t)\}, A(t) := \{\exists s \le t : U(s) < 0\}$$

The basis for simulating event A(t) is the following proposition:

Lemma 3.1. If we let $\psi(u,t) = A(t) := \{\exists s \le t : U(s) < 0\}$ then $A(t) = \bigcup_{i=0}^{N_t}\{U(T_i) < 0\}$.

Prove:

Without losing generality, we assume $N_t \ge 1$, we set

$$< T_{j-1}, T_j > := \begin{cases} (0, T_1) & \text{if } j = 1, \\ [T_{j-1}, T_j) & \text{if } j = 2 \div N_t, \\ [T_{N_t}, t] & \text{if } j = N_t + 1. \end{cases}$$

Then from (3.1) we have:

$$\bigcup_{j=1}^{N_t+1} < T_{j-1}, T_j > = (0, t], \ < T_{j-1}, T_j > \cap < T_{i-1}, T_i > = \phi (\forall i \ne j)$$

To point out that:

$$U(s) = U(T_{j-1}) (\forall s \in < T_{j-1}, T_j >, j = 1 \div N_t + 1),$$

And $U(s) = U(T_o) = u > 0$, $\forall s \in < T_o, T_1 >$.

Let $A_j(t) := \{\exists s \in < T_{j-1}, T_j > : U(s) < 0\} (\forall j = 1 \div N_t + 1)$.

Then

$$A(t) = \bigcup_{j=1}^{N_t+1} A_j(t) = \bigcup_{j=2}^{N_t+1} A_j(t)$$

because $A_1(t) := \{\exists s \in < T_o, T_1 > : U(s) < 0\} = \phi$.

On the other hand:

$$\{U(T_{j-1}) < 0\} \subset A_j(t) \subset \{U(T_{j-1}) < 0\}$$
$$\Rightarrow A_j(t) = \{U(T_{j-1}) < 0, \forall j = 2 \div N^2(t) + 1\}$$

Then

$$A(t) = \bigcup_{j=2}^{N_t+1}\{U(T_{j-1}) < 0\} = \bigcup_{j=1}^{N_t}\{U(T_j) < 0\} \quad \square.$$

From Lemma 3.1, the ruin probability at (2.4) is estimated as:

$$\psi(u,t) = P\{A(t)\} \approx \frac{M}{N}; A(t)$$
$$:= \{\exists s \le t : U(s) < 0\} = \bigcup_{i=0}^{N_t}\{U(T_i) < 0\} \quad (3.2)$$

Where M is the number of occurrences of event A(t) in N simulations and M is determined by the following algorithm.

Algorithm 3.2.

Input: initial capital u, cost rate r, time t, the number of simulations N, parameter μ, $G = \{x_1, x_2, ..., x_n, ...\}$; $P = [p_{ij}]_{n \times n}$ is transition matrix

Output: Risk probability $\psi(u,t)$

Steps of the algorithm:

First of all, assign M = 0, $T_o = 0$, $U(T_o) := u$.

Step A. (in the $n = \overline{1, N}$). With each i = 1, 2, … We do it as follows:

A1. Simulate the time to claim: $T_i = T_{i-1} + \tau_i$ with τ_i created according to the formula (3.1) and check inequality:

$$T_i \le t \tag{3.3a}$$

- If (3.3a) is false: terminate the n^{th} simulation of event A(t).
- If (3.3a) is true: move to step A2.

A2. Simulation of claim value X_i according to algorithm 3.1 to calculate (see (2.1)):

$$U(T_i) = U(T_{i-1}) + r(T_i - T_{i-1}) - X_i$$

and check inequality:

$$U(T_i) \ge 0 \tag{3.3b}$$

- If (3.3b) is false: terminate the simulation at the n^{th} time of event A(t) and assign M: = M + 1
- If (3.3a) is true: Move back to step A1 with i: = i + 1

Notice that: the loop will stop when i = N_t (xem (3.1)) and finish the n^{th} simulation of event A(t).

Step B. After simulating N times event A(t) (repeat N times step A, approximately calculate the probability of risk: $\Psi(u, t) = \frac{M}{N}$.

3.2. Algorithm to Simulate Ruin Probability for the Model (2.2)

To describe the method, we consider the model (2.2) with the assumption that: series of amounts $\{X_i\}_{i\ge1}$ and the series of the claim amount $\{Y_j\}_{j\ge1}$ are homogeneous Markov chains.

