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Structural Properties of the Essential Ideal Graph of $\mathbb{Z}_n$

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Abstract Let S be a commutative ring with unity. The essential ideal graph of S, denoted by $\mathcal{E}_S$, is a graph with vertex set consisting of all nonzero proper ideals of A and two vertices P and Q are adjacent whenever $P+Q$ is an essential ideal. An *essential ideal* P of a ring S is an ideal P of S ($P \trianglelefteq S$), having nonzero intersection with every other ideal of S. The set $Max(S)$ contains all the maximal ideals of S. The *Jacobson radical* of S, $J(S)$, is the set of intersection of all maximal ideals of S. The comaximal ideal graph of S, denoted by $\mathcal{C}(S)$, is a simple graph with vertices as proper ideals of A not contained in $J(S)$ and the vertices P and Q are associated with an edge whenever $P+Q=S$. In this paper, we study the structural properties of the graph $\mathcal{E}_S$ by using the ring theoretic concepts. We obtain a characterization for $\mathcal{E}_S$ to be isomorphic to the comaximal ideal graph $\mathcal{C}(S)$. Moreover, we derive the structure theorem of $\mathcal{E}_{\mathbb{Z}_n}$ and determine graph parameters like clique number, chromatic number and independence number. Also, we characterize the perfectness of $\mathcal{E}_{\mathbb{Z}_n}$ and determine the values of n for which $\mathcal{E}_{\mathbb{Z}_n}$ is split and claw-free, Eulerian and Hamiltonian. In addition, we show that the finite essential ideal graph of any non-local ring is isomorphic to $\mathcal{E}_{\mathbb{Z}_n}$ for some n.

Keywords Essential Ideal Graph of a Commutative Ring, Co-maximal Ideal Graph, Matching, Perfect Graph, Clique Number, Chromatic Number

1 Introduction

In 1998, Beck initiated the concept of identifying a graph structure from a commutative ring structure and studied a graph formed out of zero divisors of a commutative ring[8]. Later Anderson and Livingston[3] modified the definition and named the graph as zero divisor graph. In subsequent years, many researchers followed this work and a large variety of graphs were defined and studied[4, 5]. Most of these graphs are defined by taking all or specific elements of the ring as vertices and their properties are considered for describing adjacency. Later attempts were done by taking ideals of the ring as vertices and their specific relations are considered for describing the edges. This was a novel and interesting approach in algebraic graph theory since the structure of an ideal is closely related to that of the corresponding ring. For more details one can refer [1, 2, 9, 10, 18]. M. Ye and T. Wu[18] set out to explore the comaximal ideal graph of a commutative ring $\mathcal{C}(S)$ in 2012. It is a simple graph with vertex set consisting of the proper ideals of A not contained in $J(S)$ and an edge is made between the vertices P and Q if and only if they are comaximal. In 2016, Azadi et al.[6] investigated the perfection and planarity of the comaximal ideal graph.

Lately, J. Amjadi[2] commenced the study of essential ideal graph $\mathcal{E}_S$, of a commutative ring S with unity. He establised the structural properties of $\mathcal{E}_S$ with reference to the ring properties. The vertex set of $\mathcal{E}_S$ is comprised of all nonzero proper ideals of S. The vertices P and Q are associated with an edge if and only if the ideal $P+Q$ is an essential ideal.

First we recall the basic definitions from ring theory. The *Annihilator* of Y, denoted by $Ann(Y)$, of a ring S is $Ann(Y)=\{a \in S|\ aY=0\}$; provided Y is any element or subset of S. A ring S that has one and only one maximal ideal is classed as a *local ring*. A ring S is called *uniserial* if for every two ideals $P,\ Q$ of S, we have $P \subseteq Q$ or $Q \subseteq P$. An element $s \in S$ is called an *idempotent* if $s^2=s$. If S has a nontrivial idempotent S ($i.e,\ s \neq 0,\ 1$) then $S \simeq Ss \times S(1-s)$. One can refer [7, 14] for farther definitions and results.

Next we recall the following definitions from graph theory. Let a simple graph $G=(V,E)$ consists of vertex set $V(G)$ and edge set $E(G)$. For a vertex v in $V(G)$, the *degree* of

v denoted by $deg(v)$, is defined as the number of edges incident to v. In a graph G, a closed trail of length 3 or more is termed as a *circuit*. A circuit in which no vertex is repeated, except for the initial and the final , is called a *cycle*. The *complement* $\overline{G}$ of a graph G is a graph whose vertex set is equal to $V(G)$ and two vertices x and y are associated with an edge in $\overline{G}$ if and only if x and y are nonadjacent in G. A graph G in which there exists an edge between every pair of vertices is termed as a *complete graph* K_n. A $k-partite$ *graph* G is the one in which the vertex set is partitioned into k partite sets $X_1,\ X_2,\ \cdots,\ X_k$ in such a way that any two vertices $x,\ y$ of $V(G)$ are associated with an edge whenever x and y are in different partite sets. If, in addition, there exists an edge between any two vertices of different partite sets, then G is *complete k-partite*. The *complete bipartite* graph having part sizes m_1 and m_2 is assigned as K_{m_1,m_2} and P_n denotes a path of order n. A subset H of $V(G)$ is proposed to be *independent* if no two vertices of H are adjacent in G. The *independence number* $\beta(G)$ is set as, $\beta(G) = max\{|H| : H \subseteq V(G)$ is an independent set of $G\}$. An independent set M of edges of G is defined to be a *matching* in G. The *Matching number* $\beta_1(G)$ is set as $\beta_1(G) = max\{|M| : M \subseteq E(G)$ is a matching in $G\}$. A graph G of order n is said to have a perfect matching if and only if n is even and $\beta_1(G) = n/2$.

In a connected graph G, a vertex $x \in V(G)$ is said to be a *cut-vertex* if $G - x$ is disconnected. A *vertex(edge) cover* of G is a set of vertices(edges) that are required to cover all edges(vertices) of G. The minimum size of a vertex cover is known as the *vertex covering number* $\alpha(G)$ and the minimum size of an edge cover is known as an *edge covering number* $\alpha_1(G)$. A subset H of $V(G)$ is said to be a $clique$ in G, if H induces a complete subgraph. The order of a largest clique in G is termed as its $clique\ number$, $\omega(G)$. For a graph G, the *Chromatic number* $\chi(G)$ is defined to be the minimum number of colors needed to assign a color to each vertex in such a fashion that adjacent vertices get different colors. A graph G is called *weakly perfect* if $\chi(G) = \omega(G)$ and is called *perfect* if $\chi(T) = \omega(T)$, for any induced subgraph T of G. A *split graph* is a graph in which the vertex set is decomposed into the disjoint union of an independent set and a clique (either of which may be empty). The graph $K_{1,3}$ is termed as a *claw*. If no subgraph of G induces $K_{1,3}$, then it is said to be a *claw-free* graph. A graph operation on two graphs G_1 and G_2 in which each vertex of G_1 is joined to every vertex of G_2 is said to be the *join*, $G_1 \vee G_2$, of two graphs G_1 and G_2. A circuit in G comprising of all edges of G is termed as an *Eulerian circuit* and a connected graph having an Eulerian circuit is termed as an *Eulerian graph*. A cycle in G comprising of all vertices of G is called a *Hamiltonian cycle* and a graph having a *Hamiltonian cycle* is termed as a *Hamiltonian graph*. We write K_∞ to represent the complete graph on infinite vertices. For further definitions and notations in graph theory one can refer [11, 17].

From the definition of the comaximal ideal graph, defined in [18], it is clear that $\mathcal{C}(S)$ is a subgraph of $\mathcal{E}_S$. But the following examples show that both graphs are not the same.

Example 1.1. *If $S = \mathbb{Z}_8$, then S has exactly two nonzero proper ideals, $\langle 2\rangle$ and $\langle 4\rangle$, of which $\langle 4\rangle$ is contained in $\langle 2\rangle$. Also both of them are essential ideals. Hence $\mathcal{E}_S$ is K_2 and $\mathcal{C}(S)$ is the empty graph.*

Example 1.2. *If $S = \mathbb{Z}_{12}$, the nonzero proper ideals of S are $\langle 2\rangle$, $\langle 4\rangle$, $\langle 3\rangle$ and $\langle 6\rangle$. Of these $\langle 2\rangle$ is the only essential ideal of $\mathbb{Z}_{12}$. Thus $\mathcal{E}_S$ is $K_4 - e$. The Jacobson radical is the ideal $\langle 6\rangle$. Hence $\mathcal{C}(S)$ is the path P_3.*

This motivates the study of the essential ideal graph of a commutative ring with unity.

