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Estimating the Entropy and Residual Entropy of a Lomax Distribution under Generalized Type-II Hybrid Censoring

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Abstract The Lomax distribution (or Pareto II) was first introduced by K. S. Lomax in 1954. It can be readily applied to a wide range of situations including applications in the analysis of the business failure life time data, economics, and actuarial science, income and wealth inequality, size of cities, engineering, lifetime, and reliability modeling.

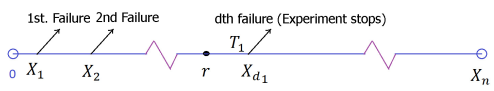
In his pioneering paper, Shannon 1948 defined the notion of entropy as a mathematical measure of information, which is sometimes called Shannon entropy in his honor. He laid the groundwork for a new branch of mathematics in which the notion of entropy plays a fundamental role over different areas of applications such as statistics, information theory, financial analysis, and data compression. [Ebrahimi and Pellerey 14] introduced the residual entropy function because the entropy shouldn't be applied to a system that has survived for some units of time, and therefore, the residual entropy is used to measure the ageing and characterize, classify and order lifetime distributions. In this paper, the estimation of the entropy and residual entropy of a two parameter Lomax distribution under a generalized Type-II hybrid censoring scheme are introduced. The maximum likelihood estimation for the entropy is provided and the Bayes estimation for the residual entropy is obtained. Simulation studies to assess the performance of the estimates with different sample sizes are described, finally conclusions are discussed.

Keywords The Entropy, Residual Entropy, Lomax Distribution, Generalized Type-II Hybrid Censoring, The Maximum Likelihood and Bayes Estimation

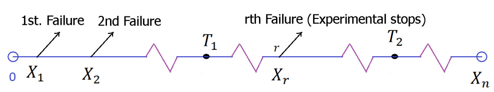
1 Introduction

Censoring schemes are commonly used in reliability qualification and reliability acceptance tests. Type-I censoring and Type-II censoring are the most popular and widely used in practice. While each Type has its merits, it has its drawbacks and limitations. Hybrid censoring schemes (HCS) are mixtures of Type-I and Type-II censoring and can be described as follows. Consider a life testing experiment in which a random sample of n units, from a distribution with a cumulative distribution function $cdf\ F(x)$ and probability density function $pdf\ f(x)$, are put on the test such that successive failure times are recorded as $X_{(1:n)}, \ldots, X_{(n:n)}$. The life-test is terminated either at a prefixed number of failures r $(r < n)$ or a prefixed time, T, whichever is reached first. Let $T^* = min(T, X_r)$ be the actual time of the termination of the experiment. Referred to this scheme as Type-I hybrid censoring scheme (Type-I HCS). The disadvantage of Type-I HCS is that there is a possibility that very few failures may occur before time T. In that case, the efficiency of the estimator(s) might be low. For this reason, [Childs et al, 2] proposed a new HCS that terminate the experiment at the random time $T^* = max(T, X_r)$, and is referred to as Type-II hybrid censoring scheme (Type-II HCS). In this scheme, more than r failures may be observed at the termination time, and this will result in a more efficient estimation procedure. On other hand, the termination time is a random variable, which is consider a disadvantage. So, [Chandrasekar et al, 3] proposed the generalized Type-II HCS (G Type-II HCS) as a modification of Type-II HCS. The reason behind the proposed modification is to fix the underlying disadvantages inherent in both Type-I and Type-II HCS. G Type-II HCS can be described as follows. Assume n items are put on a test. Fix $r \in \{1, 2, \ldots, n\}$, and $T_1, T_2 \in (0, \infty)$, where $T_1 < T_2$. Then we are faced with one of three situations:

-If the r^{th} failure occurs before the time point T_1, terminate the experiment at T_1.
-If the r^{th} failure occurs between T_1 and T_2, terminate the experiment at x_r.
-Otherwise, terminate the experiment at T_2.
The following is a schematic diagram of the three possibilities
Case I



Case II



Case III The likelihood functions for the three different cases

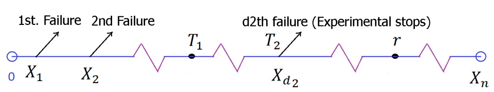


Figure 1. Schematic representation of the G Type-II HCS.

are as follows, see [Chandrasekar et al, 3].