Let $N_s^k \equiv N^k(s)(k = \overline{1,2})$ the Poisson process with intensity μ_k, represents the number of receiving times (when k = 1) and the number of payments (when k = 2) in period (0, s]. Let T_i^k, T_i^k the receiving time (when = 1) and claim payment (when k= 2) in the ith time. Then from (3.1), we have:

$$N_s^k \equiv N^k(s) := \max\left\{ i : \sum_{j=0}^{i} \tau_j^k := T_i^k \le s \right\}; \qquad \tau_o^k = T_o^k = 0\,(k = \overline{1,2}) \tag{3.3}$$

$$\tau_j^k := \frac{-\ln v_j^k}{\mu_k},\ v_j^k \sim U(0,\ 1)\ \ (\forall j \ge 1, k = \overline{1,2}) \tag{3.4}$$

In which, for each $k = \overline{1,2}$, $v_j^k\ (j \ge 1)$ are independent random numbers. Then we can determine capital process $U(T_j^2)\,(j \ge 1)$ of the insurance company at the time of claim T_j^2, through the following proposition:

Lemma 3.2. 1) With the above assumptions, if $N^2(t) > 0$ and $N^1(T_{j-1}^2) < N^2(T_j^2)\,(\forall j \ge 1)$ then it is almost sure (a.s) that:

$$\left.\begin{aligned} &0 < T_1^1 < \dots < T^1_{N^1(T_1^2)} \le T_1^2 < T^1_{N^1(T_1^2)+1} \\ &< \dots < T^1_{N^1(T_{j-1}^2)} \le T_{j-1}^2 < T^1_{N^1(T_{j-1}^2)+1} \\ &< \dots < T^1_{N^1(T_j^2)} \le T_j^2 < T^1_{N^1(T_j^2)+1} \\ &< \dots < T^1_{N^1(T^2_{N^2(t)})} \le T^2_{N^2(t)} \le t \end{aligned}\right\} \tag{3.5}$$

Then we have:

$$\begin{aligned} &U(T_j^2) = U(T_{j-1}^2) + X(T_j^2) - Y_j\ \ (j = 1 \div N^2(t)), \\ &U(T_o^2) = u; \end{aligned} \tag{3.6}$$

Where

$$X(T_j^2) = \begin{cases} 0 & \text{if } N^1(T_{j-1}^2) = N^1(T_j^2) \\ \sum\limits_{i=N^1(T_{j-1}^2)+1}^{N^1(T_j^2)} X_i & \text{if } N^1(T_{j-1}^2) < N^1(T_j^2) \end{cases} \tag{3.7}$$

2) In case $N^2(t) = 0$, we have:

$$U(\tau) \ge 0\,(\forall \tau \le t) \tag{3.8}$$

Prove:

1) From the non-trivial properties of random variables

$\tau_j^k \sim E(\mu_k)\ (\forall j \ge 1)$ we infer: $\tau_j^k > 0$ (a.s), $\forall j \ge 1$ then from (3.3) we have:

$$\begin{aligned} &0 < T_o^2 < T_1^2 < \dots < T_{j-1}^2 < T_j^2 < \\ &\dots < T^2_{N^2(t)} \le t < T^2_{N^2(t)+1}\ (a.s) \end{aligned} \tag{3.9}$$

Therefore, when considering the definition of $N^1(s)$ (in (3.3)) with, respectively, value $s = T_j^2\ (j = 1 \div N^2(t))$, we easily obtain (3.5).

Also, when using (3.3) with k = 2 and $s = T_j^2$, we also have:

$$T^2_{N^2(T_j^2)} = T_j^2 \Rightarrow N^2(T_j^2) = j\ (j = 1 \div N^2(t)). \tag{3.10}$$

On this basis we have the representation of $U(\tau)$ in (2.3) with $\tau = T_j^2$ in the form:

$$U(T_j^2) = u + \sum_{i=0}^{N^1(T_j^2)} X_i - \sum_{i=0}^{j} Y_i\ \ (1 \le j \le N^2(t)). \tag{3.11}$$

When replacing j in the above formula by $j-1 \ge 1$, we have

$$U(T_{j-1}^2) = u + \sum_{i=0}^{N^1(T_{j-1}^2)} X_i - \sum_{i=0}^{j-1} Y_i \quad (3.12)$$

$$(2 \le j \le N^2(t))$$

For each $j = 2 \div N^2(t)$, we rely on equations (3.6) and (3.12) to represent (3.11) in the form:

$$U(T_j^2) = \begin{cases} U(T_{j-1}^2) + X(T_j^2) - Y_j & \text{if } N^1(T_{j-1}^2) < N^1(T_j^2) \\ U(T_{j-1}^2) - Y_j & \text{if } N^1(T_{j-1}^2) = N^1(T_j^2) \end{cases}$$

