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Unique Common Tripled Fixed Point for Three Mappings in $\mathscr{G}_b$-metric Spaces

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Abstract It has been a century since the Banach fixed point theorem was established, and because of this, the result is the progenitor in some ways. This seems essential to revisit fixed point theorems in specific and in light of most of those. Those are numerous and prevalent in mathematics, as we will demonstrate. Fixed point theorems can be noticed in advanced mathematics, economics, micro-structures, geometry, dynamics, computational mathematics, and differential equations. $\mathscr{G}$-metric space is the broaden and extrapolate the paradigm of the concept of metric space. The characteristic of a $\mathscr{G}$-metric space, in essence, is to comprehend the topological features of three points ratherthan two points via the perimeter of a triangle, where the metric indicates the distance between two points. The domain of $\mathscr{G}_b$-metric space is significantly larger than that of the class of $\mathscr{G}$-metric space. Hence we utilised this generalized space inorder to obtain common tripled fixed point for three mappings using rational type contractions in the setting of $\mathscr{G}_b$-metric spaces. Recently, Khomadram et al have developed coupled fixed point theorems in $\mathscr{G}_b$-metric spaces via rational type contractions. The main aim of our paper is to broaden and extrapolate the paradigm of Khomadram's results into tripled fixed point theorems. Therefore, examples are offered to support our findings.

Keywords Gb-metric Space, Gb-Cauchy Sequence,Gb-convergent Sequence and Tripled Fixed Point

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1 Introduction

"Topological metric space theory" originates from the vast area of non-linear functional analysis. Fixed point theorem is a qualitative result which concerns with finding conditions on the frame of a non-empty set and the choice of mapping on that particular set, in order to obtain a fixed point usually. Many problems in problems in engineering and applied sciences are made usually in the structure of differential and integral equations.Fixed point theorems discover many applications in proving the existence of unique solutions for various existed differential and integral equations that appear in the study of problems in heat and mass transfer, fluid mechanics, chemical and electrochemical process, molecular physics and in many other fields.

Initially in 1909, Frechet[10] introduced the notion of metric as a distance function. In 1989, Bakthin[5] introduced the concept of b-metric space as a generalization of metric space. Mustafa and Sims [20] led to the enactment of $\mathscr{G}$-metric space in 2006 to broaden and extrapolate the paradigm of metric space. In assertion, understanding the geometric features of three points rather than two points through the use of the perimeter of a triangle is the characteristic of a $\mathscr{G}$-metric space. In 2014, Aghajani et al[2]introduced a $\mathscr{G}_b$-metric space as a different generalization of metric space using the concepts of b-metric space and $\mathscr{G}$-metric space.
The goal of this study is to show that unique common tripled fixed point theorems for three mappings via rational type contractive conditions which are broaden and extrapolate the paradigm of the results of Khomadram et al [14] in the setting of $\mathscr{G}_b$-metric spaces. Metric spaces are the medium where a significant amount of functional analysis ideas and conclusions are articulated. This article on $\mathscr{G}$-metric space appears to skim through the fundamentals of the problem so very rapidly. The authors of this paper aim to take a relaxed approach to the theory of G-metric spaces. The following definitions and propositions are used to derive our main results.

Definition 1.1 ([2]) Let $\mathscr{M}$ be a nonempty set and $\mathscr{G}_b : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ and $s \geq 1$ be a real number such that

($\mathscr{G}1$) $\mathscr{G}_b(a,b,c) \geq 0$ for all $a,b,c \in \mathscr{M}$ with $\mathscr{G}_b(a,b,c) = 0$ if $a = b = c$,

($\mathscr{G}2$) $\mathscr{G}_b(a,a,b) > 0$ for all $a,b \in \mathscr{M}$ with $a \neq b$,

($\mathscr{G}3$) $\mathscr{G}_b(a,a,b) \leq \mathscr{G}_b(a,b,c)$ for all $a,b,c \in \mathscr{M}$ with $c \neq b$,

($\mathscr{G}4$) $\mathscr{G}_b(a,b,c) = \mathscr{G}_b(a,c,b) = \mathscr{G}_b(b,a,c) = \mathscr{G}_b(c,a,b) = \mathscr{G}_b(b,c,a) = \mathscr{G}_b(c,b,a)$ for all $a,b,c \in \mathscr{M}$

($\mathscr{G}5$) $\mathscr{G}_b(a,b,c) \leq s[\mathscr{G}_b(a,w,w) + \mathscr{G}_b(w,b,c)]$ for all $a,b,c,w \in \mathscr{M}$

Then the pair $(\mathscr{M},\mathscr{G}_b)$ is called a $\mathscr{G}_b$-metric space with $\mathscr{G}_b$-metric $\mathscr{G}_b$ on $\mathscr{M}$. Axioms ($\mathscr{G}4$) and ($\mathscr{G}5$) are referred to as the *symmetry* and the *rectangle inequality* (of $\mathscr{G}_b$) respectively.

Given a $\mathscr{G}_b$-metric space $(\mathscr{M},\mathscr{G}_b)$, define

$$\rho_{\mathscr{G}}(a,b) = \mathscr{G}_b(a,b,b) + \mathscr{G}_b(a,a,b) \text{ for all } a,b,c \in \mathscr{M}. \tag{1.1}$$

Then it is seen in [2] that $\rho_{\mathscr{G}}$ is a b-metric on $\mathscr{M}$ and that the family of all $\mathscr{G}_b$-balls $\{B_G(a,r) : a \in \mathscr{M}, r > 0\}$ is the base topology, called the $\mathscr{G}_b$-metric topology $\tau(\mathscr{G}_b)$ on $\mathscr{M}$, where $B_{\mathscr{G}}(a,r) = \{b \in \mathscr{M} : \mathscr{G}_b(a,b,b) < r\}$. Further, it was shown that the $\mathscr{G}_b$-metric topology coincides with the metric topology induced by the b-metric $\rho_{\mathscr{G}}$, this enables us to convert numerous notions form b-metric to $\mathscr{G}_b$-metric space with ease.

Definition 1.2 ([2]) A $\mathscr{G}_b$-metric space $(\mathscr{M},\mathscr{G}_b)$ is said to be symmetric if $\mathscr{G}_b(a,b,b) = \mathscr{G}_b(b,a,a)$ for all $a,b \in \mathscr{M}$.

Definition 1.3 ([2]) A sequence $\langle a_\sigma \rangle$ in a $\mathscr{G}_b$-metric space $(\mathscr{M},\mathscr{G}_b)$ is said to be $\mathscr{G}_b$-convergent to $a \in \mathscr{M}$ if for every $\epsilon > 0$ there is a positive integer N such that $\mathscr{G}_b(a_\sigma, a_\xi, a) < \epsilon$ for all $\xi, \sigma \geq N$.

Proposition 1.1. ([2]) Let $(\mathscr{M},\mathscr{G}_b)$ be a $\mathscr{G}_b$-metric space.Then the followings are equivallent.

(i) $\langle a_\sigma \rangle_{n=1}^{\infty}$ is $\mathscr{G}_b$-convergent to a

(ii) $\mathscr{G}_b(a_\sigma, a_\sigma, a) \to 0$ as $\sigma \to \infty$

(iii) $\mathscr{G}_b(a_\sigma, a, a) \to 0$ as $\sigma \to \infty$.

Definition 1.4 ([2]) A sequence $\langle a_\sigma \rangle$ in a $\mathscr{G}_b$-metric space $(\mathscr{M},\mathscr{G}_b)$ is said to be $\mathscr{G}_b$-Cauchy if for every $\epsilon > 0$ there is a positive integer N such that $\mathscr{G}_b(x_\sigma, x_\xi, x_\eta) < \epsilon$ for all $\sigma, \xi, \eta \geq N$.

Proposition 1.2. ([2]) Let $(\mathscr{M}, \mathscr{G}_b)$ be a $\mathscr{G}_b$-metric space.Then followings are equivallent.

(i) $\langle a_\sigma \rangle$ is $\mathscr{G}_b$-Cauchy.

(ii) for every $\epsilon > 0$ there is a positive integer N such that $\mathscr{G}_b(a_\sigma, a_\xi, a_\xi) < \epsilon$ for all $\xi, \sigma \geq N$.