$$L(x) = \begin{cases} Z_1 & D_1 = r, r+1, ..., n, \\ Z_2 & D_2 = r, \\ Z_3 & D_2 = 0, 1, ..., r-1 \end{cases} \tag{1}$$

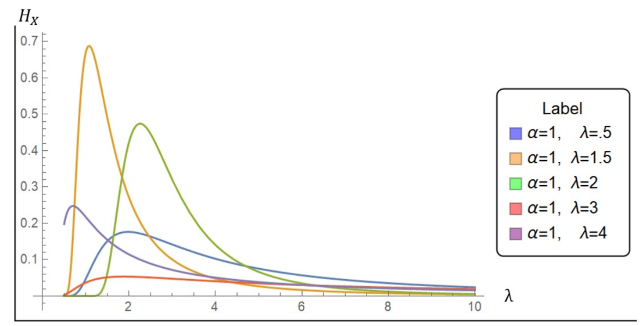


Figure 2. of lomax distrbution for given values of λ

where

$$Z_1 = \frac{n!}{(n-D_1)!}(\prod_{j=1}^{D_1} f(x_{(j)}))(1-F(T_1))^{n-D_1},$$

$$Z_2 = \frac{n!}{(n-r)!}(\prod_{j=1}^{r} f(x_{(j)}))(1-F(x_r))^{n-r},$$

$$Z_3 = \frac{n!}{(n-D_2)!}(\prod_{j=1}^{D_2} f(x_{(j)}))(1-F(T_2))^{n-D_2},$$

D_j denote the number of failures up to time T_j, $j \in (1, 2)$.

[Shannon 5]defined the notion of entropy as a mathematical measure of information which measures the average reduction of uncertainty of random variable X following a discrete distribution. [Shannon 5] defined the entropy as

$$H(f) = -E(\log f(x)) \tag{2}$$

Sometimes we write H_X instead of $H(f)$ to indicate our interest in the entropy as a measure of information contained in the random variable X.

Consider a Lomax distribution with the cdf:

$$F(x) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{3}$$

pdf:

$$f(x) = \frac{\alpha}{\lambda}(1 + \frac{x}{\lambda})^{-(\alpha+1)}, \quad x > 0, \quad \alpha, \lambda > 0, \tag{4}$$

and the reliability function:

$$\bar{F}(x) = (1 + \frac{x}{\lambda})^{-(\alpha)}, \tag{5}$$

where λ is the scale parameter and α is a shape parameter.

From equation (2) the entropy Lomax distribution is given by see [Mahmoud, et al 11]

$$H_X = -\log\frac{\alpha}{\lambda} + \frac{(\alpha+1)}{\alpha}. \quad (6)$$

The wider the spread of a distribution, the lower the value of its entropy. Conversely, the narrower the spread, the higher the value of the entropy. [Ebrahimi and Pellerey 14], considered a dynamic version of the classic Shannon entropy and defined the entropy of the residual lifetime by

$$H(f,t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} log \frac{f(x)}{\bar{F}(t)} dx. \quad (7)$$

Where $\bar{F}(x) = Pr(X > t)$ is the reliability function of X. In particular, $H(f,0) = H(X)$, the residual entropy is used to measure the ageing and characterize, classify and order lifetime distributions see [Ebrahimi and Pellerey 14], [Ebrahimi and Kirmani 13], [Ebrahimi 15,16] and [Oluyede 4]. Many authors worked on the estimation of entropy and residual entropy see [Hall and Morton 18], [Belzunce et al. 7], [Mahmoud, et al. 11], [Ahmad 10], [Cho, et al. 23], [Morabbi, et al. 8] and [Proops 9].
From equation (7) and after some calculations, the residual entropy function associated with Lomax distribution is:

$$H_{Lomax}(t) = \frac{(\alpha+1)}{\alpha} + \log((\frac{\alpha}{\lambda})(1+\frac{t}{\lambda})) \quad (8)$$

In the analysis of lifetime data, the quality of its statistical procedure used depends, to a high extent, on the assumed probability model or distribution. The proper choice of a parametric model provides useful information about reliability measures and failure characteristics of devices under study resulting in sound conclusions and decisions. Among models successfully use in analysing a wide range of real-world situations is the Lomax model. It has been extensively used in reliability modelling and life testing, see [Balkema and de Haan 1]. Lomax distribution arises as a Limiting distribution of residual lifetime at old age, see [Balkema and de Haan 1]. It is a continuous long-tailed distribution proposed as a heavy-tailed alternative to the exponential, Weibull, and long-tailed gamma distributions, see [Bryson 12]. Furthermore, Lomax distribution is related to Burr family of distributions, see [Tadikamalla 19], and can be obtained as a special case from compound gamma distributions, see [Durbey 21]. This distribution plays a fundamental role in statistics, probability and some related areas, such as socioeconomics, engineering, medical and biological sciences, and actuarial science, see [Johnson et al. 17] and [Kleiber and Kotz 6].