That mean, we have (3.6) for all $j = 2 \div N^2(t)$. Moreover, since $T_o^2 = 0, N^1(T_o^2) = 0$ (see (3.3)) so $U(T_o^2) = U(0) = u$. Then since X_o = 0 when considering (3.11) with j =1, we can rely on (3.5) to infer:

$$U(T_1^2) = U(T_o^2) + \sum_{i=1}^{N^1(T_1^2)} X_i - Y_1 = U(T_o^2) + \sum_{i=N^1(T_o^2)+1}^{N^1(T_1^2)} X_i - Y_1$$

when $N^1(T_o^2) < N^1(T_1^2)$ a

and $U(T_1^2) = U(T_o^2) - Y_1 = U(T_o^2) - Y_1$ when $N^1(T_o^2) = N^1(T_1^2)$. So, we get (3.6) in both the case j = 1.

2) Finally, we consider the case: $N^2(t) = 0$. Since $0 \le N^2(\tau) \le N^2(t), \forall \tau \le t$ (see (3.3), $N^2(\tau) = 0\,(\forall \tau \le t)$. Then formula u_τ in (2.3) has the form:

$$U(\tau) = u + \sum_{i=0}^{N^1(\tau)} X_i - Y_o = u + \sum_{i=0}^{N^1(\tau)} X_i \ (\forall \tau \le t)$$

Since u > 0 and $X_i\,(i \ge 1)$ are non-negative random variables, from the above formula, we directly deduce (3.8).

Now we consider the risky event A(t) (up to time t) of problem (2.2):

$$\psi(u,t) = P\{A(t)\}, A(t) := \{\exists s \le t : U(s) < 0\} \quad (3.13)$$

The basis for simulating event A(t) is the following proposition:

Lemma 3.3. In the conditions of Lemma 3.2, we have the following conclusions:

1. If $N^2(t) \ge 1$, then

$$A(t) = B(t) := \bigcup_{j=1}^{N^2(t)} \{U(T_j^2) < 0\} \quad (3.14)$$

Then event A(t) will not occur, if:

$$U(T_j^2) \ge 0\,(\forall j = 1 \div N^2(t)) \quad (3.15)$$

2. Event A(t) also does not occur, if:

$$N^2(t) = 0 \Leftrightarrow \tau_1^2 = \frac{-\ln v_1^2}{\mu_2} > t\,,\ (v_1^2 \sim U(0,\ 1)) \quad (3.16)$$

Prove:

In the case of $N^2(t) \ge 1$, we assign

$$< T_{j-1}^2, T_j^2 > := \begin{cases} (0, T_1^2) & \text{if } j = 1 \\ [T_{j-1}^2, T_j^2) & \text{if } j := 2 \div N^2(t), \\ [T_{N^2(t)}^2, t] & \text{if } j = N^2(t) + 1. \end{cases} \quad (3.17)$$

Then from (3.9) we have:

$$\bigcup_{j=1}^{N^2(t)+1} < T_{j-1}^2, T_j^2 > = (0,t], \quad (3.18)$$
$$< T_{j-1}^2, T_j^2 > \cap < T_{i-1}^2, T_i^2 > = \phi(\forall i \ne j)$$

To show that:

$$U(s) \ge U(T_{j-1}^2)\,(\forall s \in < T_{j-1}^2, T_j^2 >, \quad (3.19)$$
$$j = 1 \div N^2(t) + 1),$$

Firstly, we consider the case j= 1 meaning that (see (3.17)): $0 < s < T_1^2$. In this case, we have (see (3.3), (3.9)):

$$N^1(s) \ge 0, 0 = T_o^2 \le N^2(s) \le s < T_1^2 \Rightarrow N^2(s) = 0.$$

Therefore, from (2.10) we get:

$$U(s) = u + \sum_{i=0}^{N^1(s)} X_i \ge u = U(0) \quad (3.20)$$
$$= U(T_o^2) > 0(\forall s \in < T_o^2, T_1^2 >).$$