Proposition 1.3. ([2]) Every $\mathscr{G}_b$-convergent sequence in a $\mathscr{G}_b$-metric space $(\mathscr{M}, \mathscr{G}_b)$ is $\mathscr{G}_b$-Cauchy.

Definition 1.5 ([2]) A $\mathscr{G}_b$-metric space $(\mathscr{M}, \mathscr{G}_b)$ is said to be $\mathscr{G}_b$-complete if every $\mathscr{G}_b$-Cauchy sequence in $\mathscr{M}$ converges in it.

Proposition 1.4. ([2]) Let$(\mathscr{M}, \mathscr{G}_b)$ be a $\mathscr{G}_b$-metric space.A mapping $g : \mathscr{M} \to \mathscr{M}$ is $\mathscr{G}_b$-continuous at $a \in \mathscr{M}$ if and only if the sequence$\langle g a_\sigma \rangle$ converges to ga whenever $\langle a_\sigma \rangle$ converges to a.

Proposition 1.5. ([2]) The $\mathscr{G}_b$-metric $\mathscr{G}_b(a, b, c)$ is jointly continuous in all the three variables a, b and c.

Proposition 1.6. ([2]) If$(\mathscr{M}, \mathscr{G}_b)$ is a $\mathscr{G}_b$-metric space, then it follows that

(i) if $\mathscr{G}_b(a, b, c) = 0$ then $a = b = c$;

(ii) $\mathscr{G}_b(a, b, c) \leq s[\mathscr{G}_b(a, a, b) + \mathscr{G}_b(a, a, c)]$;

(iii) $\mathscr{G}_b(a, b, b) \leq 2s\mathscr{G}_b(b, a, a)$;

(iv) $\mathscr{G}_b(a, b, c) \leq s[\mathscr{G}_b(a, x, c) + \mathscr{G}_b(x, b, c)]$ for all $a, b, c, x \in \mathscr{M}$.

Samet and Vetro [23] introduced a fixed point of order $N \geq 3$ in the year 2010. The following is the definition of fixed point of order 3.

Definition 1.6 ([23]) Let $\mathscr{M}$ be a nonempty set. An element (a, b, c) in $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$is said to be tripled fixed point of a mapping $\mathscr{S} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, if $\mathscr{S}(a, b, c) = a$,$\mathscr{S}(b, a, c) = b$ and $\mathscr{S}(c, a, b) = b$.

Definition 1.7 ([23]) An element (a, b, c) in $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$is said to be common tripled fixed point of mappings $\mathscr{S}, \mathscr{T}$ and $\mathscr{R}$ on $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$, if $\mathscr{S}(a, b, c) = \mathscr{T}(a, b, c) = \mathscr{R}(a, b, c) = a$,$\mathscr{S}(b, a, c) = \mathscr{T}(b, a, c) = \mathscr{R}(b, a, c) = b$ and $\mathscr{S}(c, a, b) = \mathscr{T}(c, a, b) = \mathscr{R}(c, a, b) = c$.

In 2011,Berinde and Borcut [6] defined another notion of a tripled fixed point for mappings which are satisfying mixed monotone property in partially ordered metric spaces.

2 Main Results

Theorem 2.1 *Let $(\mathscr{M}, \mathscr{G}_b)$ be a complete symmetric $\mathscr{G}_b$-metric space with parameter $s \geq 1$ and $\mathscr{S}, \mathscr{T}, \mathscr{R} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be three mappings such that*

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{T}(\lambda, \kappa, \zeta), \mathscr{R}(\nu, \iota, \wp)) \\
\leq &\mathbb{K}_1 \frac{\mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}{3} \\
&+ \mathbb{K}_2 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{T}(\lambda, \kappa, \zeta), \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}, \\
&+ \mathbb{K}_3 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{T}(\lambda, \kappa, \zeta), \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)} \\
&+ \mathbb{K}_4 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{T}(\lambda, \kappa, \zeta), \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)} \\
&+ \mathbb{K}_5 \frac{\mathscr{G}_b(x, x, \mathscr{S}(x, y, z)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)} \\
&+ \mathbb{K}_6 \frac{\mathscr{G}_b(x, x, \mathscr{S}(x, y, z)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, b) + \mathscr{G}_b(z, \zeta, \wp)}
\end{aligned} \tag{2.1}$$

$$+\mathbb{K}_7\frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(z,\zeta,\wp)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_8\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(x,\lambda,\nu)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_9\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(y,\kappa,\iota)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_{10}\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(z,\zeta,\wp)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_{11}\frac{\mathscr{G}_b(\nu,\nu,\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(x,\lambda,\nu)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_{12}\frac{\mathscr{G}_b(\nu,\nu,\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(y,\kappa,\iota)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$
$$+\mathbb{K}_{13}\frac{\mathscr{G}_b(\nu,\nu,\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(z,\zeta,\wp)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}$$

for all $x,y,z,\lambda,\kappa,\zeta,\nu,\iota,\wp\in\mathscr{M}$ and non negative real numbers $\mathbb{K}_1,\mathbb{K}_2,\mathbb{K}_3,\mathbb{K}_4,\mathbb{K}_5,\mathbb{K}_6,\mathbb{K}_7,\mathbb{K}_8,$ $\mathbb{K}_9,\mathbb{K}_{10},\mathbb{K}_{11},\mathbb{K}_{12},\mathbb{K}_{13}$ with $0\le\mathbb{K}_1+\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4+3(\mathbb{K}_5+\mathbb{K}_6+\mathbb{K}_7)+\mathbb{K}_8+\mathbb{K}_9+\mathbb{K}_{10}+\mathbb{K}_{11}+\mathbb{K}_{12}+\mathbb{K}_{13}<1$. Then $\mathscr{S},\mathscr{T}$ and $\mathscr{R}$ have a unique common tripled fixed point in $\mathscr{M}$.

Proof. Let x_0,y_0 and z_0 be any three elements in $\mathscr{M}$. We construct three sequences $\langle x_\sigma\rangle_{\sigma=1}^{\infty}$,$\langle y_\sigma\rangle_{\sigma=1}^{\infty}$ and $\langle z_\sigma\rangle_{\sigma=1}^{\infty}$ in $\mathscr{M}$ as follows:

$$x_{3\phi+1}=\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\ y_{3\phi+1}=\mathscr{S}(y_{3\phi},z_{3\phi},x_{3\phi}),\ z_{3\phi+1}=\mathscr{S}(z_{3\phi},x_{3\phi},y_{3\phi})$$
$$x_{3\phi+2}=\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\ y_{3\phi+2}=\mathscr{T}(y_{3\phi+1},z_{3\phi+1},x_{3\phi+1}),\ z_{3\phi+2}=\mathscr{T}(z_{3\phi+1},x_{3\phi+1},y_{3\phi+1})$$
$$x_{3\phi+3}=\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}),\ y_{3\phi+3}=\mathscr{R}(y_{3\phi+2},z_{3\phi+2},x_{3\phi+2}),\ z_{3\phi+3}=\mathscr{R}(z_{3\phi+2},x_{3\phi+2},y_{3\phi+2})$$
$$\text{for } \phi=0,1,2,3,...$$

Now using (2.1) with $x=x_{3\phi}$,$y=y_{3\phi}$,$z=z_{3\phi}$,$\lambda=x_{3\phi+1}$,$\kappa=y_{3\phi+1}$,$\zeta=z_{3\phi+1}$,$\nu=x_{3\phi+2}$,$\iota=y_{3\phi+2}$ and $\wp=z_{3\phi+2}$, we get

$$\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})=\mathscr{G}_b(\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\mathscr{S}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))$$
$$\le\mathbb{K}_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}$$
$$+\mathbb{K}_2\frac{\mathscr{G}_b(\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_3\frac{\mathscr{G}_b(\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_4\frac{\mathscr{G}_b(\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_5\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_6\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_7\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_8\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$
$$+\mathbb{K}_9\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}$$

$$
\begin{aligned}
&+\mathbb{K}_{10}\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{11}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{12}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{13}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{K}_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}\\
&+\mathbb{K}_2\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_3\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_4\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_5\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_6\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_7\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_8\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_9\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{10}\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{11}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{12}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}\\
&+\mathbb{K}_{13}\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})\cdot\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{1+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}
\end{aligned}
$$

$$
\begin{aligned}
\leq&\mathbb{K}_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}\\
&+\mathbb{K}_2\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})+\mathbb{K}_3\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})+\mathbb{K}_4\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})\\
&+\mathbb{K}_5\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})+\mathbb{K}_6\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})+\mathbb{K}_7\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\\
&+\mathbb{K}_8\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})+\mathbb{K}_9\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})+\mathbb{K}_{10}\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\\
&+\mathbb{K}_{11}\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})+\mathbb{K}_{12}\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})+\mathbb{K}_{13}\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})
\end{aligned}
$$