The primary purpose of the present paper is to estimating the entropy and residual entropy of a Lomax distribution under G Type-II HCS. To the best of our knowledge, no attempt has been made to estimate the entropy and residual entropy of a Lomax distribution under G Type-II HCS. In Section 2, we obtain the maximum likelihood estimation. In Section 3, we obtain the Bayes estimation for the residual entropy. Simulation studies to assess the performance of the estimates with different sample sizes are described in Section 4, while conclusions are discussed in Section 5.

2 The maximum likelihood estimation

Assume n items with lifetime distribution that are $i.i.d.$ Lomax random variables with cdf (3) and pdf (4), put on a test. Assume also that we subject the experiment to G Type-II HCS described in section 1. Let D denotes the number of failures that occur by time point T, then based on the G Type-II HCS, the likelihood functions of α and λ are given by:
Case I

$$L(x) = \frac{n!}{(n-D_1)!}(\frac{\alpha}{\lambda})^{D_1} \prod_{i=1}^{D_1} \left(1+\frac{x_{(i)}}{\lambda}\right)^{-(\alpha+1)} * \left(1+\frac{T_1}{\lambda}\right)^{-\alpha(n-D_1)}$$

Case II

$$L(x) = \frac{n!}{(n-r)!}((\frac{\alpha}{\lambda})^r \prod_{i=1}^{r}(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)})(1+\frac{x_{(r)}}{\lambda})^{-\alpha(n-r)},$$

Case III

$$L(x) = \frac{n!}{(n-D_2)!}(\frac{\alpha}{\lambda})^{D_2} \prod_{i=1}^{D_2} \left(1+\frac{x_{(i)}}{\lambda}\right)^{-(\alpha+1)} * (1+\frac{T_2}{\lambda})^{-\alpha(n-D_2)}$$

Therefore, Cases I, II and III can be combined and can be written as:

$$L(x) = \frac{n!}{(n-U)!}((\frac{\alpha}{\lambda})^U \prod_{i=1}^{U}(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)})(1+\frac{Q}{\lambda})^{-\alpha(n-U)} \quad (9)$$

where $U = D_1$ and $Q = T_1$ for Case I, $U = r$ and $Q = x_r$ for Case II and $U = D_2$ and $Q = T_2$ for Case III.

The logarithm of (9) can be written as:

$$\ln L \propto U \ln\lambda - U\ln\alpha - (\alpha+1)\Sigma_{i=1}^{r_1} \ln(1+\frac{x_{(i)}}{\lambda}) - \alpha(n-U)\ln(1+\frac{Q}{\lambda}). \quad (10)$$

Taking derivatives with respect to α and λ of (10)

$$\frac{\partial \ln L}{\partial\alpha} = \frac{U}{\alpha} - \Sigma_{i=1}^{U} \ln(1+\frac{x_{(i)}}{\lambda}) - (n-U)\ln(1+\frac{Q}{\lambda}) \quad (11)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{U}{\lambda} + (\alpha+1)\Sigma_{i=1}^{U} \ln\left(1+\frac{x_{(i)}}{\lambda(\lambda+x_{(i)})}\right) + \alpha(n-U)(\frac{Q}{\lambda(\lambda+Q)}) \quad (12)$$

From (11) we obtain:

$$\widehat{\alpha} = \frac{U}{\Sigma_{i=1}^{U} \ln(1+\frac{x_{(i)}}{\lambda}) - (n-U)\ln(1+\frac{Q}{\lambda})} \quad (13)$$

Using (13) in (12) $\widehat{\lambda}$ can be written as:

$$\widehat{\lambda} = \frac{U}{(\widehat{\alpha})+1)\Sigma_{i=1}^{U}(1+\frac{x_{(i)}}{\widehat{\lambda}(\widehat{\lambda}+x_{(i)})}) + (\widehat{\alpha})(n-U)(\frac{Q}{\widehat{\lambda}(\widehat{\lambda}+Q)})}$$