The rest of the paper is prepared as follows. In Section 2, we state some results from literature that are needed to prove our results in the following sections. In Section 3, we derive a characterization for the essential ideal graph of a commutative ring with unity to be ismorphic to the comaximal ideal graph. Also we compute the matching number, covering number, clique number, chromatic number and independence number of the essential ideal graph of the commutative ring $S = F_1 \times F_2 \times \cdots \times F_n,\ n \geq 2$, where each F_i is a field for $i = 1, 2, \cdots, n$. Moreover, we prove a characterization for perfectness of the essential ideal graph of $\mathcal{E}_S$, for $S = F_1 \times F_2 \times \cdots \times F_n,\ n \geq 2$. In Section 4, we compute the degree of vertices of $\mathcal{E}_{\mathbb{Z}_n}$ and characterize the Eulerian, Hamiltonian, split and claw-free graphs of $\mathcal{E}_{\mathbb{Z}_n}$. We also prove that $\mathcal{E}_{\mathbb{Z}_n}$ is perfect if and only if the number of distinct prime factors of n is less than or equal to 4. In Section 5, we check the conditions on rings for which their essential ideal graphs are isomorphic and show that the finite essential ideal graph of any non-local ring is isomorphic to $\mathcal{E}_{\mathbb{Z}_n}$ for some n.

Throughout this paper, S denotes a commutative ring with nonzero unity that is not a field and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

2 Preliminaries

The results given below are useful in this paper.

Theorem 2.1. *[11] A nontrivial graph G is Eulerian if and only if each vertex of G has even degree.*

Lemma 2.2. *[11] A graph G fails to be Hamiltonian, if it holds a cut vertex.*

Theorem 2.3. *[17] If G is a simple graph with at least 3 vertices and $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 2.4. *[11] For a graph G of order n containing no isolated vertices,*

(i) $\alpha_1(G) + \beta_1(G) = n$.

(ii) $\alpha(G) + \beta(G) = n$.

Theorem 2.5. *[15] A graph is split if and only if no induced subgraph is either a cycle on four or five vertices, or a pair of disjoint edges.*

Theorem 2.6. *[12](Strong Perfect Graph Theorem) Let G and $\overline{G}$ denotes a graph and its complement respectively. Then G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

Corollary 2.7. *[14] Let n be a positive integer and let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be its factorization into powers of distinct primes. Then $\mathbb{Z}/n\mathbb{Z} \simeq (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})$.*

Observation 2.8. *[2] Let S be a commutative ring with unity. In $\mathcal{E}_S$,*

(i) every proper essential ideal turn into a universal vertex.

(ii) The induced subgraph $\mathcal{E}_S(Max(S))$ becomes a clique.

(iii) For an arbitrary ideal P of S, $P+Ann(P)$ is an essential ideal .

Lemma 2.9. *[2] Let S be a commutative ring with unity. If S contains no proper essential ideal, then $J(S) = (0)$.*

Proposition 2.10. *[18] Let S be a ring such that $|Max(S)| \geq 2$. Then for any proper ideal P such that $P \nsubseteq J(S)$, P is a vertex of $\mathcal{C}(S)$.*

Theorem 2.11. *[13] Let S be a ring and $s \in S \setminus (J(S) \cup U(S))$. If the chain $\cdots Ss^2 \subseteq Ss$ is finite, then S has a nontrivial idempotent.*

Lemma 2.12. *[13] Let (S, m) be a local ring containing finitely many ideals. Then the following hold:*

(i) If the residue field S/m is infinite , then m is a principal ideal and every ideal of S is a power of m. In particular, S is a uniserial ring.

(ii) If the residue field S/m is finite, then S is a finite ring.

Theorem 2.13. *[18] Let S be a ring and let $G = C(S)$. Then the ensuing three numbers are equal.*

(i) The number $|Max(S)|$ of maximal ideals of the ring S.

(ii) The clique number $\omega(G)$ of G.

(iii) The chromatic number $\chi(G)$ of G.

Theorem 2.14. *[6] If $|Max(S)| \leq 4$, then $\mathcal{C}(S)$ is a perfect graph.*

3 Properties of the Essential Ideal Graph

The following theorem gives a characterization for the essential ideal graph of a commutative ring with unity to be isomorphic to the comaximal ideal graph.

Theorem 3.1. *Let S be a commutative ring with unity. Then $\mathcal{E}_S \cong \mathcal{C}(S)$ if and only if S has no proper essential ideal.*

Proof: Assume that S has no proper essential ideal. Then by Lemma 2.9 and Proposition 2.10, both vertex sets $V(\mathcal{E}_S)$ and $V(\mathcal{C}(S))$ are equal. Now, let P and Q be any two adjacent vertices of $\mathcal{E}_S$. Then $P + Q = S$. Also, there is an isomorphism (the identity map) from $V(\mathcal{E}_S)$ to $V(\mathcal{C}(S))$ preserving the adjacency. Hence $\mathcal{E}_S \cong \mathcal{C}(S)$.

For the converse, assume that S has a proper essential ideal, say P. Then, P is a universal vertex of $\mathcal{E}_S$. Since $\mathcal{E}_S \cong \mathcal{C}(S)$, we have $|V(\mathcal{E}_S)| = |V(\mathcal{C}(S))|$ and $\phi : V(\mathcal{E}_S) \to V(\mathcal{C}(S))$ be the isomorphism that preserves adjacency. Then $\phi(P)$ is a universal vertex of $\mathcal{C}(S)$. Being a proper ideal of S, P is contained in a maximal ideal M_i, for some i . Now, $\phi(P) + \phi(M_i) = \phi(M_i)$, $\phi(M_i) \neq S$. Thus, $\phi(P)$ is not adjacent to $\phi(M_i)$ in $\mathcal{C}(S)$, for some i and hence $\phi(P)$ is cannot be a universal vertex of $\mathcal{C}(S)$. This contradiction assures the converse.

Corollary 3.2. *Let $S = F_1 \times F_2 \times \cdots \times F_n$, $n \geq 2$, where each F_i is a field for $1 \leq i \leq n$. Then $\mathcal{E}_S \cong \mathcal{C}(S)$.*

Proposition 3.3. *Let S be a non-local ring. If S has no nontrivial idempotent, then $\mathcal{E}_S$ contains an induced subgraph isomorphic to K_∞.*

Proof: Let I_0 and K_0 be two nonzero proper ideals of S such that $I_0 + K_0 = S$. This ensures the existence of $i \in I_0$ and $k \in K_0$ such that $i + k = 1$. Let $I_n = Si^n$ and $K_m = Sk^m$. Clearly $I_n + K_m = S$, for all $n, m \geq 0$. Since S has no nontrivial idempotent, by Theorem 2.11, the chains $\cdots I_2 \subseteq I_1 \subseteq I_0$ and $\cdots K_2 \subseteq K_1 \subseteq K_0$ are infinite. Hence the subgraph induced by the vertices $\{I_n\}_{n\geq 0}$ and $\{K_m\}_{m\geq 0}$ of $\mathcal{E}_S$ is isomorphic to K_∞.

The following observation is required to prove our next result.

Observation 3.4. *Let $S = S_1 \times S_2 \times \cdots \times S_n$, $n \geq 2$, where each S_i is a commutative ring with unity for $1 \leq i \leq n$.Then $\mathcal{E}_{S_i}$ is isomorphic to an induced subgraph of $\mathcal{E}_S$.*

Theorem 3.5. *Let S be a non-local ring such that $\mathcal{E}_S$ does not contain K_∞ as a subgraph. Then $S \cong S_1 \times S_2 \times \cdots \times S_n$, where each S_i is a local ring with finitely many ideals.*

Proof: Since K_∞ is not a subgraph of $\mathcal{E}_S$, S has a nontrivial idempotent by Theorem 3.3. Hence S must be a direct product of finitely many rings. Since $\mathcal{E}_A[Max(S)]$ forms a complete subgraph in $\mathcal{E}_S$ and also by hypothesis, we have $|Max(S)| < \infty$. Let n be the largest number such that $S \cong S_1 \times S_2 \times \cdots \times S_n$, where each S_i has no nontrivial idempotent. Moreover, since the essential ideal graph of each factor is a subgraph of $\mathcal{E}_S$, we conclude that each S_i must be a local ring. If $n \geq 2$ and S_i has infinitely many ideals, then $\mathcal{E}_S$ must contain K_∞ which is a contradiction. Thus each S_i is a local ring with finitely many ideals.
By Lemma 2.12 and Theorem 3.5, we constitute the following corollary.

Corollary 3.6. *Let S be a non-local ring such that $\mathcal{E}_S$ is finite. Then $S \cong S_1 \times S_2 \times \cdots \times S_n$, where each S_i is a finite local ring or uniserial ring, for $1 \leq i \leq n$.*

In the following, we let $S = F_1 \times F_2 \times \cdots \times F_n$, $n \geq 2$ and discuss the structural properties like perfect matching, independence, perfectness, etc of $\mathcal{E}_S$.

Theorem 3.7. *For the essential ideal graph $\mathcal{E}_S$, the following holds:*

(i) The matching number $\beta_1(\mathcal{E}_S) = 2^{n-1} - 1$.

(ii) $\mathcal{E}_S$ has a perfect matching.

Proof:(i) Firstly we partition $V(\mathcal{E}_S)$ as given below.