That means, we obtained (3.19) with j = 1. Next, we consider case $j = 2 \div N^2(t)$, in which (see (3.17)): $T_{j-1}^2 \le s < T_j^2$. Then from (3.9) and (3.3) we have: $N^2(s) = N^2(T_{j-1}^2) = j-1, N^1(s) \ge N^1(T_{j-1}^2)$. Therefore, from (2.0), (3.10) and (3.12) we deduce:

$$U(s) \ge u + \sum_{i=0}^{N^1(T_{j-1}^2)} X_i - \sum_{i=0}^{j-1} Y_i = U(T_{j-1}^2)\,(\forall s \in < T_{j-1}^2, T_j^2 >).$$

And we obtain (3.19) with $j = 2 \div N^2(t)$. Finally, case $j = N^2(t) + 1$, where $s \in [T_{N^2(t)}^2, t]$. When $T_{N^2(t)}^2 = t$ then (3.19) is obvious. When $T_{N^2(t)}^2 < t$ then from (3.9) we have $T_{N^2(t)}^2 \le s \le t < T_{N^2(t)+1}^2$ and from the above case, we obtain (3.19) in both cases. Then the formula (3.19) is completely proved.

To prove (3.14), firstly, we let:

$$A_j(t) := \{\exists s \in < T_{j-1}^2, T_j^2 > : U(s) < 0\} \quad (3.21)$$
$$(\forall j = 1 \div N^2(t) + 1).$$

In which (see (3.20)):

$$A_1(t) := \left\{\exists s \in (0, T_1^2) : U(s) < 0\right\} = \phi$$
.

Then from (3.13) and (3.18), it is easy to see that:

$$A(t) = \bigcup_{j=1}^{N^2(t)+1} A_j(t) = \bigcup_{j=2}^{N^2(t)+1} A_j(t) \qquad (3.22)$$

But from (3.19) and (3.21) we also find:

$$\left\{U(T_{j-1}^2) < 0\right\} \subset A_j(t) \subset \left\{U(T_{j-1}^2) < 0\right\}$$
$$\Rightarrow A_j(t) = \left\{U(T_{j-1}^2) < 0, \forall j = 2 \div N^2(t) + 1\right\},$$

On this basis (3.21) and (3.22) we get:

$$A(t) = \bigcup_{j=1}^{N^2(t)+1} A_j(t) = \bigcup_{j=2}^{N^2(t)+1} \left\{U(T_{j-1}^2) < 0\right\} = \bigcup_{j=1}^{N^2(t)} \left\{U(T_j^2) < 0\right\},$$

So, (3.14) is proven. When letting:

$$B_j(t) := \left\{U(T_j^2) < 0\right\}$$
$$\Leftrightarrow \overline{B_j(t)} := \left\{U(T_j^2) \ge 0\right\} (\forall j = 1 \div N^2(t) + 1),$$

We rely on (3.14) and the De Morgan duality rule to infer:

$$\overline{A(t)} = \overline{B(t)} = \bigcap_{j=1}^{N^2(t)} \overline{B_j(t)} = \left\{U(T_j^2) \ge 0, \forall j = 1 \div N^2(t)\right\}$$
.

Therefore, in condition (3.15) event A(t) will not occur and conclusion number 1. is completely proved.

To prove the rest, we rely on (3.4) and (3.5) to deduce the equivalence of the following events:

$$\left\{N^2(t) = 0\right\} = \left\{\tau_1^2 = \frac{-\ln v_1^2}{\mu_2} > t\right\}, \quad v_1^2 \sim U(0, 1).$$

When the above event has occurred, from (3.8) and (3.13) we find that event A(t) will not happen and we get the conclusion number 2.

Since random variables $U(T_j^2)$ can be simulated by Lemma 3.2, so random event A(t) can also be simulated according to Lemma 3.3. Therefore, we can approximate the solution of the problem (2.10) in the following form:

$$\psi(u, t) = P\left\{A(t)\right\} \approx \frac{M}{N} \qquad (3.23)$$

Where M is the number of occurrences of event A(t) in N simulations and determined by the following algorithm:

Algorithm 3.3.

Input: initial capital u, time t, number of simulations N, parameter μ_1, parameter μ_2, $G = \{x_1, x_2, ..., x_n, ...\}$;

P = $[p_{ij}]_{n \times n}$ is transition matrix of $\{X_i\}$; Q = $[q_{ij}]_{n \times n}$ is transition matrix of $\{Y_i\}$

Output: Risk probability $\psi(u, t)$

Comment: For the problem of determining the risk probability of this model, we only need to calculate and check the condition that capital receives negative values at the time of claim as in Lemma 3.2 and Lemma 3.3.