The estimate of α and λ for case I, case II and III in a G Type-I HCS can be written as:

$$\widehat{\alpha} = \begin{cases} \varphi_{11} & D_1 = r, r+1, ..., n, \\ \varphi_{12}, & D_2 = r, \\ \varphi_{13} & D_2 = 0, 1, ..., r-1, \end{cases} \quad (14)$$

where

$$\varphi_{11} = \frac{D_1}{\Sigma_{i=1}^{D_1} \ln(1+\frac{x_{(i)}}{\widehat{\lambda}}) - (n-D_1)\ln(1+\frac{T_1}{\widehat{\lambda}})},$$

$$\varphi_{12} = \frac{r}{\Sigma_{i=1}^{r} \ln(1+\frac{x_{(i)}}{\widehat{\lambda}}) - (n-r)\ln(1+\frac{x_{(r)}}{\widehat{\lambda}})},$$

$$\varphi_{13} = \frac{D_2}{\Sigma_{i=1}^{D_2} \ln(1+\frac{x_{(i)}}{\widehat{\lambda}}) - (n-D_2)\ln(1+\frac{T_2}{\widehat{\lambda}})}.$$

and

$$\widehat{\lambda} = \begin{cases} \varphi_{21}, & D_1 = r, r+1, ..., n, \\ \varphi_{22}, & D_2 = r, \\ \varphi_{23}, & D_2 = 0, 1, ..., r-1, \end{cases} \quad (15)$$

where

$$\varphi_{21} = \frac{D_1}{(\widehat{\alpha}+1)\Sigma_{i=1}^{D_1}(1+\frac{x_{(i)}}{\widehat{\lambda}(\widehat{\lambda}+x_{(i)})}) + (\widehat{\alpha})(n-D_1)(\frac{T_1}{\widehat{\lambda}(\widehat{\lambda}+T_1)})},$$

$$\varphi_{22} = \frac{r}{(\widehat{\alpha}+1)\Sigma_{i=1}^{r}(1+\frac{x_{(i)}}{\widehat{\lambda}(\widehat{\lambda}+x_{(i)})}) + (\widehat{\alpha})(n-r)(\frac{x_{(r)}}{\widehat{\lambda}(\widehat{\lambda}+x_{(r)})})},$$

$$\varphi_{23} = \frac{D_2}{(\widehat{\alpha}+1)\Sigma_{i=1}^{D_2}(1+\frac{x_{(i)}}{\widehat{\lambda}(\widehat{\lambda}+x_{(i)})}) + (\widehat{\alpha})(n-D_2)(\frac{T_2}{\widehat{\lambda}(\widehat{\lambda}+T_2)})},$$

Equation (15) can be solved numerically to obtain $\widehat{\lambda}$ and substitute it in equation (14) to obtain $\widehat{\alpha}$. We used Mathematica 11 to solve equation (15). After we obtain $\hat{\alpha}$ and $\hat{\lambda}$ we use equation (6) to get the MLEs of the entropy Lomax distribution as:

$$\hat{H}_X = -\log\frac{\hat{\alpha}}{\hat{\lambda}} + \frac{(\hat{\alpha}+1)}{\hat{\alpha}}.$$

3 The Bayes estimation for the residual entropy for the Lomax distribution under generalized Type II hybrid censored sample

[Ashour et al 20] considered the Bayesian inference for the two-parameter α and λ on a HCS. They assumed that α and λ have the joint prior density as

$$\pi(\alpha,\lambda) = \pi(\alpha)\pi(\lambda) \propto (\alpha)^{\delta-1}(\lambda)^{-1} \quad (16)$$

Let $X = (x_1, \ldots, x_U)$ be the observed sample by the end the experiment. Based on the above joint prior distribution, the joint density of the α, λ and X can be written as follows.

$$\pi(\alpha,\lambda,x) = \alpha^{U+\delta-1}\lambda^{-(U+1)} \prod_{i=1}^{U} \left(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)}\right) * \left(1+\frac{Q}{\lambda}\right)^{-\alpha(n-U)} \quad (17)$$

The posterior distribution of α and λ, given X, is obtained as:

$$\pi(\alpha,\lambda|x) = \frac{\pi(\alpha,\lambda,x)}{\int_0^\infty \int_0^\infty \pi(\alpha,\lambda,x)d\alpha d\lambda}$$