Since $\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\leq\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})$, $\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\leq\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})$ and $\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})\leq\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})$, we have

$$
\mathscr{G}_b(x_{3\phi+1},x_{3\phi+2},x_{3\phi+3})\leq\mathbb{K}_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}
$$

$$+ (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4) \cdot \mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) + (\mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7) \cdot \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2})$$
$$+ (\mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13}) \cdot \mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3})$$

so that

$$\begin{aligned}\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) \leq & \frac{\frac{\mathbb{K}_1}{3} + \mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2}) \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}\mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) \leq & \frac{\frac{\mathbb{K}_1}{3} + \mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \leq & \frac{\frac{\mathbb{K}_1}{3} + \mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \frac{\frac{\mathbb{K}_1}{3}}{[1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})]} \cdot \\ & \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) \end{aligned} \tag{2.4}$$

Adding (2.2),(2.3) and (2.4),we have

$$\begin{aligned} \mathscr{G}_b(x_{3\phi+1}, & x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \\ & \leq \frac{\mathbb{K}_1 + 3(\mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7)}{1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})} \cdot \\ & \qquad [\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]. \end{aligned}$$

or

$$\begin{aligned} \mathscr{G}_b(x_{3\phi+1}, & x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \\ & \leq c \cdot [\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]. \end{aligned} \tag{2.5}$$

where $c = \dfrac{\mathbb{K}_1 + 3(\mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7)}{1 - (\mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13})}$ and $0 \leq c < 1$ in view of choice of $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3, \mathbb{K}_4, \mathbb{K}_5, \mathbb{K}_6, \mathbb{K}_7, \mathbb{K}_8, \mathbb{K}_9, \mathbb{K}_{10}, \mathbb{K}_{11}, \mathbb{K}_{12}$ and $\mathbb{K}_{13}$.
Also, it can be proved that

$$\mathscr{G}_b(x_{3\phi+2}, x_{3\phi+3}, x_{3\phi+4}) + \mathscr{G}_b(y_{3\phi+2}, y_{3\phi+3}, y_{3\phi+4}) + \mathscr{G}_b(z_{3\phi+2}, z_{3\phi+3}, z_{3\phi+4})$$

$$\leq c \cdot [\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3})].$$
$$\leq c^2 \cdot [\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})].$$

Proceeding like this, we get

$$\begin{aligned}
\mathscr{G}_b(x_\sigma,&x_{\sigma+1}, x_{\sigma+2}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+2}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+2}) \\
&\leq c \cdot [\mathscr{G}_b(x_{\sigma-1}, x_\sigma, x_{\sigma+1}) + \mathscr{G}_b(y_{\sigma-1}, y_\sigma, y_{\sigma+1}) + \mathscr{G}_b(z_{\sigma-1}, z_\sigma, z_{\sigma+1})]. \\
&\leq c^2 \cdot [\mathscr{G}_b(x_{\sigma-2}, x_{\sigma-1}, x_\sigma) + \mathscr{G}_b(y_{\sigma-2}, y_{\sigma-1}, y_\sigma) + \mathscr{G}_b(z_{\sigma-2}, z_{\sigma-1}, z_\sigma)]. \\
&\qquad \vdots \\
&\leq c^\sigma \cdot [\mathscr{G}_b(x_0, x_1, x_2) + \mathscr{G}_b(y_0, y_1, y_2) + \mathscr{G}_b(z_0, z_1, z_2)] \text{ where } \sigma = 3\phi + 2.
\end{aligned}$$

Let $\Delta_\sigma = \mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+2}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+2}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+2})$. Then the above inequality becomes as $\Delta_\sigma \leq c\Delta_{\sigma-1} \leq c^2\Delta_{\sigma-2} \leq ... \leq c^\sigma \Delta_0$ for $\sigma \in \mathbb{N}$

Suppose $\xi > \sigma$. Now using repeated application of rectangle inequality of $\mathscr{G}_b$-metric and the inequality$\Delta_\sigma \leq c^\sigma \Delta_0$, we have

$$\begin{aligned}
\mathscr{G}_b(x_\sigma,&x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq s[\mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(x_{\sigma+1}, x_\xi, x_\xi)] \\
&+ s[\mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+1}) + \mathscr{G}_b(y_{\sigma+1}, y_\xi, y_\xi)] + s[\mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+1}) + \mathscr{G}_b(z_{\sigma+1}, z_\xi, z_\xi)] \\
\leq& s[\mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+1}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+1})] \\
&+ s^2[\mathscr{G}_b(x_{\sigma+1}, x_{\sigma+2}, x_{\sigma+2}) + \mathscr{G}_b(y_{\sigma+1}, y_{\sigma+2}, y_{\sigma+2}) + \mathscr{G}_b(z_{\sigma+1}, z_{\sigma+2}, z_{\sigma+2})] \\
&+ \cdots s^{\xi-\sigma}[\mathscr{G}_b(x_{\xi-1}, x_{\xi-1}, x_\xi) + \mathscr{G}_b(y_{\xi-1}, y_{\xi-1}, y_\xi) + \mathscr{G}_b(z_{\xi-1}, z_{\xi-1}, z_\xi)] \\
=& s\Delta_\sigma + s^2\Delta_{\sigma+1} + s^3\Delta_{\sigma+2} + \cdots + s^{\xi-\sigma}\Delta_{\xi-1} \\
\leq& [sc^\sigma + s^2c^{\sigma+1} + \cdots + s^{\xi-\sigma}c^{\xi-1}]\Delta_0 \\
\leq& sc^\sigma \cdot \frac{1}{1-c}\Delta_0 \text{ for } \xi > \sigma
\end{aligned}$$

or

$$\mathscr{G}_b(x_\sigma, x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq s.c^n \cdot \frac{1}{1-c}\Delta_0 \text{ for } \xi > \sigma. \tag{2.6}$$

Since $0 \leq c < 1$,$c^\sigma \to 0$ as $\sigma \to \infty$.

Now applying limit as $\sigma \to \infty$ with $\xi > \sigma$ in the inequality (2.6), we have $\mathscr{G}_b(x_\sigma, x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq 0$ which follows that $\langle x_\sigma\rangle_{\sigma=1}^\infty$,$\langle y_\sigma\rangle_{\sigma=1}^\infty$ and $\langle z_\sigma\rangle_{\sigma=1}^\infty$ are $\mathscr{G}_b-$cauchy sequences in $\mathscr{M}$.

Since $(\mathscr{M}, \mathscr{G}_b)$ is a complete $\mathscr{G}_b$-metric space, there exist $x, y, z \in \mathscr{M}$ such that $x_\sigma \to x$,$y_\sigma \to y$ and $z_\sigma \to z$ as $\sigma \to \infty$.

Now we prove that (x, y, z) is a tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.