$X_1 = \{F_1 \times F_2 \times \cdots \times F_{i_1-1} \times (0) \times F_{i_1+1} \times \cdots \times F_n,$ $(0)\times(0)\times\cdots\times(0)\times F_{i_1}\times(0)\times\cdots\times(0);\ 1 \le i_1 \le n\}$

$X_2 = \{F_1 \times F_2 \times \cdots \times F_{i_1-1} \times (0) \times F_{i_1+1} \times \cdots \times F_{i_2-1} \times (0) \times F_{i_2+1} \times \cdots \times F_n, (0) \times (0) \times \cdots \times (0)\times F_{i_1}\times(0)\times\cdots\times(0)\times F_{i_2}\times(0)\times\cdots\times(0);\ i_1 \neq i_2$ and $1 \le i_1, i_2 \le n\}$

Similarly,

$X_k = \{F_1\times F_2\times\cdots\times F_{i_1-1}\times(0)\times F_{i_1+1}\times\cdots\times F_{i_2-1}\times (0) \times F_{i_2+1} \times \cdots \times F_{i_k-1} \times (0) \times F_{i_k+1} \times \cdots \times F_n, (0) \times (0) \times \cdots \times (0) \times F_{i_1} \times (0) \times \cdots \times (0) \times F_{i_2} \times (0) \times \cdots \times (0) \times F_{i_k} \times (0) \times \cdots \times (0)$, each i_j are distinct and $1 \le i_j \le n$ for $1 \le j \le k\}$

for each $1 \le k \le \lfloor n/2 \rfloor$ and $|S_k| = 2\binom{n}{k}$.

We can easily verify that $V(\mathcal{E}_S) = X_1 \cup X_2 \cup \cdots \cup X_{\lfloor n/2 \rfloor}$. Now let M_k be a matching in the subgraph of $\mathcal{E}_S$ induced by the set of vertices X_k, denoted as $\mathcal{E}_S[X_k]$, for $1 \le k \le \lfloor n/2 \rfloor$. Then $|M_k| \le \binom{n}{k}$, for $1 \le k \le \lfloor n/2 \rfloor$. Next we claim that $|M_k| = \binom{n}{k}$, for $1 \le k \le \lfloor n/2 \rfloor$. In $\mathcal{E}_S[X_k]$, the vertices $F_1 \times F_2 \times \cdots \times F_{i_1-1} \times (0) \times F_{i_1+1} \times \cdots \times F_{i_2-1} \times (0) \times F_{i_2+1} \times \cdots \times F_{i_k-1} \times (0) \times F_{i_k+1} \times \cdots \times F_n$ and $(0) \times (0) \times \cdots \times (0) \times F_{i_1} \times (0) \times \cdots \times (0) \times F_{i_2} \times (0) \times \cdots \times (0) \times F_{i_k} \times (0) \times \cdots \times (0)$ are necessarily adjacent for each $1 \le k \le \lfloor n/2 \rfloor$. This constitutes $\binom{n}{k}$ independent edges in $\mathcal{E}_S[X_k]$ for each $k = 1, 2, \cdots, \lfloor n/2 \rfloor$. Moreover we conclude that M_k is the maximum possible matching in $\mathcal{E}_S[X_k]$ for each k, since each edge saturates exactly two vertices incident with it.

Now let $M = M_1 \cup M_2 \cup \cdots \cup M_{\lfloor n/2 \rfloor}$. Clearly, M is a matching in $\mathcal{E}_S$ as each M_k contains independent set of edges of $\mathcal{E}_S$. Also $|M| = |M_1| + |M_2| + \cdots + |M_{\lfloor n/2 \rfloor}| = 2^{n-1} - 1$ which proves that M provides a maximum matching in $\mathcal{E}_S$. Hence $\beta_1(\mathcal{E}_S) = 2^{n-1} - 1$.

(ii) Here, $|V(\mathcal{E}_S)| = 2^n - 2$ is even and from Theorem 3.7 (i), it follows that $\beta_1(\mathcal{E}_S) = 2^{n-1} - 1 = |V(\mathcal{E}_S)|/2$. Hence $\mathcal{E}_S$ has a perfect matching.

Example 3.8. *For $S = F_1 \times F_2 \times F_3$, $\mathcal{E}_S$ is given below.*

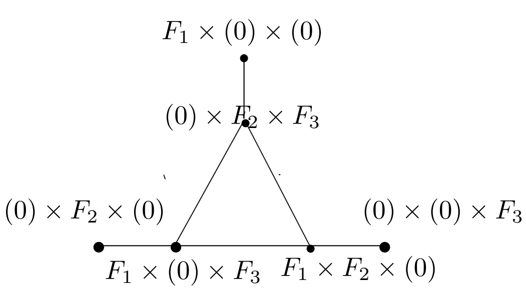

***Figure 1**. Perfect matching in $\mathcal{E}_S$*

Since $n = 3$, there is only a single set in the partition, given by,

$S_1 = \{(0) \times F_2 \times F_3, F_1 \times (0) \times F_3, F_1 \times F_2 \times (0), F_1 \times (0) \times (0), (0) \times F_2 \times (0), (0) \times (0) \times F_3\}$.

Hence, matching number $\beta_1(\mathcal{E}_S) = 3 = 2^{3-1} - 1$.

Theorem 3.9. *Let $S = F_1 \times F_2 \times \cdots \times F_n$, $n \ge 2$, each F_i is a field for $1 \le i \le n$. Then $\mathcal{E}_S$ is perfect if and only if $n \le 4$.*

Proof: By Theorem 2.13 and Corollary 3.2, it is easy to identify that $\mathcal{E}_S$ is weakly perfect and $\omega(\mathcal{E}_S) = \chi(\mathcal{E}_S) = |Max(S)| = n$. Also Theorem 2.14 assures that $\mathcal{E}_S$ is perfect if n is less than or equal to 4.

For the converse, we prove the contrapositive. Let $n \ge 5$. Then $C : F_1 \times F_2 \times (0) \times (0) \times F_5 \times \cdots \times F_n - (0) \times (0) \times F_3 \times F_4 \times F_5 \times \cdots \times F_n - F_1 \times F_2 \times (0) \times F_4 \times (0) \times F_6 \times \cdots \times F_n - F_1 \times (0) \times F_3 \times (0) \times F_5 \times \cdots \times F_n - (0) \times F_2 \times F_3 \times F_4 \times (0) \times \cdots \times F_n - F_1 \times F_2 \times (0) \times (0) \times F_5 \times \cdots \times F_n$ is an induced odd-5-cycle in $\mathcal{E}_S$. Hence by Theorem 2.6, $\mathcal{E}_S$ is not perfect.

Proposition 3.10. *For the essential ideal graph $\mathcal{E}_S$, the following three numbers are equal.*

(i) The independence number, $\beta(\mathcal{E}_S)$.

(ii) The vertex covering number, $\alpha(\mathcal{E}_S)$.

(iii) The edge covering number, $\alpha_1(\mathcal{E}_S)$.

Proof: (i) Firstly we divide the vertex set $V(\mathcal{E}_S)$ in the following way.

Let $V_1 = \{(0)\times(0)\times\cdots\times(0)\times F_i\times(0)\times\cdots\times(0),\ 1 \le i \le n\}$

$V_2 = \{(0) \times \cdots \times (0) \times F_i \times (0) \times \cdots \times (0) \times F_j \times (0) \times \cdots \times (0),\ i \neq j \ and\ 1 \le i, j \le n\}$

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$V_{n-1} = \{F_1 \times F_2 \times \cdots \times F_{i-1} \times (0) \times F_{i+1} \times \cdots \times F_n,\ 1 \le i \le n\}$

Clearly $V_{n-1} = Max(S)$ and $|V_k| = \binom{n}{k}$ for $k = 1, 2, \cdots, n-1$. Also $V(\mathcal{E}_S) = V_1 \cup V_2 \cup \cdots \cup V_{n-1}$. From these sets, we construct another partition of $V(\mathcal{E}_S)$ into n independent sets $K_1, K_2, \cdots, K_n$ as follows.

Construction of $K_1, K_2, \cdots, K_n$: Let the vertices in V_{n-1} be named as M_i for $1 \le i \le n$. Since $\mathcal{E}_S[Max(S)]$ forms a clique in $\mathcal{E}_S$, we place the vertices $M_1, M_2 \cdots, M_n$ in $K_1, K_2, \cdots, K_n$ respectively. To make K_1 independent it should contain the ideals having (0) in the first position. Hence K_1 contains $\binom{n-1}{n-2}$ vertices of V_1, $\binom{n-1}{n-3}$ vertices of V_2 and so on.