Steps of the algorithm:

Firstly, let M = 0, $T_o^2 = T_o^1 = 0, U(T_o^2) = u$

Step A. With each j =1,2, … we perform the following steps:

A1. Simulate the time to claim T_j^2 (after the time of claiming T_{j-1}^2 in the previous time) by this formula: $T_j^2 := T_{j-1}^2 - \frac{\ln v_j^2}{\mu_2}$, v_j^2 ~ U(0, 1), and check the inequality:

$$T_1^2 \le t \qquad (3.23a)$$

- If (3.23a) is false: terminate the n[th] time simulation of event A(t).
- If (3.23a) is true: simulate Y_j depending on homogeneous Markov chain according to algorithm 3.1 and we move to step A2.

A2. Simulate the time to claim $T_i^1\ (i = N^1(T_{j-1}^2) + 1 \div N^1(T_j^2))$ according to the iterative formula:

$$T_i^1 := T_{i-1}^1 - \frac{\ln v_i^1}{\mu_1}, \quad V_i^1 \sim U(0, 1)$$

Where $N^1(T_j^2) = N^1(T_{j-1}^2)$ when $T^1_{N^1(T_{j-1}^2)+1} > T_j^2$. Otherwise, $N^1(T_j^2)$ is selected from the condition: $T^1_{N^1(T_{j-1}^2)} < T^1_{N^1(T_{j-1}^2)+1} < ... < T^1_{N^1(T_j^2)} \le T_j^2 < T^1_{N^1(T_j^2)+1}$.

A3. Simulate X_i depending on homogeneous Markov chain according to algorithm 3.1 ($i = N^1(T_{j-1}^2) + 1 \div N^1(T_j^2)$), so as to:

A4. Calculate $U(T_j^2)$ according to formula (3.6) and check inequality:

$$U(T_j^2) \ge 0 \qquad (3.23b)$$

- If (3.23a) is true: Move back to step A1, with j : = j + 1.
- If (3.23b) is false: terminate the n[th] time simulation of event A(t) and assign M: = M +1.

Step B: After simulating N times event B(t) (repeat N times step A), approximately calculate the ruin probability: $\Psi(u, t) = \frac{M}{N}$.

Notice 3.1. The aforementioned loop will stop with $j = N^2(t) : T^2_{N^2(t)} \le t < T^2_{N^2(t)+1}$. Then we finish the n[th] time simulation of event A(t) (see (3.15)). In case $N^2(t) = 0$ (see (3.16), the n^{th} time simulation of event A(t) will end immediately at step A1 with j = 1.

4. Numerical Experiment Results

4.1. Simulation Ruin Probability of the Model (2.1)

With input data: initial capital takes values: u = 2; u = 3; u = 4; u = 5; u = 6; u = 7; time t gets values: t = 4, t = 6, t = 10; number of simulations N = 1000; interest rate r = 0,088; parameter μ = 2,5.

*The claim process $\{X_n\}$ follows a homogeneous Markov chain with transition matrix, X_n are taken from G = {0; 1; 2; 3; ....}.

$$P = \begin{bmatrix} 0,1 & 0,15 & 0,3 & 0,4 \\ 0,1 & 0,15 & 0,4 & 0,5 \\ 0,1 & 0,4 & 0,3 & 0,1 \\ 0,1 & 0,2 & 0,4 & 0,5 \end{bmatrix}; \qquad (4.1)$$

We have compiled calculation software in Python environment to demonstrate algorithm 3.2, when running this program, we obtain simulation results of ruin probability for model (2.1) with hypothesis (4.1) given in table 4.1 below:

Table 4.1. Simulating the ruin probability of the model (2.1) with assumption (4.1)

Initial capital	Number of simulations	Interest rate	Ruin Probability $\psi(u,t)$		
u	N	r	t = 4	t = 6	t = 10
2	1000	0.08	0,5420	0,6150	0,6250
3			0,3080	0,4120	0,5192
4			0,1450	0,2412	0,3154
5			0,0840	0,1502	0,2442
6			0,0230	0,1152	0,1582
7			0,0142	0,0421	0,1276

4.2. Simulation Ruin Probability of the Model (2.3)

With input data: u = 2; u = 3; u = 4; u = 5; u = 6; u = 7; time t gets values: t = 4, t = 6, t = 10; number of simulations N = 1000; parameter $\mu_1 = 4$; parameter $\mu_2 = 2$;