Useing Equation (9) and Equation (16) the joint posterior density functions of α and λ under G Type-II HCS will be:

$$\pi(\alpha,\lambda|x) \propto \frac{A}{B}$$

where

$$A = \Big(\alpha^{U+\delta-1}\lambda^{-(U+1)} * \Big(\prod_{i=1}^{U}(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)}\Big) * (1+\frac{Q}{\lambda})^{-\alpha(n-U)}\Big)$$

and

$$B = \int_0^\infty \int_0^\infty \alpha^{U+\delta-1}\lambda^{-(U+1)} * \prod_{i=1}^{U}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-(\alpha+1)} * \left(1+\frac{Q}{\lambda}\right)^{-\alpha(n-U)} d\alpha d\lambda$$

The Bayes estimation of α and λ is obtained as:

$$H_B(x) \propto \frac{\int_0^\infty \int_0^\infty H(f,t)\pi(\alpha,\lambda,x)d\alpha d\lambda}{\int_0^\infty \int_0^\infty \pi(\alpha,\lambda,x)d\alpha d\lambda} \tag{18}$$

Using Equation (8) and Equation (17) the Bayes estimation for the residual entropy for the Lomax distribution under G Type-II HCS will be:

$\tilde{H}_B(x) = \frac{V}{W}$ where

$$V = \Big(\int_0^\infty \int_0^\infty \frac{\alpha+1}{\alpha} + \log(\frac{\alpha}{\lambda}(1+\frac{t}{\lambda})\alpha^{U+\delta-1}) * \lambda^{-(U+1)} \Big(\prod_{i=1}^{U}(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)}\Big)(1+\frac{Q}{\lambda})^{-\alpha(n-U)}d\alpha d\lambda\Big)$$

$$W = \Big(\int_0^\infty \int_0^\infty \alpha^{U+\delta-1}\lambda^{-(U+1)}\Big(\prod_{i=1}^{U} 1+(\frac{x_{(i)}}{\lambda})^{-(\alpha+1)}\Big) ((1+\frac{Q}{\lambda})^{-\alpha(n-U)})d\alpha d\lambda\Big)$$

$\tilde{H}_B(x)$ will be computed numerically after substituting $\hat{\lambda}$ and $\hat{\alpha}$ obtained from solving Equations (14) and (15).

4 Illustrative Example

For illustration purposes, we use the data set given by [Nelson 22] that explains the results of a life test experiment in which certain of a type electrical insulating material was subjected to constant voltage stress. The observed failure times (in minutes) are as follows: 0.27, 0.4, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.8, 53.24, 82.85, 89.29, 100.58, 215.1. We shall try to fit the Lomax data based on G-Type-II HCS. We take case I ($T_1 = 3$, $T_2 = 10$, and $r = 7$), case II ($T_1 = 7$, $T_2 = 10$, and $r = 5$), and case III ($T_1 = 8$, $T_2 = 12$, and $r = 6$).

The estimation of the entropy of the G-Type-II HCS is presented in Table 1.

5 Simulation Study

In this section, we present the results of two simulation studies that were carried out for the following.
Again Mathematica 11 will be used to compute Equation (20).

1- To assess the performance of MLE estimation of the Lomax entropy. The assessment is carried out through measures of the entropy and the mean square error (MSE) of entropy under different choices of the G Type-II HCS and different combinations of n, T_1, T_2 and r values. In each case the process was replicated N = 1000 times for a particular G Type-II HCS. The MLEs for the entropy were obtained as described before in Section 2. We were able to express α in terms of λ as in formula (9) therefore obtaining the MLE estimates is attained by solving the equation (10). The computational system Mathematica 11 was used to solve equation (10) in λ. We substituted these values in (9) to obtain the values of α. These values of α and λ constitute their maximum likelihood estimates. We substituted these values in (13) to obtain the MLE estimates of the entropy of the Lomax distribution under G Type-II HCS in Tables 2 and 3.

2- To assess the performance of the Bayes estimation for the residual entropy of the Lomax distribution. The assessment is carried out through measures of the residual entropy, Relative Bias (RB) and the mean square error (MSE) of residual entropy under different choices of the G Type-II HCS and different combination of n, T_1, T_2 and r values. In each case process was replicated $N = 1000$ times for a particular G Type-II HCS. The Bayes estimation for the residual entropy were obtained as described before in formula (19) in Tables 4 and 5.