Again using (2.1) with $x = x_\sigma$,$y = y_\sigma$,$z = z_\sigma$,$\sigma = x$,$\kappa = y$,$\zeta = z$,$\nu = x$,$\iota = y$ and $\wp = z$, we get

$$\begin{aligned}
\mathscr{G}_b(x, \mathscr{S}(x,y,z),\mathscr{S}(x,y,z)) &\leq s[\mathscr{G}_b(x, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(x_{\sigma+1}, \mathscr{S}(x,y,z), \mathscr{S}(x,y,z))] \qquad (2.7) \\
&\leq s\mathscr{G}_b(x, x_{\sigma+1}, x_{\sigma+1}) + s\big[\mathbb{K}_1 \frac{\mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)}{3} \\
&\quad + \mathbb{K}_2 \frac{\mathscr{G}_b(S(x_\sigma, x, x), \mathscr{S}(y_\sigma, y, y), \mathscr{S}(z_\sigma, z, z)) \cdot \mathscr{G}_b(x_\sigma, x, x)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&\quad + \mathbb{K}_3 \frac{\mathscr{G}_b(\mathscr{S}(x_\sigma, x, x), \mathscr{S}(y_\sigma, y, y), \mathscr{S}(z_\sigma, z, z)) \cdot \mathscr{G}_b(y_\sigma, v, b)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&\quad + \mathbb{K}_4 \frac{\mathscr{G}_b(\mathscr{S}(x_\sigma, x, x), \mathscr{S}(y_\sigma, y, y), \mathscr{S}(z_\sigma, z, z)) \cdot \mathscr{G}_b(z_\sigma, w, c)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&\quad + \mathbb{K}_5 \frac{\mathscr{G}_b(x_\sigma, x_\sigma, \mathscr{S}(x_\sigma, y_\sigma, z_\sigma)) \cdot \mathscr{G}_b(x_\sigma, x, x)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{K}_6 \frac{\mathscr{G}_b(x_\sigma, x_\sigma, \mathscr{S}(x_\sigma, y_\sigma, z_\sigma)) \cdot \mathscr{G}_b(y_\sigma, y, y)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_7 \frac{\mathscr{G}_b(x_\sigma, x_\sigma, \mathscr{S}(x_\sigma, y_\sigma, z_\sigma)) \cdot \mathscr{G}_b(z_\sigma, z, z)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_8 \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(x_\sigma, x, x)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_9 \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(y_\sigma, y, y)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_{10} \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(z_\sigma, z, z)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_{11} \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(x_\sigma, x, x)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_{12} \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(y_\sigma, y, y)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)} \\
&+ \mathbb{K}_{13} \frac{\mathscr{G}_b(x, x, S(x, y, z)) \cdot \mathscr{G}_b(z_\sigma, z, z)}{1 + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)}]
\end{aligned}
$$

Since $x_\sigma \to x$,$y_\sigma \to y$ and $z_\sigma \to z$, from the inequality(2.7) it follows that $\mathscr{G}_b(x, \mathscr{S}(x,y,z), \mathscr{S}(x,y,z)) = 0$ which implies that $\mathscr{S}(x,y,z) = x$ and also $\mathscr{S}(y,x,z) = y$,$\mathscr{S}(z,x,y) = z$.
Similarly, it can be shown that $\mathscr{T}(x,y,z) = x, \mathscr{T}(y,x,z) = y, \mathscr{T}(z,x,y) = z, \mathscr{R}(x,y,z) = x, \mathscr{R}(y,x,z) = y$ and $\mathscr{R}(z,x,y) = z$.
That is (x,y,z) is a common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.

Uniqueness:

Suppose (l,m,n) is another common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.
That is, $\mathscr{S}(l,m,n) = \mathscr{T}(l,m,n) = \mathscr{R}(l,m,n) = l, \mathscr{S}(m,l,n) = \mathscr{T}(m,l,n) = \mathscr{R}(m,l,n) = m$ and $\mathscr{S}(n,l,m) = \mathscr{T}(n,l,m) = \mathscr{R}(n,l,m) = n$
Using inequality(2.1),we have

$$
\begin{aligned}
\mathscr{G}_b(x,l,l) = \mathscr{G}_b(\mathscr{S}(x,y,z), \mathscr{T}(l,m,n), \mathscr{R}(l,m,n))
&\le \mathbb{K}_1 \frac{\mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)}{3} \\
&+ \mathbb{K}_2 \frac{\mathscr{G}_b(\mathscr{S}(x,y,z), \mathscr{T}(l,m,n), \mathscr{R}(l,m,n)) \cdot \mathscr{G}_b(x,l,l)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_3 \frac{\mathscr{G}_b(\mathscr{S}(x,y,z), \mathscr{T}(l,m,n), \mathscr{R}(l,m,n)) \cdot \mathscr{G}_b(y,m,m)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_4 \frac{\mathscr{G}_b(\mathscr{S}(x,y,z), \mathscr{T}(l,m,n), \mathscr{R}(l,m,n) \cdot \mathscr{G}_b(z,n,n)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_5 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(x,l,l)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_6 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(y,m,m)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_7 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(z,n,n)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_8 \frac{\mathscr{G}_b(l,l,\mathscr{T}(l,m,n)) \cdot \mathscr{G}_b(x,l,l)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_9 \frac{\mathscr{G}_b(l,l,\mathscr{T}(l,m,n)) \cdot \mathscr{G}_b(y,m,m)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)} \\
&+ \mathbb{K}_{10} \frac{\mathscr{G}_b(l,l,\mathscr{T}(l,m,n)) \cdot \mathscr{G}_b(z,n,n)}{1 + \mathscr{G}_b(x,l,l) + \mathscr{G}_b(y,m,m) + \mathscr{G}_b(z,n,n)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbb{K}_{11}\frac{\mathscr{G}_b(l,l,\mathscr{R}(l,m,n))\cdot\mathscr{G}_b(x,l,l)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{12}\frac{\mathscr{G}_b(l,l,\mathscr{R}(l,m,n))\cdot\mathscr{G}_b(y,m,m)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{13}\frac{\mathscr{G}_b(l,l,\mathscr{R}(l,m,n))\cdot\mathscr{G}_b(z,n,n)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{K}_1\frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3}+\mathbb{K}_2\frac{\mathscr{G}_b(x,l,l)\cdot\mathscr{G}_b(x,l,l)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_3\frac{\mathscr{G}_b(x,l,l)\cdot\mathscr{G}_b(y,m,m)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_4\frac{\mathscr{G}_b(x,l,l)\cdot\mathscr{G}_b(z,n,n)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_5\frac{\mathscr{G}_b(x,x,x)\cdot\mathscr{G}_b(x,l,l)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_6\frac{\mathscr{G}_b(x,x,x)\cdot\mathscr{G}_b(y,m,m)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_7\frac{\mathscr{G}_b(x,x,x)\cdot\mathscr{G}_b(z,n,n)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_8\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(x,l,l)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_9\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(y,m,m)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{10}\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(z,n,n)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{11}\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(x,l,l)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{12}\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(y,m,m)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}\\
&+ \mathbb{K}_{13}\frac{\mathscr{G}_b(l,l,l)\cdot\mathscr{G}_b(z,n,n)}{1+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}
\end{aligned}
$$

which implies

$$\mathscr{G}_b(x,l,l)\le\mathbb{K}_1\frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3}+(\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4)\mathscr{G}_b(x,l,l) \tag{2.8}$$

Similarly

$$\mathscr{G}_b(y,m,m)\le\mathbb{K}_1\frac{\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)+\mathscr{G}_b(x,l,l)}{3}+(\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4)\mathscr{G}_b(y,m,m) \tag{2.9}$$

and

$$\mathscr{G}_b(z,n,n)\le\mathbb{K}_1\frac{\mathscr{G}_b(z,n,n)+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)}{3}+(\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4)\mathscr{G}_b(z,n,n) \tag{2.10}$$

Adding (2.8),(2.9) and (2.10), we get

$$
\begin{aligned}
\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)\le(\mathbb{K}_1+\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4)[&\mathscr{G}_b(x,l,l)\\
&+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]
\end{aligned}
$$

or

$$(1-\mathbb{K}_1-\mathbb{K}_2-\mathbb{K}_3-\mathbb{K}_4)\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)\leq 0 \tag{2.11}$$

Since $\mathbb{K}_1+\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4<1$, we have $\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)=0$ which implies that $x=l, y=m$ and $z=n$.
Thus, (x,y,z) is a unique common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$. □