Moreover, $|K_1| = 1 + \binom{n-1}{n-2} + \binom{n-1}{n-3} + \cdots + \binom{n-1}{1}$. To be nonadjacent with M_2, K_2 consists of all ideals with (0) in the second position. But of these, all ideals having (0) in the first position are included in the set K_1. Thus for vertices in K_2, ideals having F_1 in the first position and (0) in the second position are considered. Hence K_2 consists of $\binom{n-2}{n-2}$ vertices of V_1, $\binom{n-2}{n-3}$ vertices of V_2 and so on. In this manner we construct $K_3, K_4, \cdots, K_{n-1}$ and K_n, where $|K_{n-1}| = 1 + \binom{1}{1}$ and $|K_n| = 1$.

$$
\begin{aligned}
&|K_1|+|K_2|+\cdots+|K_n| \\
&\quad = 1+\tbinom{n-1}{n-2}+\tbinom{n-1}{n-3}+\cdots+\tbinom{n-1}{1}+1+ \\
&\qquad \tbinom{n-2}{n-2}+\tbinom{n-2}{n-3}+\cdots+\tbinom{n-2}{1}+\cdots+1+\tbinom{1}{1}+1 \\
&\quad = [1+\tbinom{n-1}{n-2}+\tbinom{n-1}{n-3}+\cdots+\tbinom{n-1}{1}+1]+[1+ \\
&\qquad \tbinom{n-2}{n-2}+\tbinom{n-2}{n-3}+\cdots+\tbinom{n-2}{1}]+\cdots+[1+\tbinom{1}{1}] \\
&\quad = 2^n-2 = |V(\mathcal{E}_S)|.
\end{aligned}
$$

It is clear from this construction that the sets $K_1, K_2, \cdots K_n$ are independent and $|K_1| > |K_2| > \cdots > |K_n|$. Thus K_1 is the independent set of maximum cardinality in $\mathcal{E}_S$. Hence $\beta(\mathcal{E}_S) = |K_1| = 1+\binom{n-1}{n-2}+\binom{n-1}{n-3}+\cdots+\binom{n-1}{1} = 2^{n-1}-1$.
(ii) Since $|V(\mathcal{E}_S)| = 2^n-2$ and from (i), $\beta(\mathcal{E}_S) = 2^{n-1}-1$. Then by Theorem 2.4, the vertex covering number, $\alpha(\mathcal{E}_S) = 2^{n-1}-1$.
(iii) Follows from Theorems 2.4 and 3.7.

Proposition 3.11. *For the essential ideal graph $\mathcal{E}_S$, the following holds:*

(i) $\mathcal{E}_S$ is not Eulerian.

(ii) $\mathcal{E}_S$ is not Hamiltonian.

Proof:(i) For each i, deg($(0)\times(0)\times\cdots\times(0)\times F_i\times(0)\times\cdots\times(0)$)$=1$ is odd, $1\le i\le n$. Hence by the characterization of Eulerian graphs, $\mathcal{E}_S$ is not Eulerian.
(ii) Since the vertex $(0)\times(0)\times\cdots\times(0)\times F_i\times(0)\times\cdots\times(0)$ is adjacent only to the vertex $F_1\times F_2\times\cdots\times F_{i-1}\times(0)\times F_{i+1}\times\cdots\times F_n$ for each $i=1,2,\cdots,n$; the latter is a cut vertex in $\mathcal{E}_S$. Hence $\mathcal{E}_S$ is not Hamiltonian.

Theorem 3.12. *Let $S = S_1\times S_2\times\cdots\times S_n,\ n\ge 2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1\le i\le n$. Then the essential ideal graph $\mathcal{E}_S \simeq H\vee K_m$ where H is an n-partite graph and K_m is a complete graph of order $m=\prod_{i=1}^{n}(m_i+1)-1$.*

Proof: Let $X = \{I_1\times I_2\times\cdots\times I_n : I_j \trianglelefteq S_j$ and $I_j\ne(0)$ for $1\le j\le n\}\setminus\{S_1\times S_2\times\cdots\times S_n\}$

$X_1 = \{(0)\times I_2\times\cdots\times I_n : I_j\trianglelefteq S_j$ and at least one $I_j\ne(0)$ for $2\le j\le n\}$

$X_2 = \{I_1\times(0)\times I_3\times\cdots\times I_n : I_1\ne(0)$ and $I_j\trianglelefteq S_j$ for $3\le j\le n\}$

$X_i = \{I_1\times I_2\times\cdots\times I_{i-1}\times(0)\times I_{i+1}\times\cdots\times I_n : I_j\ne(0)$ for $1\le j\le i-1$ and $I_j\trianglelefteq S_j$ for $i+1\le j\le n\}$

for $1\le i\le n$.
This construction assures that the sets $X_1, X_2, \cdots X_n$ are independent and the set X contains all the essential ideals of the ring S, which induces a clique in $\mathcal{E}_S$. Also $V(\mathcal{E}_S) = X\cup X_1\cup X_2\cup\cdots\cup X_n$. Since each vertex of X_i is adjacent to at least one vertex of X_j for some $j,\ i\ne j$ and $i,j\in\{1,2,\cdots,n\}$, the induced subgraph $\mathcal{E}_S(X_1\cup X_2\cup\cdots\cup X_n)$ forms an n-partite graph H. But H cannot be complete n-partite as there are vertices which are nonadjacent in X_i and X_j, for some distinct i and j. For example, the vertices $(0)\times(0)\times S_3\times\cdots\times A_n$ in X_1 and $S_1\times(0)\times S_3\times\cdots\times A_n$ in X_2 are nonadjacent in $\mathcal{E}_S$.

Here $|X| = \prod_{j=1}^{n}(m_j+1)-1$

$|X_1| = \prod_{j=2}^{n}(m_j+2)-1$

$|X_2| = (m_1+1)\prod_{j=3}^{n}(m_j+2)$

Similarly $|X_i| = \prod_{j=1}^{i-1}(m_j+1)\prod_{k=i+1}^{n}(m_k+2)$, for $2\le i\le n$.

Since X consists of all the essential ideals of S, each vertex in X is adjacent to every vertex of X_j for $1\le j\le n$. Hence $\mathcal{E}_S\simeq H\vee K_m$.

Corollary 3.13. *Let $S = S_1\times S_2\times\cdots\times S_n,\ n\ge 2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1\le i\le n$. Then $\mathcal{E}_S$ is weakly perfect.*

Proof: By Theorem 3.12, $\mathcal{E}_S\simeq H\vee K_m$ where H is an n-partite graph and K_m is a complete graph of order $m=\prod_{i=1}^{n}(m_i+1)-1$. Since there are n independent sets and a clique of order m, we have $\omega(\mathcal{E}_S)\le\chi(\mathcal{E}_S)\le m+n$, where m is the size of the set $X=\{I_1\times I_2\times\cdots\times I_n : I_j\trianglelefteq S_j$ and $I_j\ne(0)$ for $1\le j\le n\}\setminus\{S_1\times S_2\times\cdots\times S_n\}$ as explained in the proof of the Theorem 3.12. But the subgraph induced by the set $X\cup\{(0)\times S_2\times\cdots\times S_n,\ S_1\times(0)\times S_3\times\cdots\times S_n,\ \cdots,S_1\times S_2\times\cdots\times S_{n-1}\times(0)\}$ is also a clique in $\mathcal{E}_S$. Therefore $\chi(\mathcal{E}_S)\ge\omega(\mathcal{E}_S)\ge m+n$. Combining both the inequalities, we get $\omega(\mathcal{E}_S)=\chi(\mathcal{E}_S)=m+n=\prod_{i=1}^{n}(m_i+1)+n-1$.

Corollary 3.14. *Let $S = S_1\times S_2\times\cdots\times S_n,\ n\ge 2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1\le i\le n$. Then the independence number $\beta(\mathcal{E}_S) = max\{\prod_{j=2}^{n}(m_j+2)-1,\ \prod_{j=1}^{i-1}(m_j+1)\prod_{k=i+1}^{n}(m_k+2),\ 2\le i\le n\}$.*

Proposition 3.15. *Let $S=S_1\times S_2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1\le i\le 2$. Then the matching number, $\beta_1(\mathcal{E}_S)=\lfloor\frac{|V(\mathcal{E}_S)|}{2}\rfloor$.*

Proof: By Theorem 3.12 , $\mathcal{E}_S\simeq H\vee K_m$ where $H=K_{l_1,l_2}$ with $l_1=m_2+1$, $l_2=m_1+1$ and K_m is a complete graph of order $m=\prod_{i=1}^{2}(m_i+1)-1$.
Case 1: $m<l_1+l_2$
Then $m_1=m_2=1$ and hence $\mathcal{E}_S\simeq K_3\vee K_{2,2}$ as shown in Fig. 2(c). Therefore,
$\beta_1(\mathcal{E}_S)=3=\lfloor\frac{|V(\mathcal{E}_S)|}{2}\rfloor$.
Case 2: $m=l_1+l_2$
Then either $m_1=1,\ m_2=2$ or vice versa. Hence $\mathcal{E}_S\simeq K_5\vee K_{3,2}$. The subgraph K_5 contains 2 independent edges and $K_{3,2}$ also contains 2 independent edges. Moreover the remaining vertices of K_5 and $K_{3,2}$ constitute one more independent edge. Hence $\beta_1(\mathcal{E}_S)=5=\lfloor\frac{|V(\mathcal{E}_S)|}{2}\rfloor$.
Case 3: $m>l_1+l_2$

Since $\mathcal{E}_S$ is the join of K_m and K_{l_1,l_2}, there are at least $l_1 + l_2$ independent edges. If $m - (l_1 + l_2) \geq 1$, then these vertices induce a complete subgraph of $\mathcal{E}_S$, which has $\lfloor \frac{m-(l_1+l_2)}{2} \rfloor$ independent edges. Thus $l_1 + l_2 + \lfloor \frac{m-(l_1+l_2)}{2} \rfloor \leq \beta_1(\mathcal{E}_S)$. Also, this matching saturates every vertex of $\mathcal{E}_S$. Hence $\beta_1(\mathcal{E}_S) = \lfloor \frac{m+l_1+l_2}{2} \rfloor = \lfloor \frac{|V(\mathcal{E}_S)|}{2} \rfloor$.