In Table 2, when λ is fixed, and α is increase the RB and MSE are decrease, and when fixed n, r and T_1, and the time T_2 is increase we observe the MSE and RB are increases, also, when the sample size n increase, the values of $\hat{H}(x)$, MSE and RB are decreases.

In Table 3, when α is fixed, and λ is increase the RB and MSE are decrease, and when the sample size n increase, the values of $\hat{H}(x)$, MSE and RB are decreases.

In Table 4, when λ is fixed, and α is increase the SD is increase, the RB and MSE are decrease, and when the time T_2 is increase the SD is increase, the RB and MSE are decrease.

In Table 5, for a fixed n, α and r is increase the $\tilde{H}(B)$ is decrease. When n, r are increases the SD is decrease, when T_1 is increase the SD is decrease. Also, T_2 is fixed and n, T_1 are increase, the MSE and $RBias$ are decrease.

6 Summary

In this article, entropy estimates for the Lomax distribution were computed using the MLE of α and λ based on G Type-II HCS. The estimates were assessed in terms of their mean square error (MSE), and relative bias $(RBias)$ of entropy. Also, we performed simulation studies with different sample sizes focusing on the residual entropy estimate of the Lomax distribution under the G Type-II HCS. The estimates we were assessed in terms of their standard deviation (SD), the mean square error (MSE), and relative bias (RB), of residual entropy under different chaise of the G Type-II HCS.

7 Tables

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Table 1. Estimation of entropy as an example

	T_1	T_2	r	$\widehat{H}$	$RBiase\widehat{H}$	$MSE\widehat{H}$	$RRMSE\widehat{H}$
CaseI	3	10	7	3.98843	0.687759	0.0880509	0.125567
CaseII	7	10	5	3.96291	0.676962	0.0853080	0.123596
CaseIII	8	12	6	3.63759	0.434944	0.0405231	0.079409

Table 2. Entropy estimates $\hat{H}$ and its MSEs and relative bias, for selected values of n,T_1,T_2 and r.

α	λ	n	r	T_1	T_2	$\widehat{H}$	MSE	$RBias$
6	10	50	20	1.5	3	1.7023	0.1001	0.1987
					4	1.8754	0.1000	0.1999
					5	1.7976	0.1112	0.2131
				2	3	1.5395	0.0091	0.0659
					4	1.8200	0.0141	0.2602
					5	1.7719	0.1073	0.2268
		100	30	1.5	3	1.7702	0.1056	0.2216
					4	1.8012	0.1055	0.2143
					5	1.8098	0.1076	0.2215
				2	3	1.2189	0.7314	0.2772
					4	1.2383	0.7765	0.2975
					5	2.2156	0.7943	0.3245
		150	50	1.5	3	1.7954	1.1089	0.2214
					4	1.8345	1.1079	0.2314
					5	1.8023	1.1083	0.2012
				2	3	1.2215	0.2981	0.0039
					4	1.2630	0.3201	0.0101
					5	2.2287	0.3653	0.0124
		200	60	1.5	3	1.7732	1.1077	0.2122
					4	1.7743	1.1077	0.2012
					5	1.7744	1.1077	0.2223
				2	3	1.2752	0.2238	0.0390
					4	1.2387	0.2181	0.0601
					5	2.2126	0.3067	0.0640
8		50	20	1.5	3	1.7773	1.0762	0.0221
					4	1.7782	1.0781	0.0223
					5	1.7823	1.0788	0.0231
				2	3	1.6889	0.0047	0.0276
					4	1.7707	0.0064	0.0475
					5	1.7719	0.1073	0.2268
		100	30	1.5	3	1.7843	1.0771	0.0221
					4	1.7855	1.0769	0.0231
					5	1.7930	1.0775	0.0237
				2	3	1.4478	0.8013	0.2779
					4	1.5024	0.8752	0.2900
					5	1.9947	0.9942	0.3145
		150	50	1.5	3	1.8112	1.0783	0.0231
					4	1.7999	1.0779	0.0234
					5	1.8166	1.0792	0.0233
				2	3	1.1836	0.7298	0.2789
					4	1.1954	0.8121	0.2980
					5	1.3625	0.9650	0.3312
		200	60	1.5	3	1.8099	1.0775	0.0241
					4	1.8145	1.0781	0.0239
					5	1.8255	1.0785	0.0239
				2	3	1.0996	0.7238	0.2936
					4	1.1106	0.8010	0.3176
					5	1.9998	0.8367	0.3675