Example 2.1. Let $\mathscr{M}=[0,1]$. Define $\mathscr{G}_b:\mathscr{M}\times\mathscr{M}\times\mathscr{M}\to[0,\infty)$ by $\mathscr{G}_b(\alpha,\beta,\gamma)=(|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|)^2$, where $s=2$. Clearly, $(\mathscr{M},\mathscr{G}_b)$ is a complete $\mathscr{G}_b$-metric space.
Let $\mathscr{S}(x,y,z)=\frac{2x-4y+6z+20}{24}$,$\mathscr{T}(\lambda,\kappa,\zeta)=\frac{3\lambda-6\kappa+9\zeta+30}{36}$ and
$\mathscr{R}(\nu,\iota,\wp)=\frac{4\nu-8\iota+12\wp+40}{48}$ for $x,y,z,\lambda,\kappa,\zeta,\nu,\iota,\wp\in\mathscr{M}$.
Now

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(\lambda,\kappa,\zeta),\mathscr{R}(\nu,\iota,\wp))=\big(|(x,y,z)-(\lambda,\kappa,\zeta)|+|(\lambda,\kappa,\zeta)-(\nu,\iota,\wp)|\\
&\qquad+|(\nu,\iota,\wp))-(x,y,z)|\big)^2\\
&=\Big(\Big|\tfrac{2x-4y+6z+20}{24}-\tfrac{3\lambda-6\kappa+9\zeta+30}{36}\Big|+\Big|\tfrac{3\lambda-6\kappa+9\zeta+30}{36}-\tfrac{4\nu-8\iota+12\wp+40}{48}\Big|\\
&\qquad+\big|\tfrac{4\nu-8\iota+12\wp+40}{48}-\tfrac{2x-4y+6z+20}{24}\big|\Big)^2\\
&\leq\Big(\tfrac{1}{12}[|x-\lambda|+|\lambda-\nu|+|\nu-x|]+\tfrac{1}{6}[|y-\kappa|+|\kappa-\iota|+|\iota-y|]\\
&\qquad+\tfrac{1}{4}[|z-\zeta|+|\zeta-\wp|+|\wp-z|]\Big)^2\\
&\leq\Big(\tfrac{1}{4}[|x-\lambda|+|\lambda-\nu|+|\nu-x|]+\tfrac{1}{4}[|y-\kappa|+|\kappa-\iota|+|\iota-y|]\\
&\qquad+\tfrac{1}{4}[|z-\zeta|+|\zeta-\wp|+|\wp-z|]\Big)^2\\
&\leq\tfrac{3}{16}\big[(|x-\lambda|+|\lambda-\nu|+|\nu-x|)^2+(|y-\kappa|+|\kappa-\iota|+|\iota-y|)^2\\
&\qquad+(|z-\zeta|+|\zeta-\wp|+|\wp-z|)^2\big]\\
&=\tfrac{3}{16}[\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]\\
&=\tfrac{9}{16}\cdot\tfrac{[\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}{3}
\end{aligned}$$

$$\begin{aligned}
&\leq\mathbb{K}_1\frac{\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}{3}\\
&\quad+\mathbb{K}_2\frac{\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(\lambda,\kappa,\zeta),\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(x,\lambda,\nu)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_3\frac{\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(\lambda,\kappa,\zeta),\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(y,\kappa,\iota)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_4\frac{\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(\lambda,\kappa,\zeta),\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(z,\zeta,\wp)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_5\frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(x,\lambda,\nu)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_6\frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(y,\kappa,\iota)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,b)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_7\frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(z,\zeta,\wp)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_8\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(x,\lambda,\nu)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}\\
&\quad+\mathbb{K}_9\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(y,\kappa,\iota)}{1+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}
\end{aligned}$$

$$+ \mathbb{K}_{10} \frac{\mathscr{G}_b(\lambda, \lambda, \mathscr{T}(\lambda, \kappa, \zeta)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{11} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{12} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{13} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{R}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$

where $\mathbb{K}_1 = 9/16$ and $\mathbb{K}_2 = \mathbb{K}_3 = \mathbb{K}_4 = ... = \mathbb{K}_{13} = 0$
so that $0 \leq \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_3 + \mathbb{K}_4 + 3(\mathbb{K}_5 + \mathbb{K}_6 + \mathbb{K}_7) + \mathbb{K}_8 + \mathbb{K}_9 + \mathbb{K}_{10} + \mathbb{K}_{11} + \mathbb{K}_{12} + \mathbb{K}_{13} < 1$. Thus all the conditions of Theorem 2.1 are satisfied. Therefore, $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$ have a unique common coupled fixed point, namely $(1, 1, 1)$.

Corollary 2.1. Let $(\mathscr{M}, \mathscr{G}_b)$ be a complete symmetric $\mathscr{G}_b$-metric space with parameter $s \geq 1$ and $\mathscr{S} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be a mapping such that

$$\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{S}(\lambda, \kappa, \zeta), \mathscr{S}(\nu, \iota, \wp)) \tag{2.12}$$
$$\leq \mathbb{K}_1 \frac{\mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}{3}$$
$$+ \mathbb{K}_2 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{S}(\lambda, \kappa, \zeta), \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_3 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{S}(\lambda, \kappa, \zeta), \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_4 \frac{\mathscr{G}_b(\mathscr{S}(x, y, z), \mathscr{S}(\lambda, \kappa, \zeta), \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_5 \frac{\mathscr{G}_b(x, x, \mathscr{S}(x, y, z)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_6 \frac{\mathscr{G}_b(x, x, \mathscr{S}(x, y, z)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, b) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_7 \frac{\mathscr{G}_b(x, x, \mathscr{S}(x, y, z)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_8 \frac{\mathscr{G}_b(\lambda, \lambda, \mathscr{S}(\lambda, \kappa, \zeta)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_9 \frac{\mathscr{G}_b(\lambda, \lambda, \mathscr{S}(\lambda, \kappa, \zeta)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{10} \frac{\mathscr{G}_b(\lambda, \lambda, \mathscr{S}(\lambda, \kappa, \zeta)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{11} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(x, \lambda, \nu)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{12} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(y, \kappa, \iota)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$
$$+ \mathbb{K}_{13} \frac{\mathscr{G}_b(\nu, \nu, \mathscr{S}(\nu, \iota, \wp)) \cdot \mathscr{G}_b(z, \zeta, \wp)}{1 + \mathscr{G}_b(x, \lambda, \nu) + \mathscr{G}_b(y, \kappa, \iota) + \mathscr{G}_b(z, \zeta, \wp)}$$

for all $x, y, z, \lambda, \kappa, \zeta, \nu, \iota, \wp \in \mathscr{M}$ and non negative real numbers $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3, \mathbb{K}_4, \mathbb{K}_5, \mathbb{K}_6, \mathbb{K}_7, \mathbb{K}_8,$
$\mathbb{K}_9, \mathbb{K}_{10}, \mathbb{K}_{11}, \mathbb{K}_{12}$,$\mathbb{K}_{13}$ with $0 \leq \mathbb{K}_1+\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4+3(\mathbb{K}_5+\mathbb{K}_6+\mathbb{K}_7)+\mathbb{K}_8+\mathbb{K}_9+\mathbb{K}_{10}+\mathbb{K}_{11}+\mathbb{K}_{12}+\mathbb{K}_{13} < 1$.
Then $\mathscr{S}$ has a unique tripled fixed point in $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$.

Theorem 2.2 *Let $(\mathscr{M},\mathscr{G}_b)$ be a complete symmetric $\mathscr{G}_b$-metric space with parameter $s \geq 1$ and $\mathscr{S},\mathscr{T},\mathscr{R} : \mathscr{M}\times\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ be three mappings such that*

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(\lambda,\kappa,\zeta),\mathscr{R}(\nu,\iota,\wp)) \leq \chi_1\frac{\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}{3}\\
&+\chi_2\frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))}{1+s[\mathscr{G}_b(x,x,\mathscr{T}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}\\
&+\chi_3\frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{T}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(\nu,\nu,\mathscr{R}(\nu,\iota,\wp))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{T}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{R}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,c)]}\\
&+\chi_4\frac{\mathscr{G}_b(\nu,\nu,\mathscr{R}(\nu,\iota,\wp))\cdot\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,x,\mathscr{R}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}
\end{aligned} \tag{2.13}$$

for all $x,y,z,\lambda,\kappa,\zeta,\nu,\iota,\wp\in\mathscr{M}$ and non negative real numbers χ_1,χ_2,χ_3 and χ_4 with $0\leq\chi_1+\chi_2+\chi_3+\chi_4<1$.
Then $\mathscr{S},\mathscr{T}$ and $\mathscr{R}$ have a unique common tripled fixed point in $\mathscr{M}\times\mathscr{M}\times\mathscr{M}$.