Corollary 3.16. *Let $S = S_1 \times S_2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1 \leq i \leq 2$. Then $\mathcal{E}_S$ has a perfect matching if and only if at least one m_i is even.*

Proposition 3.17. *Let $S = S_1 \times S_2 \times S_3$, where each S_i is a uniserial ring with m_i nonzero proper ideals, $1 \leq i \leq 3$. If all the m_i's are equal, then $\beta_1(\mathcal{E}_S) = \lfloor \frac{|V(\mathcal{E}_S)|}{2} \rfloor$.*

Proof: Firstly we divide the vertex set of $\mathcal{E}_S$ into sets $X, X_1, X_2, \cdots, X_n$ as in the proof of Theorem 3.12.
Case 1: $m_i = 1$, for $i = 1, 2$ and 3.
Let I_j be the nonzero proper ideal in S_j, $1 \leq j \leq 3$. Hence

$$X = \{I_1 \times I_2 \times I_3,\ I_1 \times I_2 \times S_3,\ I_1 \times S_2 \times I_3,\ S_1 \times I_2 \times I_3,\ I_1 \times S_2 \times S_3,\ S_1 \times I_2 \times S_3,\ S_1 \times S_2 \times I_3\}$$

$$X_1 = \{(0) \times I_2 \times I_3,\ (0) \times I_2 \times S_3,\ (0) \times S_2 \times S_3,\ (0) \times S_2 \times I_3,\ (0) \times (0) \times I_3,\ (0) \times (0) \times S_3,\ (0) \times I_2 \times (0),\ (0) \times S_2 \times (0)\}$$

$$X_2 = \{I_1 \times (0) \times I_3,\ S_1 \times (0) \times I_3,\ I_1 \times (0) \times S_3,\ S_1 \times (0) \times S_3,\ I_1 \times (0) \times (0),\ S_1 \times (0) \times (0)\}$$

$$X_3 = \{I_1 \times I_2 \times (0),\ I_1 \times S_2 \times (0),\ S_1 \times I_2 \times (0),\ S_1 \times S_2 \times (0)\}$$

Let $|X| = m$ and $|X_i| = k_i$ for $i = 1, 2, 3$. Thus $m = 7$, $k_1 = 8$, $k_2 = 6$ and $k_3 = 4$. In order to find the maximum matching in $\mathcal{E}_S$, first we find the maximum possible independent edges between X, X_1 and X_2, X_3. Obviously, there are seven independent edges between X and X_1 and the four vertices of X_3 can be matched to X_2. Now the remaining two vertices in X_2 and one vertex in X_1, which are not saturated yet, will contribute one independent edge. Hence the matching number, $\beta_1(\mathcal{E}_S) \geq 12$. But $\beta_1(\mathcal{E}_S) \leq \lfloor \frac{|V(\mathcal{E}_S)|}{2} \rfloor = 12$.
Case 2: $m_i = 2$, for $i = 1, 2$ and 3.
Proof is analogous to case 1.
Case 3: $m_1 = m_2 = m_3$ and $m_i \geq 3$ for $i = 1,\ 2,\ 3$
Obviously, $|X| > |X_1| + |X_2| + |X_3|$. Then there are at least $k_1 + k_2 + k_3$ independent edges. If $m - (k_1 + k_2 + k_3) > 1$, these vertices induce a complete subgraph in $\mathcal{E}_S$. Hence $k_1 + k_2 + k_3 + \lfloor \frac{m-(k_1+k_2+k_3)}{2} \rfloor \leq \beta_1(\mathcal{E}_S)$. But $\beta_1(\mathcal{E}_S) \leq \lfloor \frac{|V(\mathcal{E}_S)|}{2} \rfloor = \lfloor \frac{m+k_1+k_2+k_3}{2} \rfloor$.

Example 3.18. *Let $S = \mathbb{Z}_{810000}$. That is, $m_1 = m_2 = m_3 = 3$. Then the total number of vertices, $T = \prod_{i=1}^{3}(m_i + 2) - 2 = 123$. Also, $m = 63$, $k_1 = 24$, $k_2 = 20$ and $k_3 = 16$. Clearly $m > k_1 + k_2 + k_3$. Hence there is a matching M of cardinality at least 60. More over, $m - (k_1 + k_2 + m_3) = 3$ and this provides one more independent edge to M. Hence $\beta_1(\mathcal{E}_{\mathbb{Z}_{810000}}) \geq 61$. But, $\beta_1(\mathcal{E}_{\mathbb{Z}_{810000}}) \leq \lfloor \frac{123}{2} \rfloor$.*

Observation 3.19. *Let $S = S_1 \times S_2 \times \cdots \times S_n$, $n \geq 2$, where each S_i is a uniserial ring with m_i nonzero proper ideals, for $1 \leq i \leq n$. Then the matching number, $\beta_1(\mathcal{E}_S) = \lfloor \frac{|V(\mathcal{E}_S)|}{2} \rfloor$.*

4 Characterization of Eulerian, perfect, split and claw-free graphs of $\mathcal{E}_{\mathbb{Z}_n}$

In this section, we compute the degree of vertices of $\mathcal{E}_{\mathbb{Z}_n}$ and hence characterize the Eulerian graphs of $\mathcal{E}_{\mathbb{Z}_n}$. We also take into account the problem of characterizing the ring $\mathbb{Z}_n$ for which the graph $\mathcal{E}_{\mathbb{Z}_n}$ is perfect, split and claw-free.

For any composite integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(k,\ \alpha_i) \in \mathbb{N}$, $(k, \alpha_1) \neq (1, 1)$, p_i's are distinct primes $(1 \leq i \leq k)$.

Proposition 4.1. *In the graph $\mathcal{E}_{\mathbb{Z}_n}$, the following are identical.*

(i) $\mathcal{E}_{\mathbb{Z}_n}$ is a complete graph.

(ii) $diam(\mathcal{E}_{\mathbb{Z}_n}) = 1$.

(iii) $k = 1$ and $\alpha_1 > 1$ or $k = 2$ and $\alpha_1 = \alpha_2 = 1$.

Theorem 4.2. *For the graph $\mathcal{E}_{\mathbb{Z}_n}$,*

(i) $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$, where H is a k-partite graph and K_m is a complete graph of order $m = \prod_{i=1}^{k} \alpha_i - 1$.

(ii) $\mathcal{E}_{\mathbb{Z}_n}$ is weakly perfect, $\omega(\mathcal{E}_{\mathbb{Z}_n}) = \chi(\mathcal{E}_{\mathbb{Z}_n}) = \prod_{i=1}^{k} \alpha_i + k - 1$.

Proof: By Corollary 2.7, $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}$. Then, applying Theorem 3.12 and Corollary 3.13, we get the required results.

Corollary 4.3. *For the graph $\mathcal{E}_{\mathbb{Z}_n}$, the independence number $\beta(\mathcal{E}_{\mathbb{Z}_n}) = max\{\prod_{j=2}^{k}(\alpha_j + 1) - 1,\ \prod_{j=1}^{i-1} \alpha_j \prod_{r=i+1}^{k}(\alpha_r + 1),\ 2 \leq i \leq k\}$.*

Proposition 4.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then the following holds:*

(i) $\beta_1(\mathcal{E}_{\mathbb{Z}_n}) = \lfloor \frac{|V(\mathcal{E}_{\mathbb{Z}_n})|}{2} \rfloor$, if $k = 2$.

(ii) $\mathcal{E}_{\mathbb{Z}_n}$ has a perfect matching, if $k = 2$ and at least one α_i is odd, $1 \leq \alpha_i \leq 2$.

Proof: (i) Follows directly from Proposition 3.15.
(ii) Follows from (i) and Corollary 3.16.

Example 4.5. *Let $n = 36$. Then $p = 2, q = 3, \alpha_1 = 2, \alpha_2 = 2, k = 2$ and $m = 2 \times 2 - 1 = 3$. By Theorem 4.2, $\mathcal{E}_{\mathbb{Z}_{36}} \cong K_3 \vee K_{2,2}$ and $\omega(\mathcal{E}_{\mathbb{Z}_{36}}) = \chi(\mathcal{E}_{\mathbb{Z}_{36}}) = 2 \times 2 + 2 - 1 = 5$. By Corollary 4.3 the independence number $\beta(\mathcal{E}_{\mathbb{Z}_{36}}) = max\{2, 2\} = 2$ and by Proposition 4.4, the matching number $\beta_1(\mathcal{E}_{\mathbb{Z}_{36}}) = \lfloor \frac{|V(\mathcal{E}_{\mathbb{Z}_{36}})|}{2} \rfloor = 3$. These are clear from the graph of $\mathcal{E}_{\mathbb{Z}_{36}}$ shown in Fig. 2C.*

Now, to find the degree of vertices of $\mathcal{E}_{\mathbb{Z}_n}$, we calculate the degree of corresponding vertices of $V(\mathcal{E}_S)$, $S = \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}$. This is possible since the vertex sets $V(\mathcal{E}_{\mathbb{Z}_n})$ and $V(\mathcal{E}_S)$ are in one-one correspondence by Corollary 2.7.