Continuance Table 2

α	λ	n	r	T_1	T_2	$\widehat{H}$	MSE	$RBias$
11		50	20	1.5	3	1.8177	1.0778	0.0237
					4	1.8099	1.0784	0.0239
					5	1.8100	1.0786	0.0239
				2	3	1.7944	0.0088	0.0497
					4	1.6529	0.0554	0.1245
					5	1.7106	0.6162	0.9416
		100	30	1.5	3	1.7765	1.0076	0.0043
					4	1.7543	1.0095	0.0085
					5	1.7469	1.0074	0.0039
				2	3	0.9875	0.7224	0.2252
					4	1.0024	0.7432	0.2085
					5	1.9024	0.7686	0.2974
		150	50	1.5	3	1.7499	1.0068	1.0038
					4	1.7398	1.0047	1.0048
					5	1.7003	1.0043	1.0026
				2	3	0.9763	0.7034	0.1962
					4	1.0432	0.7243	0.1999
					5	1.8764	0.7342	0.2129
		200	60	1.5	3	1.6999	1.0041	0.9881
					4	1.6800	1.0038	0.9677
					5	1.6289	1.0022	0.9623
				2	3	1.0987	0.7345	0.1854
					4	1.2098	0.7453	0.1929
					5	1.2468	0.7643	0.2590

Table 3. Entropy estimates $\hat{H}$ and its MSEs and relative bias, for selected values of n,T_1,T_2 and r.

α	λ	n	r	T_2	T_1	$\widehat{H}$	MSE	$RBias$
9	8	50	35	6	2	1.6582	0.0337	0.2595
					3	1.6480	0.0349	0.2645
					4	2.1463	0.0866	0.0415
				7	2	1.6602	0.0351	0.2311
					3	1.6299	0.0334	0.0889
					4	1.6015	0.0309	0.0488
		100	60	7	2	1.3796	0.0012	0.1868
					3	1.1009	0.0021	0.2873
					4	1.0235	0.0023	0.2037
				7.5	2	1.3310	0.0022	0.1611
					3	1.0999	0.0019	0.1433
					4	1.0447	0.0016	0.1293
		150	100	8	2	1.4278	0.0057	0.1818
					3	1.2281	0.0053	0.2412
					4	1.1563	0.0056	0.2144
				9	2	1.3217	0.0076	0.0910
					3	1.1732	0.0034	0.2807
					4	1.0383	0.0032	0.2305
		200	120	9	2	1.7722	0.0045	0.4527
					3	1.8121	0.0038	0.3988
					4	1.8243	0.0036	0.3377
				10	2	1.5290	0.0023	0.3697
					3	1.0891	0.0032	0.2990
					4	1.0371	0.0042	0.2358

Continuance Table 3

α	λ	n	r	T_2	T_1	$\widehat{H}$	MSE	$RBias$
	11	50	35	6	2	1.3897	0.0124	0.2026
					3	1.2242	0.0683	0.1933
					4	1.2026	0.0291	0.0392
				7	2	1.4429	0.0226	0.1182
					3	1.2399	0.0103	0.0188
					4	1.1877	0.0092	0.0098
		100	60	7	2	1.3945	0.0012	0.0971
					3	1.0510	0.0023	0.0750
					4	1.0343	0.0071	0.0210
				7.5	2	1.3086	0.0013	0.0984
					3	1.1195	0.0011	0.0634
					4	1.0111	0.0009	0.0132
		150	100	8	2	1.2667	0.0095	0.0101
					3	1.1599	0.0088	0.0044
					4	1.0414	0.0066	0.0009
				9	2	1.4027	0.0013	0.0820
					3	1.1311	0.0033	0.0029
					4	1.0307	0.0054	0.0008
		200	120	9	2	1.9311	0.0041	0.1544
					3	1.8221	0.0029	0.0835
					4	1.7153	0.0018	0.6424
				10	2	1.4423	0.0021	0.0870
					3	1.9753	0.0076	0.0283
					4	1.9945	0.0087	0.0110
	12	50	35	6	2	1.1289	0.1249	0.3426
					3	1.2424	0.2687	0.3333
					4	1.3026	0.2918	0.0409
				7	2	1.1198	0.1198	0.2288
					3	1.0911	0.9987	0.1243
					4	1.0321	0.0188	0.1199
		100	60	7	2	1.3491	0.0032	0.2342
					3	1.0409	0.0054	0.0485
					4	0.9898	0.0089	0.0151
				7.5	2	1.2987	0.0021	0.0188
					3	1.1066	0.0016	0.0138
					4	0.7754	0.0010	0.0115
		150	100	8	2	1.3387	0.0089	0.1481
					3	1.2976	0.0071	0.0923
					4	1.1991	0.0069	0.0901
				9	2	1.2834	0.0043	0.1094
					3	1.0343	0.0044	0.0655
					4	1.0045	0.0064	0.0897
		200	120	9	2	1.4467	0.0066	0.0432
					3	1.3976	0.0028	0.0222
					4	1.2298	0.0013	0.0018
				10	2	1.3665	0.0043	0.0444
					3	1.0523	0.0045	0.1349
					4	1.0215	0.0034	0.4647