Proof. Let x_0,y_0 and z_0 be any three elements in $\mathscr{M}$. We construct three sequences $\langle x_\sigma\rangle_{\sigma=1}^{\infty}$,$\langle y_\sigma\rangle_{\sigma=1}^{\infty}$ and $\langle z_\sigma\rangle_{\sigma=1}^{\infty}$ in $\mathscr{M}$ as follows:

$$\begin{aligned}
&x_{3\phi+1}=\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\ y_{3\phi+1}=\mathscr{S}(y_{3\phi},z_{3\phi},x_{3\phi}),\ z_{3\phi+1}=\mathscr{S}(z_{3\phi},x_{3\phi},y_{3\phi})\\
&x_{3\phi+2}=\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\ y_{3\phi+2}=\mathscr{T}(y_{3\phi+1},z_{3\phi+1},x_{3\phi+1}),\ z_{3\phi+2}=\mathscr{T}(z_{3\phi+1},x_{3\phi+1},y_{3\phi+1})\\
&x_{3\phi+3}=\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}),\ y_{3\phi+3}=\mathscr{R}(y_{3\phi+2},z_{3\phi+2},x_{3\phi+2}),\ z_{3\phi+3}=\mathscr{R}(z_{3\phi+2},x_{3\phi+2},y_{3\phi+2})\\
&\qquad\text{for } \phi=0,1,2,3,...
\end{aligned}$$

Now using (2.13) with $x=x_{3\phi}$,$y=y_{3\phi}$,$z=z_{3\phi}$,$\lambda=x_{3\phi+1}$,$\kappa=y_{3\phi+1}$,$\zeta=z_{3\phi+1}$,$\nu=x_{3\phi+2}$,$\iota=y_{3\phi+2}$ and $\wp=z_{3\phi+2}$, we get

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}),\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}),\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\\
&\leq\chi_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}\\
&\quad+\chi_2\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))\cdot\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))}{\begin{aligned}1+s[&\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))+\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))\\&+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})]\end{aligned}}\\
&\quad+\chi_3\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))\cdot\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},R(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))}{\begin{aligned}1+s[&\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{T}(x_{3\phi+1},y_{3\phi+1},z_{3\phi+1}))+\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},R(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\\&+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},b)+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})]\end{aligned}}\\
&\quad+\chi_4\frac{\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\cdot\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))}{\begin{aligned}1+s[&\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},\mathscr{S}(x_{3\phi},y_{3\phi},z_{3\phi}))+\mathscr{G}_b(x_{3\phi},x_{3\phi},\mathscr{R}(x_{3\phi+2},y_{3\phi+2},z_{3\phi+2}))\\&+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})]\end{aligned}}\\
&=\chi_1\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})}{3}\\
&\quad+\chi_2\frac{\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+1})\cdot\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})}{\begin{aligned}1+s[&\mathscr{G}_b(x_{3\phi},x_{3\phi},x_{3\phi+2})+\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+1})+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})\\&+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})]\end{aligned}}\\
&\quad+\chi_3\frac{\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+2})\cdot\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+3})}{\begin{aligned}1+s[&\mathscr{G}_b(x_{3\phi+2},x_{3\phi+2},x_{3\phi+2})+\mathscr{G}_b(x_{3\phi+1},x_{3\phi+1},x_{3\phi+3})+\mathscr{G}_b(x_{3\phi},x_{3\phi+1},x_{3\phi+2})\\&+\mathscr{G}_b(y_{3\phi},y_{3\phi+1},y_{3\phi+2})+\mathscr{G}_b(z_{3\phi},z_{3\phi+1},z_{3\phi+2})]\end{aligned}}
\end{aligned}$$

$$+\chi_4 \frac{\mathscr{G}_b(x_{3\phi+2}, x_{3\phi+2}, x_{3\phi+3}) \cdot \mathscr{G}_b(x_{3\phi}, x_{3\phi}, x_{3\phi+1})}{1 + s[\mathscr{G}_b(x_{3\phi+2}, x_{3\phi+2}, x_{3\phi+1}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi}, x_{3\phi+3}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]}$$

$$\leq \chi_1 \frac{\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})}{3}$$

$$+\chi_2 \frac{\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) \cdot \mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3})}{1 + s[\mathscr{G}_b(x_{3\phi}, x_{3\phi}, x_{3\phi+2}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]}$$

$$+\chi_3 \frac{\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) \cdot \mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3})}{1 + s[\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+1}, x_{3\phi+3}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]}$$

$$+\chi_4 \frac{\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) \cdot \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2})}{1 + s[\mathscr{G}_b(x_{3\phi+2}, x_{3\phi+2}, x_{3\phi+1}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi}, x_{3\phi+3}) + \mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]}$$

$$\leq \chi_1 \frac{\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})}{3} + (\chi_2 + \chi_3 + \chi_4) \cdot \mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3})$$

So that,

$$\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) \leq \frac{\frac{\chi_1}{3}}{[1 - (\chi_2 + \chi_3 + \chi_4)]} \cdot \big[\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})\big] \tag{2.14}$$

Similarly,

$$\mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) \leq \frac{\frac{\chi_1}{3}}{[1 - (\chi_2 + \chi_3 + \chi_4)]} \cdot \big[\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})\big] \tag{2.15}$$

and

$$\mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \leq \frac{\frac{\chi_1}{3}}{[1 - (\chi_2 + \chi_3 + \chi_4)]} \cdot \big[\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})\big] \tag{2.16}$$

Adding (2.14),(2.15) and (2.16),we have

$$\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \leq \frac{\chi_1}{[1 - (\chi_2 + \chi_3 + \chi_4)]} \cdot \big[\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})\big].$$

or

$$\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3}) \leq h \cdot [\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})]. \tag{2.17}$$

where $h = \dfrac{\chi_1}{[1-(\chi_2+\chi_3+\chi_4)]}$ and $0 \leq h < 1$ in view of choice of χ_1, χ_2, χ_3 and χ_4.
Also, it can be proved that

$$\begin{aligned}\mathscr{G}_b(x_{3\phi+2},&x_{3\phi+3}, x_{3\phi+4}) + \mathscr{G}_b(y_{3\phi+2}, y_{3\phi+3}, y_{3\phi+4}) + \mathscr{G}_b(z_{3\phi+2}, z_{3\phi+3}, z_{3\phi+4})\\ &\leq h \cdot [\mathscr{G}_b(x_{3\phi+1}, x_{3\phi+2}, x_{3\phi+3}) + \mathscr{G}_b(y_{3\phi+1}, y_{3\phi+2}, y_{3\phi+3}) + \mathscr{G}_b(z_{3\phi+1}, z_{3\phi+2}, z_{3\phi+3})].\\ &\leq h^2 \cdot [\mathscr{G}_b(x_{3\phi}, x_{3\phi+1}, x_{3\phi+2}) + \mathscr{G}_b(y_{3\phi}, y_{3\phi+1}, y_{3\phi+2}) + \mathscr{G}_b(z_{3\phi}, z_{3\phi+1}, z_{3\phi+2})].\end{aligned}$$

Proceeding like this, we get

$$\begin{aligned}\mathscr{G}_b(x_\sigma,&x_{\sigma+1}, x_{\sigma+2}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+2}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+2})\\ &\leq h \cdot [\mathscr{G}_b(x_{\sigma-1}, x_\sigma, x_{\sigma+1}) + \mathscr{G}_b(y_{\sigma-1}, y_\sigma, y_{\sigma+1}) + \mathscr{G}_b(z_{\sigma-1}, z_\sigma, z_{\sigma+1})].\\ &\leq h^2 \cdot [\mathscr{G}_b(x_{\sigma-2}, x_{\sigma-1}, x_\sigma) + \mathscr{G}_b(y_{\sigma-2}, y_{\sigma-1}, y_\sigma) + \mathscr{G}_b(z_{\sigma-2}, z_{\sigma-1}, z_\sigma)].\\ &\leq h^\sigma \cdot [\mathscr{G}_b(x_0, x_1, x_2) + \mathscr{G}_b(y_0, y_1, y_2) + \mathscr{G}_b(z_0, z_1, z_2)], \text{ where } \sigma = 3\phi + 2.\end{aligned}$$