Let $T = |V(\mathcal{E}_S)|$ and $I = I_1 \times I_2 \times \cdots \times I_k$ be an arbitrary ideal of S, $I_i \trianglelefteq \mathbb{Z}_{p_i^{\alpha_i}}$. The vertex of $\mathcal{E}_S$ corresponding to the ideal P of S is denoted by v_I. Then,

$$T = \prod_{i=1}^{k}(\alpha_i + 1) - 2.$$

(i) Let $I = I_1 \times I_2 \times \cdots \times I_{i-1} \times (0) \times I_{i+1} \times \cdots \times I_k$, $I_j \neq 0$ for $j \neq i$ and $i, j \in \{1, 2, \cdots, k\}$ and N_I denotes the number of vertices to which I is nonadjacent.

$$N_I = (\alpha_1 + 1) \times \cdots (\alpha_i - 1) \times (\alpha_i + 1) \times \cdots \times (\alpha_k + 1) - 1$$

$$= \prod_{\substack{t=1 \\ t \neq i}}^{n} (\alpha_t + 1) - 1.$$

Hence,

$$deg\ v_I = T - N_I = \prod_{i=1}^{k}(\alpha_i + 1) - 2 - [\prod_{\substack{t=1 \\ t \neq i}}^{n} (\alpha_t + 1) - 1]$$

$$= \alpha_i \prod_{\substack{t=1 \\ t \neq i}}^{n} (\alpha_t + 1) - 1.$$

Then the total number of vertices having same degree as that of v_I, denoted as T_{v_I} is given by,

$$T_{v_I} = \sum_{i=1}^{k} \prod_{\substack{t=1 \\ t \neq i}}^{n} \alpha_t.$$

(ii) If $I = I_1 \times \cdots \times I_{i-1} \times (0) \times I_{i+1} \times \cdots \times I_{j-1} \times (0) \times I_{j+1} \times \cdots \times I_k$, $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$.
Then,

$$deg\ v_I = \alpha_i \alpha_j \prod_{\substack{t=1 \\ t \neq i,\ j}}^{n} (\alpha_t + 1) - 1$$

and

$$T_{v_I} = \sum_{\substack{i,j=1 \\ i < j}}^{k} \prod_{\substack{t=1 \\ t \neq i, j}}^{k} \alpha_t.$$

Similarly, for $I = (0) \times (0) \times \cdots (0) \times I_i \times (0) \times \cdots \times (0)$, $I_i \neq (0)$, $i \in \{1, 2, \cdots, k\}$,

$$deg\ v_I = (\alpha_i + 1) \prod_{\substack{t=1 \\ t \neq i}}^{n} \alpha_t - 1$$

and

$$T_{v_I} = \sum_{i=1}^{k} \alpha_i.$$

(iii) From the Proof of Theorem 3.12, it is obvious that an essential ideal of $\mathcal{E}_S$ is of the form $I = I_1 \times I_2 \times \cdots \times I_k$, $I_j \neq (0)$ for $1 \leq j \leq k$. Thus v_I is universal and hence $deg\ v_I = T - 1$. Also there are $\prod_{i=1}^{k} \alpha_i - 1$ vertices having the same degree as that of v_I.

Lemma 4.6. *If $n = p_1 p_2 \cdots p_k$, p_i's are distinct primes , then $\mathcal{E}_{\mathbb{Z}_n}$ is not Eulerian.*

Proof: Follows directly from Proposition 3.11.

Lemma 4.7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where k, $\alpha_i \in \mathbb{N}$, $(k, \alpha_1) \neq (1, 1)$, p_i's are distinct primes($1 \leq i \leq k$). Then $\mathcal{E}_{\mathbb{Z}_n}$ is not Eulerian if,*

(i) at least one α_i is odd.

(ii) all α_i's are even and $k > 1$.

Proof: (i) If at least one α_i is odd, then the total number of vertices $T = \prod_{i=1}^{k}(\alpha_i + 1) - 2$ is even and consequently the vertices corresponding to the essential ideals of $\mathbb{Z}_n$ are of odd degree. Thus, $\mathcal{E}_{\mathbb{Z}_n}$ is not Eulerian.
(ii) If all α_i's are even,

$$deg\ v_I = \alpha_i \prod_{\substack{t=1 \\ t \neq i}}^{n} (\alpha_t + 1) - 1$$

must be odd for the ideal $I = I_1 \times \cdots \times I_{i-1} \times (0) \times I_{i+1} \times \cdots \times I_k$, $i \in \{1, 2, \cdots, k\}$.

The following theorem describes a characterization of Eulerian graphs of $\mathcal{E}_{\mathbb{Z}_n}$.

Theorem 4.8. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where k, $\alpha_i \in \mathbb{N}$, p_i's are distinct primes for $1 \leq i \leq k$.Then $\mathcal{E}_{\mathbb{Z}_n}$ is Eulerian if and only if $n = p^{\alpha}$, $\alpha > 2$ is even.*

Proof: Let $n = p^{\alpha}$, $\alpha > 2$ is even. Then by Proposition 4.1, $\mathcal{E}_{\mathbb{Z}_n}$ is a complete graph on odd number of vertices. Hence $\mathcal{E}_{\mathbb{Z}_n}$ is Eulerian.
Converse follows from Lemmas 4.6 and 4.7.

Proposition 4.9. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is Hamiltonian if $k = 2$ and at least one $\alpha_i > 1$ for $i = 1, 2$.*

Proof: By Theorem 4.2, $\mathcal{E}_{\mathbb{Z}_n} \simeq H \vee K_m$ where $H = K_{\alpha_1, \alpha_2}$ and K_m is a complete graph of order $m = \prod_{i=1}^{2} \alpha_i - 1$. The total number of vertices of $\mathcal{E}_{\mathbb{Z}_n}$, denoted by T is given by, $T = (\alpha_1 + 1)(\alpha_2 + 1) - 2$. Now, assume that $\alpha_1 \leq \alpha_2$. Then each vertex v_I of the set X_1(as explained in the proof of the Theorem 3.12) has minimum degree given by, $deg\ v_I = \alpha_1(\alpha_2 + 1) - 1$. Also, $deg\ v_I \geq T/2$. Hence by Theorem 2.3, $\mathcal{E}_{\mathbb{Z}_n}$ is Hamiltonian.

Proposition 4.10. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geq 2$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is not Hamiltonian if all α_i's are 1, $1 \leq i \leq k$.*

Proof: Follows directly from Proposition 3.11.

Observation 4.11. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geq 2$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is Hamiltonian if $k \geq 3$ and not all α_i's are 1, $1 \leq i \leq k$.*

In the next theorem we derive the necessary and sufficient condition for $\mathcal{E}_{\mathbb{Z}_n}$ to be perfect.

Theorem 4.12. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(k,\ \alpha_i) \in \mathbb{N}$, p_i's are distinct primes ($1 \leq i \leq k$) be the unique factorization of a composite integer $n > 1$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is perfect if and only if $k \leq 4$.*

Proof: If all the α_i's are 1, this is obvious from Theorem 3.9 . Hence assume that not all α_i's are 1. Suppose $k \leq 4$. Obviously, for $k = 1$, $\mathcal{E}_{\mathbb{Z}_n}$ is perfect by Proposition 4.1.
Case 1: Let $k = 2$.

Then, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}}$. Hence by Theorem 3.12, $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$ where H is a complete bipartite graph and K_m is a complete graph of order $m = \prod_{i=1}^{2} \alpha_i - 1$. Let C be any odd five cycle in $\mathcal{E}_{\mathbb{Z}_n}$. Since any bipartite graph doesn't contain odd cycles, C consists of vertices of both K_m and H. But, each vertex of K_m is universal and hence C contains chords, a contradiction. Also, as $\overline{\mathcal{E}_{\mathbb{Z}_n}}$ is the disjoint union of K_{m_1}, K_{m_2} and m isolated vertices, there is no induced 5 cycle in $\overline{\mathcal{E}_{\mathbb{Z}_n}}$.
Case 2: Let $k = 3$.