*The premium process $\{X_n\}$ follows a homogeneous Markov chain with transition matrix, X_n are taken from G = {0; 1; 2; 3; ....}.

$$P = \begin{bmatrix} 0,1 & 0,15 & 0,3 & 0,4 \\ 0,1 & 0,15 & 0,4 & 0,5 \\ 0,1 & 0,4 & 0,3 & 0,1 \\ 0,1 & 0,2 & 0,4 & 0,5 \end{bmatrix}; \qquad (4.2)$$

*The claim process {Yn} follows a homogeneous Markov chain with transition matrix, Yn are taken from G = {0; 1; 2; 3; ....}.

$$P = \begin{bmatrix} 0,1 & 0,1 & 0,25 & 0,4 \\ 0,15 & 0,15 & 0,4 & 0,5 \\ 0,1 & 0,35 & 0,3 & 0,2 \\ 0,16 & 0,2 & 0,4 & 0,5 \end{bmatrix}; \qquad (4.3)$$

We have compiled calculation software in Python environment to demonstrate algorithm 3.3, when running this program, we obtain simulation results of ruin probability for model (2.3) with hypothesis (4.2) and (4.3) given in table 4.3 below:

Table 4.2. Simulating the ruin probability of the model (2.3) with assumptions (4.2) and (4.3)

Initial capital	Ruin Probability $\psi(u,t)$		
u	t = 4	t = 6	t = 10
2	0,034	0,086	0,185
3	0,013	0,048	0,088
4	0,011	0,034	0,064
5	0,008	0,028	0,042
6	0,006	0,013	0,026
7	0,004	0,008	0,012

5. Conclusions

This paper study two models: i)the claim amounts is an homogeneous Markov chain in classical model; ii) premiums amounts, claim amounts are homogeneous Markov chains in the general model does not have effects of interests.

The sequence of random variables in the article is considered as a series of Markov dependent random variables. The main results of this paper are Lemma 3.1, Lemma 3.2 and Lemma 3.3, which have built mathematical formulas for the simulation of the probability of insurance models considered in this article.

This paper has built the theoretical basis of simulation for (2.1) modes and (2.3) model. We have built algorithms 3.2 and 3.3 to simulate ruin probability for model (2.1) and model (2.3) with a series of regression dependent random variables. From the results of approximately calculating the ruin probability for model (2.1) given in table 4.1and model (2.3) given in table 4.2 shows the conformity of the results of quantitative research with qualitative research, specifically:

When increasing the initial capital u of insurance companies, the ruin probability will decrease. For each level of capital u, as time t increases, the ruin probability will increase.

This study is a result of a research with the title ‘Mathematical Models in Economics and Application in

Solving Some Problems of Economics and the Social Sciences' led by Dr. Phung Duy Quang, Foreign Trade University, Vietnam.

REFERENCES

[1] Chunhao Cai, Weilin Xiao. "Simulation of an integro-differential equation and application in estimation of ruin probability with mixed fractional Brownian motion", Journal Integral Equations Applications, Vo.33, No.1, 1-17 (2021). DOI: 10.1216/jie.2021.33.1.

[2] David J. Santanaa, Luis Rincón. "Approximations of the ruin probability in a discrete time risk model", Modern Stochastics: Theory and Applications,Vol.7, No.3, 221–243 (2020) DOI: https://doi.org/10.15559/20-VMSTA158.

[3] H. U. Gerber. "Ruin theory in the linear model". Insurance: Mathematics and Economics, Vo.1, 177-184 (1982). DOI: https://doi.org/10.1016/0167-6687(82)90011-7.

[4] J. Grandell. Aspects of Risk Theory. Springer, New York, 1992.

[5] Phung Duy Quang, "Upper bounds for ruin probability in a generalized risk process under rates of interest with homogenous Markov chain claims and homogenous Markov chain premiums", Applied Mathematical Sciences, Vol.8, No.29, 1445-1454 (2014). DOI: http://dx.doi.org/10.12988/ams.2014.4144.

[6] Phung Duy Quang, "Ruin Probability in a Generalised Risk Process under Rates of Interest with Homogenous Markov Chains", East Asian Journal on Applied Mathematics, Vol.4, No.3, 283 – 300 (2014). DOI: https://doi.org/10.4208/eajam.051013.230614a.