Let $\Delta_\sigma = \mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+2}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+2}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+2})$. Then the above inequality becomes as $\Delta_\sigma \leq h\Delta_{\sigma-1} \leq h^2\Delta_{\sigma-2} \leq ... \leq h^\sigma\Delta_0$ for $\sigma \in \mathbb{N}$
Suppose $\xi > \sigma$. Now using repeated application of rectangle inequality of $\mathscr{G}_b$-metric and the inequality$\Delta_\sigma \leq h^\sigma\Delta_0$, we have

$$\begin{aligned}\mathscr{G}_b(x_\sigma,&x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq s[\mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(x_{\sigma+1}, x_\xi, x_\xi)]\\ &+ s[\mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+1}) + \mathscr{G}_b(y_{\sigma+1}, y_\xi, y_\xi)] + s[\mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+1}) + \mathscr{G}_b(z_{\sigma+1}, z_\xi, z_\xi)]\\ \leq& s[\mathscr{G}_b(x_\sigma, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(y_\sigma, y_{\sigma+1}, y_{\sigma+1}) + \mathscr{G}_b(z_\sigma, z_{\sigma+1}, z_{\sigma+1})]\\ &+ s^2[\mathscr{G}_b(x_{\sigma+1}, x_{\sigma+2}, x_{\sigma+2}) + \mathscr{G}_b(y_{\sigma+1}, y_{\sigma+2}, y_{\sigma+2}) + \mathscr{G}_b(z_{\sigma+1}, z_{\sigma+2}, z_{\sigma+2})]\\ &+ \cdots s^{m-n}[\mathscr{G}_b(x_{\xi-1}, x_{\xi-1}, x_m) + \mathscr{G}_b(y_{\xi-1}, y_{\xi-1}, y_\xi) + \mathscr{G}_b(z_{\xi-1}, z_{\xi-1}, z_\xi)]\\ =& s\Delta_n + s^2\Delta_{\sigma+1} + s^3\Delta_{\sigma+2} + \cdots + s^{\xi-\sigma}\Delta_{\xi-1}\\ \leq& [sh^\sigma + s^2h^{\sigma+1} + \cdots + s^{\xi-\sigma}h^{\xi-1}]\Delta_0\\ \leq& sh^\sigma \cdot \frac{1}{1-h}\Delta_0 \text{ for } \xi > \sigma\end{aligned}$$

or

$$\mathscr{G}_b(x_\sigma, x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq s.h^n \cdot \frac{1}{1-h}\Delta_0 \text{ for } \xi > \sigma. \tag{2.18}$$

Since $0 \leq h < 1$,$h^\sigma \to 0$ as $\sigma \to \infty$.
Now applying limit as $\sigma \to \infty$ with $\xi > \sigma$ in the inequality (2.18), we have
$\mathscr{G}_b(x_\sigma, x_\xi, x_\xi) + \mathscr{G}_b(y_\sigma, y_\xi, y_\xi) + \mathscr{G}_b(z_\sigma, z_\xi, z_\xi) \leq 0$ which follows that $\langle x_\sigma\rangle_{\sigma=1}^\infty$,$\langle y_\sigma\rangle_{\sigma=1}^\infty$ and $\langle z_\sigma\rangle_{\sigma=1}^\infty$ are $\mathscr{G}_b-$Cauchy sequences in $\mathscr{M}$.
Since $(\mathscr{M}, \mathscr{G}_b)$ is a complete $\mathscr{G}_b$-metric space, there exist $x, y, z \in \mathscr{M}$ such that $x_\sigma \to x$,$y_\sigma \to y$ and $z_\sigma \to z$.
Now we prove that (x, y, z) is a tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.
Again using (2.13) with $x = x_\sigma$,$y = y_\sigma$,$z = z_\sigma$,$\lambda = x$,$\kappa = y$,$\zeta = z$,$\nu = x$,$\iota = y$ and $\wp = z$, we get

$$\mathscr{G}_b(x, \mathscr{S}(x, y, z), \mathscr{S}(x, y, z)) \leq s[\mathscr{G}_b(x, x_{\sigma+1}, x_{\sigma+1}) + \mathscr{G}_b(x_{\sigma+1}, \mathscr{S}(x, y, z), \mathscr{S}(x, y, z))] \tag{2.19}$$

$$\leq s\mathscr{G}_b(x, x_{\sigma+1}, x_{\sigma+1}) + s\big[\chi_1\frac{\mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)}{3}\big] \tag{2.20}$$

$$+\chi_2\frac{\mathscr{G}_b(x_\sigma, x_\sigma, \mathscr{S}(x_\sigma, y_\sigma, z_\sigma)) \cdot \mathscr{G}_b(x, x, \mathscr{S}(x, y, z))}{1 + s[\mathscr{G}_b(x_\sigma, x_\sigma, \mathscr{S}(x, y, z)) + \mathscr{G}_b(x, x, \mathscr{S}(x_\sigma, y_\sigma, z_\sigma)) + \mathscr{G}_b(x_\sigma, x, x) + \mathscr{G}_b(y_\sigma, y, y) + \mathscr{G}_b(z_\sigma, z, z)]}$$

$$+\chi_3 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(x,x,\mathscr{S}(x,y,z))}{1+s[\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) + \mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) + \mathscr{G}_b(x_\sigma,x,x) + \mathscr{G}_b(y_\sigma,y,y)+\mathscr{G}_b(z_\sigma,z,z)]}$$
$$+\chi_4 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(x_\sigma,x_\sigma,\mathscr{S}(x_\sigma,y_\sigma,z_\sigma))}{1+s[\mathscr{G}_b(x,x,\mathscr{S}(x_\sigma,y_\sigma,z_\sigma)) + \mathscr{G}_b(x_\sigma,x_\sigma,\mathscr{S}(x,y,z)) + \mathscr{G}_b(x_\sigma,x,x) + \mathscr{G}_b(y_\sigma,y,y)+\mathscr{G}_b(z_\sigma,z,z)]}$$

Since $x_\sigma \to x$,$y_\sigma \to y$ and $z_\sigma \to z$, from the inequality(2.19) it follows that
$(1-s\chi_3)\mathscr{G}_b(x,\mathscr{S}(x,y,z),\mathscr{S}(x,y,z)) \leq 0$ which implies $\mathscr{S}(x,y,z)=x$.
Similarly, it can be shown that $\mathscr{S}(y,x,z)=y, \mathscr{S}(z,x,y)=z, \mathscr{T}(x,y,z)=x, \mathscr{T}(y,x,z)=y, \mathscr{T}(z,x,y)=z, \mathscr{R}(x,y,z)=x, \mathscr{R}(y,x,z)=y$ and $\mathscr{R}(z,x,y)=z$
That is (x,y,z) is a common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.