Then, $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}}$. Let the vertex set $V(\mathcal{E}_{\mathbb{Z}_n})$ be partitioned into the sets $X,\ X_1,\ X_2$ and X_3 where the vertices of X induce a complete subgraph in $\mathcal{E}_{\mathbb{Z}_n}$ and $X_1,\ X_2$, X_3 are independet sets as in the proof of Theorem 3.12. Let $C : V_1 - V_2 - V_3 - V_4 - V_5 - V_1$ be an induced odd-5-cycle in $\mathcal{E}_{\mathbb{Z}_n}$. Then C contains at least one vertex from each of the independent sets $X_1,\ X_2,\ X_3$.
Without loss of generality we persume that $v_1 \in X_3$, $v_2 \in X_2$ and $v_3 \in X_1$. Then $v_1 = I_1 \times I_2 \times (0)$, where $I_k \trianglelefteq \mathbb{Z}_{p_k^{\alpha_k}}$, $I_k \neq (0)$ for $k = 1,\ 2$ and $v_2 = I_1 \times (0) \times I_3$, $I_k \trianglelefteq \mathbb{Z}_{p_k^{\alpha_k}}$, for $k = 1,\ 3$; $I_1 \neq (0)$. Since v_3 is nonadjacent to v_1, $v_3 = (0) \times I_2 \times (0)$ where I_2 is a nonzero ideal of $\mathbb{Z}_{p_2^{\alpha_2}}$. Since X_i's are independent, C contains atmost two vertices from each set. Thus, v_4 is in X_2 or X_3. But the latter is not possible, since then $v_4 = I_1 \times I_2 \times (0)$, where $I_k \trianglelefteq \mathbb{Z}_{p_k^{\alpha_k}}$, $I_k \neq (0)$ for $k = 1,\ 2$ cannot be adjacent to v_3. Hence $v_4 \in X_2$. Then $v_5 \in X_1$. Since v_4 is nonadjacent to v_1 and v_2, $v_4 = I_1 \times (0) \times (0)$, I_1 is a nonzero ideal of $\mathbb{Z}_{p_1^{\alpha_1}}$. Similarly, $v_5 = (0) \times (0) \times I_3$, I_3 is a nonzero ideal of $\mathbb{Z}_{p_3^{\alpha_3}}$. But, this contradicts the adjacency of v_4 and v_5. Hence $\mathcal{E}_{\mathbb{Z}_n}$ contains no induced 5 cycle. The proof is analogous for any odd cycle of length greater than five in $\mathcal{E}_{\mathbb{Z}_n}$. Also from the structure of $\overline{\mathcal{E}_{\mathbb{Z}_n}}$, it is obvious that no induced subgraph of $\overline{\mathcal{E}_{\mathbb{Z}_n}}$ is an odd cycle having length greater than or equal to five.
Case 3 : Let $k = 4$.

Then, $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times \mathbb{Z}_{p_4^{\alpha_4}}$. Let $X_1,\ X_2,\ X_3$ and X_4 be the independent sets as in the proof of Theorem 3.12. Then the induced 5 cycle C does have at least one vertex from each of the independent sets or vertices from exactly three independent sets.The latter is discussed in case 2. Hence C includes at least one vertex from each of the independent sets. Without loss of generality we persume that v_1 is an ideal of X_4. That is, $v_1 = I_1 \times I_2 \times I_3 \times (0)$, where $I_k \trianglelefteq \mathbb{Z}_{p_k^{\alpha_k}}$, $I_k \neq (0)$ for $k = 1,\ 2$ and 3. Then v_2 can be an ideal of $X_1,\ X_2$ or X_3. If $v_2 \in X_3$, $v_2 = I_1 \times I_2 \times (0) \times I_4$, where $I_k \trianglelefteq \mathbb{Z}_{p_k^{\alpha_k}}$, $k = 1,\ 2,\ 4$; $I_k \neq (0)$ for $k = 1,\ 2$. Similarly v_3 can be in $X_1,\ X_2$ or X_4. Since C contains no chords, v_3 is an ideal having (0) at the fourth position. In the same way, any choice of v_4 must also contain (0) at the fourth position to be nonadjacent with v_1. But, this contradicts the adjacency of v_3 and v_4 in C. Similarly we can rule out the other the other two choices for v_2. Thus we can conclude that $\mathcal{E}_{\mathbb{Z}_n}$ has no induced 5 cycle. Similarly, we can prove that $\mathcal{E}_{\mathbb{Z}_n}$ includes no odd cycle of length greater than 5 as an induced subgraph. Also, the structure of $\overline{\mathcal{E}_{\mathbb{Z}_n}}$ emphasizes exemption of the same. Hence by Theorem 2.6, $\mathcal{E}_{\mathbb{Z}_n}$ is perfect for $k \leq 4$.

For the converse, suppose on the contrary that $k \geq 5$ in $\mathbb{Z}_n$. Then, $C : \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} - (0) \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}} \times \mathbb{Z}_{p_4^{\alpha_4}} \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} - \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times \mathbb{Z}_{p_4^{\alpha_4}} \times (0) \times \mathbb{Z}_{p_6^{\alpha_6}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} - \mathbb{Z}_{p_1^{\alpha_1}} \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}} \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} - (0) \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times \mathbb{Z}_{p_4^{\alpha_4}} \times (0) \times \mathbb{Z}_{p_6^{\alpha_6}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} - \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}$ is clearly an induced 5 cycle in $\mathcal{E}_{\mathbb{Z}_n}$. Hence $\mathcal{E}_{\mathbb{Z}_n}$ is not perfect.

Remark 4.13. *Since every split graph is perfect [16], by Theorem 4.12, we conclude that the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is not split for $k > 4$. Thus we need to check for $k \leq 4$.*

In the following lemmas we characterize, for which values of n, $\mathcal{E}_{\mathbb{Z}_n}$ is split.

Lemma 4.14. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is not a split graph if $k = 4$.*

Proof: Since $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$, H is a 4-partite graph, the set $\Gamma = \{\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times (0),\ (0) \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}} \times \mathbb{Z}_{p_4^{\alpha_4}},\ \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times \mathbb{Z}_{p_4^{\alpha_4}},\ \mathbb{Z}_{p_1^{\alpha_1}} \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}} \times \mathbb{Z}_{p_4^{\alpha_4}}\}$ induces C_4 in $\mathcal{E}_{\mathbb{Z}_n}$. Hence by Theorem 2.5, $\mathcal{E}_{\mathbb{Z}_n}$ is not a split graph.

Lemma 4.15. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $1 \leq k \leq 3$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph if and only if one of the following holds:*

(i) *$k = 1$ and $\alpha_1 > 1$.*

(ii) *$k = 2$ and at least one $\alpha_i = 1$ ($i \in \{1, 2\}$).*

(iii) *$k = 3$ and all α_i's are 1, for $1 \leq i \leq 3$.*

Proof: $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph $\Leftrightarrow (i)$, is obvious.
If (ii) holds, then by Theorem 4.2, $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$, where $H = K_{1,\alpha_2}$ and K_m is a complete graph of order $m = \alpha_2 - 1$ (assuming $\alpha_1 = 1$). Hence the vertex set of $\mathcal{E}_{\mathbb{Z}_n}$ is partitioned into $V_1 \cup V_2$, where $|V_1| = \alpha_2 + 1$ and $|V_2| = \alpha_2 - 1$ such that $\mathcal{E}_{\mathbb{Z}_n}[V_1]$ is a clique and V_2 forms an independent set. Thus $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph.
Also, if (iii) holds, from Figure 1, $\mathcal{E}_{\mathbb{Z}_n}$ is split.

Conversely assume that $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph and $k = 2$. We claim that at least one $\alpha_i = 1$, $i \in \{1, 2\}$. Assume, to the contrary, that $\alpha_1,\ \alpha_2 > 1$. Then $\mathcal{E}_{\mathbb{Z}_n} \cong H \vee K_m$, where $H = K_{\alpha_1,\alpha_2}$ and K_m is a complete graph of order $m = \alpha_1\alpha_2 - 1$, by Theorem 4.2. Since $\alpha_1,\ \alpha_2 > 1$, K_{α_1,α_2} must contain the cycle C_4 as an induced subgraph, which is a contradiction.

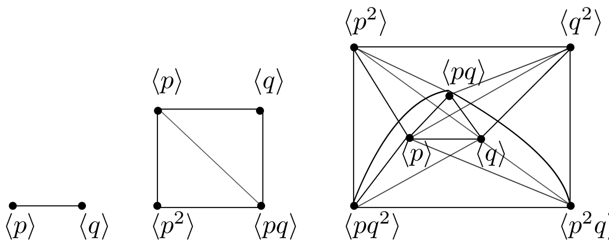
Next assume that $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph and $k = 3$. Now, we claim that all α_i's are 1, for $1 \le i \le 3$. Suppose at least one $\alpha_i > 1$, say α_1. Then by Theorem 4.2 and from the proof of Theorem 3.12, the set $\Omega = \{(0) \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}},\ \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0),\ \mathbb{Z}_{p_1^{\alpha_1}} \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}},\ (p_1) \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0)\}$ of vertices in $\mathcal{E}_{\mathbb{Z}_n}$ induces C_4. Hence $\mathcal{E}_{\mathbb{Z}_n}$ is a not a split graph, which is a contradiction.

Theorem 4.16. *The essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is a split graph if and only if $n = p_1^{\alpha_1} p_2^{\alpha_2}$ with at least one α_i is equal to 1 , $i \in \{1,\ 2\}$ or $n = p_1p_2p_3$, p_i's are distinct primes.*

Next, we specify the values of n for which $\mathcal{E}_{\mathbb{Z}_n}$ is claw-free.