Uniqueness:

Suppose (l,m,n) is another common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$.
That is, $\mathscr{S}(l,m,n)=\mathscr{T}(l,m,n)=\mathscr{R}(l,m,n)=l, \mathscr{S}(m,l,n)=\mathscr{T}(m,l,n)=\mathscr{R}(m,l,n)=m$ and $\mathscr{S}(n,l,m)=\mathscr{T}(n,l,m)=\mathscr{R}(n,l,m)=n$
Using inequality(2.13),we have

$$\begin{aligned}
&\mathscr{G}_b(x,l,l) = \mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{T}(l,m,n),\mathscr{R}(l,m,n))\\
&\leq \chi_1 \frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3}\\
&+\chi_2 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z)) \cdot \mathscr{G}_b(l,l,\mathscr{T}(l,m,n))}{1+s[\mathscr{G}_b(x,x,\mathscr{T}(l,m,n))+\mathscr{G}_b(l,l,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}\\
&+\chi_3 \frac{\mathscr{G}_b(l,l,\mathscr{T}(l,m,n)) \cdot \mathscr{G}_b(l,l,\mathscr{R}(l,m,n))}{1+s[\mathscr{G}_b(l,l,\mathscr{T}(l,m,n))+\mathscr{G}_b(l,l,\mathscr{R}(l,m,n))+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}\\
&+\chi_4 \frac{\mathscr{G}_b(l,l,\mathscr{R}(l,m,n)) \cdot \mathscr{G}_b(x,x,\mathscr{S}(x,y,z))}{1+s[\mathscr{G}_b(l,l,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,x,\mathscr{R}(l,m,n))+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}
\end{aligned}$$

$$\begin{aligned}
&= \chi_1 \frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3}\\
&+\chi_2 \frac{\mathscr{G}_b(x,x,x) \cdot \mathscr{G}_b(l,l,l)}{1+s[\mathscr{G}_b(x,x,l)+\mathscr{G}_b(l,l,x)+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}\\
&+\chi_3 \frac{\mathscr{G}_b(l,l,l) \cdot \mathscr{G}_b(l,l,l)}{1+s[\mathscr{G}_b(l,l,l)+\mathscr{G}_b(l,l,l)+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}\\
&+\chi_4 \frac{\mathscr{G}_b(l,l,l) \cdot \mathscr{G}_b(x,x,x)}{1+s[\mathscr{G}_b(l,l,x)+\mathscr{G}_b(x,x,l)+\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)]}
\end{aligned}$$

which implies

$$\mathscr{G}_b(x,l,l) \leq \chi_1 \frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3} \tag{2.21}$$

Similarly

$$\mathscr{G}_b(y,m,m) \leq \chi_1 \frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3} \tag{2.22}$$

and

$$\mathscr{G}_b(z,n,n) \leq \chi_1 \frac{\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)}{3} \tag{2.23}$$

Adding (2.21),(2.22) and (2.23), we get

$$(1-\chi_1)\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n) \leq 0 \tag{2.24}$$

Since $\chi_1 < 1$, we have $\mathscr{G}_b(x,l,l)+\mathscr{G}_b(y,m,m)+\mathscr{G}_b(z,n,n)=0$ which implies that $x=l, y=m$ and $z=n$.
Thus, (x,y,z) is a unique common tripled fixed point of $\mathscr{S}$,$\mathscr{T}$ and $\mathscr{R}$ in $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$. □

Corollary 2.2. Let $(\mathscr{M},\mathscr{G}_b)$ be a complete symmetric $\mathscr{G}_b$-metric space with parameter $s \geq 1$ and $\mathscr{S} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be a mapping such that

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{S}(\lambda,\kappa,\zeta),\mathscr{S}(\nu,\iota,\wp)) \qquad (2.25)\\
&\leq \chi_1 \frac{\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}{3}\\
&+\chi_2 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\lambda,\kappa,\zeta))}{1+s[\mathscr{G}_b(x,x,\mathscr{S}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}\\
&+\chi_3 \frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(\nu,\nu,\mathscr{S}(\nu,\iota,\wp))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{S}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}\\
&+\chi_4 \frac{\mathscr{G}_b(\nu,\nu,\mathscr{S}(\nu,\iota,\wp))\cdot\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,x,\mathscr{S}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}
\end{aligned}$$

for all $x,y,z,\lambda,\kappa,\zeta,\nu,\iota,\wp \in X$ and non negative real numbers χ_1,χ_2,χ_3 and χ_4 with $0 \leq \chi_1+\chi_2+\chi_3+\chi_4 < 1$.
Then $\mathscr{S}$ has a unique tripled fixed point in $\mathscr{M}\times\mathscr{M}\times\mathscr{M}$.

The following example illustrates the Corollary 2.2

Example 2.2. Let $\mathscr{M} = [0,1]$. Define $\mathscr{G}_b : \mathscr{M}\times\mathscr{M}\times\mathscr{M} \to [0,\infty)$ by $\mathscr{G}_b(\alpha,\beta,\gamma) = (|\alpha-\beta|+|\beta-\gamma|+|\gamma-\alpha|)^2$, where $s = 2$. Clearly, $(\mathscr{M},\mathscr{G}_b)$ is a complete $\mathscr{G}_b$-metric space.
Let $\mathscr{S}(\alpha,\beta,\gamma) = \frac{\alpha+2\beta+3\gamma}{18}$ for $\alpha,\beta,\gamma \in \mathscr{M}$.
Now

$$\begin{aligned}
&\mathscr{G}_b(\mathscr{S}(x,y,z),\mathscr{S}(\lambda,\kappa,\zeta),\mathscr{S}(\nu,\iota,\wp))) = \big(|\mathscr{S}(x,y,z)-\mathscr{S}(\lambda,\kappa,\zeta)|+|\mathscr{S}(\lambda,\kappa,\zeta)-\mathscr{S}(\nu,\iota,\wp)|\\
&\qquad\qquad +|\mathscr{S}(\nu,\iota,\wp)-\mathscr{S}(x,y,z)|\big)^2\\
&= \Big(\Big|\tfrac{x+2y+3z}{18}-\tfrac{\lambda+2\kappa+3\zeta}{18}\Big|+\Big|\tfrac{\lambda+2\kappa+3\zeta}{18}-\tfrac{\nu+2\iota+3\wp}{18}\Big|+\Big|\tfrac{\nu+2\iota+3\wp}{18}-\tfrac{x+2y+3z}{18}\Big|\Big)^2\\
&\leq \big(\tfrac{1}{18}[|x-\lambda|+|\lambda-\nu|+|\nu-x|]+\tfrac{1}{9}[|y-\kappa|+|\kappa-\iota|+|\iota-y|]\\
&\qquad\qquad +\tfrac{1}{6}[|z-\zeta|+|\zeta-\wp|+|\wp-z|]\big)^2\\
&\leq \big(\tfrac{1}{6}[|x-\lambda|+|\lambda-\nu|+|\nu-x|]+\tfrac{1}{6}[|y-\kappa|+|\kappa-\iota|+|\iota-y|]\\
&\qquad\qquad +\tfrac{1}{6}[|z-\zeta|+|\zeta-\wp|+|\wp-z|]\big)^2\\
&\leq \tfrac{1}{12}\big[(|x-\lambda|+|\lambda-\nu|+|\nu-x|)^2+(|y-\kappa|+|\kappa-\iota|+|\iota-y|)^2\\
&\qquad\qquad +(|z-\zeta|+|\zeta-\wp|+|\wp-z|)^2\big]\\
&= \tfrac{1}{12}[\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]\\
&= \tfrac{1}{4}\cdot\tfrac{[\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}{3}
\end{aligned}$$

$$\begin{aligned}
&\leq \chi_1 \frac{\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)}{3}\\
&+\chi_2 \frac{\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))\cdot\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\lambda,\kappa,\zeta))}{1+s[\mathscr{G}_b(x,x,\mathscr{S}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}\\
&+\chi_3 \frac{\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\lambda,\kappa,\zeta))\cdot\mathscr{G}_b(\nu,\nu,\mathscr{S}(\nu,\iota,\wp))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{S}(\lambda,\kappa,\zeta))+\mathscr{G}_b(\lambda,\lambda,\mathscr{S}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}\\
&+\chi_4 \frac{\mathscr{G}_b(\nu,\nu,\mathscr{S}(\nu,\iota,\wp))\cdot\mathscr{G}_b(x,x,\mathscr{S}(x,y,z))}{1+s[\mathscr{G}_b(\nu,\nu,\mathscr{S}(x,y,z))+\mathscr{G}_b(x,x,\mathscr{S}(\nu,\iota,\wp))+\mathscr{G}_b(x,\lambda,\nu)+\mathscr{G}_b(y,\kappa,\iota)+\mathscr{G}_b(z,\zeta,\wp)]}
\end{aligned}$$

where $\chi_1 = 1/4$ and $\chi_2 = 0, \chi_3 = 0, \chi_4 = 0$ such that $0 \leq \chi_1+\chi_2+\chi_3+\chi_4 < 1$. Hence all the conditions of Corollary 2.2 are satisfied. Thus, $\mathscr{S}$ has a unique tripled fixed point $(0,0,0)$ in $\mathscr{M}\times\mathscr{M}\times\mathscr{M}$.

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