Lemma 4.17. *The essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is claw free if $n = p^{\alpha}$ $(\alpha > 1)$, $pq,\ p^2q$ or p^2q^2 where p and q are distinct primes.*

Proof: Let $n = p^{\alpha}$, $\alpha > 1$. Then by Proposition 4.1, $\mathcal{E}_{\mathbb{Z}_n}$ is complete and hence contains no claw. Graphs corresponding to $n = pq,\ p^2q$ and p^2q^2 are given in Fig.2 A, B and C respectively. Obviously, $\mathcal{E}_{\mathbb{Z}_n}$ is claw free for these values.



A. $\mathcal{E}_{\mathbb{Z}_{pq}}$ B. $\mathcal{E}_{\mathbb{Z}_{p^2q}}$ C. $\mathcal{E}_{\mathbb{Z}_{p^2q^2}}$

Figure 2. Claw free graphs for $n = pq,\ p^2q$ and p^2q^2

Lemma 4.18. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$ such that at least one $\alpha_i > 2$, $1 \le i \le 2$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is not claw-free.*

Proof: Without loss of generality, take $\alpha_1 > 2$. Then by Theorem 4.2, the set $\Omega_1 = \{(0) \times \mathbb{Z}_{p_2^{\alpha_2}},\ \mathbb{Z}_{p_1^{\alpha_1}} \times (0),\ (p_1) \times (0),\ (p_1^2) \times (0)\}$ induces a claw in $\mathcal{E}_{\mathbb{Z}_n}$.

Lemma 4.19. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is claw-free if and only if all α_i's are 1, $1 \le i \le 3$.*

Proof: If all the α_i's are 1, then obviously from Figure 1, $\mathcal{E}_{\mathbb{Z}_n}$ is claw-free. Conversely let $\mathcal{E}_{\mathbb{Z}_n}$ be claw-free. We need to prove that all α_i's are 1. Suppose on the contrary that at least one $\alpha_i > 1$, say α_1. Then the set $\Omega_2 = \{\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0), \mathbb{Z}_{p_1^{\alpha_1}} \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}}, (p_1) \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}}, (0) \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}}\}$ induces a claw in $\mathcal{E}_{\mathbb{Z}_n}$.

Lemma 4.20. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ such that $k \ge 4$. Then $\mathcal{E}_{\mathbb{Z}_n}$ is not claw free.*

Proof: The set $\Omega_3 = \{(0) \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}, \mathbb{Z}_{p_1^{\alpha_1}} \times (0) \times \mathbb{Z}_{p_3^{\alpha_3}} \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}, \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times (0) \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}, \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \mathbb{Z}_{p_3^{\alpha_3}} \times (0) \times \mathbb{Z}_{p_5^{\alpha_5}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}\}$ induces $K_{1,3}$ in $\mathcal{E}_{\mathbb{Z}_n}$. Hence $\mathcal{E}_{\mathbb{Z}_n}$ is not claw free.

Theorem 4.21. *The essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is claw free if and only if n is expected in one of the forms: p^{α} $(\alpha > 1)$, $pq,\ p^2q,\ p^2q^2$ or pqr, where p, q and r are distinct primes.*

Proof: Follows immediately from Lemmas 4.17, 4.18, 4.19 and 4.20.

5 Isomorphism properties of $\mathcal{E}_S$

If $S = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $R = \mathbb{Z}_{11} \times \mathbb{Z}_{13} \times \mathbb{Z}_{17}$, then $\mathcal{E}_S \cong \mathcal{E}_R = K_1 \vee K_3$. But S and R are not isomorphic. Hence we are interested in finding rings (upto isomorphism) whose essential ideal graphs are isomorphic.

Theorem 5.1. *Let $S = \prod_{i=1}^{n} S_i \times \prod_{j=1}^{m} F_j$ and $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F'_j$ be finite commutative rings with $n + m \ge 2$, where each (S_i, m_i) and (R_i, m'_i) are local rings not field and each F_j and F'_j are fields. Let t_i and t'_i be the number of ideals in S_i and R_i respectively. Then $\mathcal{E}_S \cong \mathcal{E}_R$ if and only if $t_i = t'_i$ for all i, $1 \le i \le n$.*

Proof: If $S \cong R$, the result is obvious. Assume that $S \ncong R$ and $t_i = t'_i$ for $1 \le i \le n$. Then $|V(\mathcal{E}_S)| = |V(\mathcal{E}_R)|$. The set of ideals of S_j and R_j, denoted by $I(S_j)$ and $I(R_j)$, are $I(S_j) = \{I_{1j} = (0), I_{2j} = (m_j), I_{3j}, \cdots, I_{t_jj} = S_j\}$ and $I(R_j) = \{I'_{1j} = (0), I'_{2j} = (m'_j), I'_{3j}, \cdots, I'_{t_jj} = R_j\}$ respectively. Obviously, the map : $I_{kj} \to I'_{kj}$ is a bijection from $I(S_j)$ to $I(R_j)$. Now, define the map $\phi : V(\mathcal{E}_S) \to V(\mathcal{E}_R)$ by $\phi(\prod_{i=1}^{n} I_{ki} \times \prod_{j=1}^{m} P_j) = \prod_{i=1}^{n} I'_{ki} \times \prod_{j=1}^{m} P'_j$, where

$$P'_j = \begin{cases} F'_j & \text{if } P_j = F_j \\ (0) & \text{if } P_j = (0) \end{cases}$$

Then ϕ is well-defined and bijective. Now, let $I = \prod_{i=1}^{n} I_i \times \prod_{j=1}^{m} P_j$ and $P = \prod_{i=1}^{n} A_i \times \prod_{j=1}^{m} B_j$ be ideals of S such that I is adjacent to P in $\mathcal{E}_S$. Then $I + P$ is an essential ideal of S. Henceforth there is atleast one i for which $I_i + A_i$ is an essential ideal of S_i and $P_j + B_j = F_j$ for all $j = 1, 2 \cdots, m$. Let $\phi(I) = \prod_{i=1}^{n} I'_i \times \prod_{j=1}^{m} P'_j$ and $\phi(P) = \prod_{i=1}^{n} A'_i \times \prod_{j=1}^{m} B'_j$. Then by definition of ϕ, $I'_j + P'_j$ is an essential ideal of R_j for the same j and $P'_j + B'_j = F'_j$. Hence $\phi(I)$ is adjacent to $\phi(P)$. Hence $\mathcal{E}_S \cong \mathcal{E}_R$.

For the converse, let $\mathcal{E}_S \cong \mathcal{E}_R$ and $t_i = t'_i$ for some i. Then $|V(\mathcal{E}_S)| \neq |V(\mathcal{E}_R)|$, which is a contradiction. Hence $t_i = t'_i$ for all $i = 1, 2, \cdots n$.

Corollary 5.2. *Let S and R be two non-local rings such that $\mathcal{E}_S$ is finite. Then $\mathcal{E}_S \cong \mathcal{E}_R$ if and only if $S \cong S_1 \times S_2 \times \cdots \times S_n$ and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where S_i and R_i are local rings with same number of ideals, for $1 \le i \le n$.*

Proof: Assume that $\mathcal{E}_S \cong \mathcal{E}_R$. By Theorem 5.1, $S \cong S_1 \times S_2 \times \cdots \times S_m$ and $A \cong A_1 \times A_2 \times \cdots \times A_n$, where S_i and R_i are local rings with same number of ideals, for $1 \le i \le m$ and $1 \le j \le n$. Obviously, $m = n$ and each S_i and R_i contains the same number of ideals for otherwise $|V(\mathcal{E}_S)| \neq |V(\mathcal{E}_R)|$.

Converse part is obvious from Theorem 5.1.

Corollary 5.3. *Let S be a non-local ring such that $\mathcal{E}_S$ is a finite graph. Then $\mathcal{E}_S \cong \mathcal{E}_{\mathbb{Z}_n}$ for some natural number n.*

Proof: Follows directly from Chinese Remainder Theorem and Corollary 5.2.

Corollary 5.4. *Let $S = \prod_{i=1}^{n} F_i$ and $R = \prod_{i=1}^{n} F'_i$, where F_i and F'_i are fields and $n \ge 2$. Then $\mathcal{E}_S \cong \mathcal{E}_R$.*

6 Conclusion

In this paper, we provide a characterization for $\mathcal{E}_S$ to be isomorphic to the comaximal ideal graph of S. The structural properties like matching, perfect matching, independence, clique number and chromatic number of the essential ideal graph of some commutative rings are obtained. Moreover, a characterization theorem for the perfectness of $\mathcal{E}_{\mathbb{Z}_n}$ is established. The values of n for which the graph $\mathcal{E}_{\mathbb{Z}_n}$ is split, claw-free, Eulerian and Hamiltonian are determined. In addition, we found that the finite essential ideal graph of any non-local ring is isomorphic to $\mathcal{E}_{\mathbb{Z}_n}$ for some n.

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