Table 4. The residual entropy estimates $\sim H_B$, SD, MSE and $RBias$ for selected values of α, λ, r when $t = 20$.

6	16	50	20	2	4	2.2525	0.0844	0.1162	0.9179
					6	2.2242	0.1765	0.1102	0.8938
					8	2.3173	0.2631	0.1306	0.9731
	14				4	2.2211	0.1198	0.1199	0.9099
					6	2.2322	0.1012	0.1089	0.8933
					8	2.3021	0.0987	0.0997	0.7719
	16	100	30		4	3.2070	0.5424	0.4131	1.7307
					6	2.9217	0.5765	0.3053	1.4877
					8	2.9106	0.3792	0.3014	1.4783
	14				4	2.9976	0.4412	0.3498	1.5578
					6	2,8562	0.3876	0.3215	1.4327
					8	2.7712	0.3356	0.2278	1.3387
	16	150	50		4	2.8999	0.4873	0.2977	1.4692
					6	2.0067	0.3683	0.2034	1.2144
					8	3.0013	0.6382	0.3337	1.5556
	14				4	2.1871	0.3654	0.2967	1.3865
					6	2.1543	0.3177	0.2688	1.3277
					8	2.1057	0.2298	0.2433	1.2467
	16	200	60		4	2.4964	0.2114	0.1747	1.1256
					6	2.4013	0.1836	0.1505	1.0447
					8	2.3907	0.1615	0.1479	1.0356
	14				4	2.4633	0.2066	0.1422	0.9443
					6	2.3765	0.1765	0.1289	0.9001
					8	2.2965	0.1453	0.1099	0.7332
7	16	50	20		4	2.3363	0.1334	0.0838	0.6447
					6	2.4224	0.2381	0.1004	0.7053
					8	2.4586	0.1527	0.1077	0.7087
	14				4	2.3341	0.2219	0.0712	0.6388
					6	2.2134	0.1593	0.0499	0.6011
					8	2.2098	0.1129	0.0223	0.4657
	16	100	30		4	3.2577	0.4611	0.3375	1.2935
					6	3.1901	0.8470	0.3131	1.2461
					8	2.9712	0.4038	0.2404	1.0917
	14				4	3.2011	0.4706	0.2809	1.0733
					6	3.0008	0.4278	0.2600	1.0388
					8	2.7933	0.3659	0.1746	1.0072
	16	150	50		4	2.8934	0.4188	0.2169	1.0317
					6	2.9177	0.4273	0.2241	1.0542
					8	2.8030	0.5309	0.1911	0.9733
	14				4	2.8799	0.5502	0.2151	1.0113
					6	2.6633	0.5011	0.1616	0,9236
					8	2.4325	0.4520	0.1286	0.7213
	16	200	60		4	2.6639	0.1722	0.1546	0.8754
					6	2.6365	0.1244	0.1478	0.8561
					8	2.6202	0.3121	0.1439	0.8446
	14				4	2.4122	0.2218	0.1566	0.6611
					6	2.3211	0.1923	0.1218	0.4238
					8	2.1674	0.1813	0.0901	0.2765
9	16	50	20		4	2.6950	0.1546	0.0829	0.5104
					6	2.7802	0.1999	0.0992	0.5582
					8	2.7186	0.2303	0.0873	0.5237
	14				4	2.7709	0.1677	0.0771	0.4256
					6	2.6755	0.1542	0.0638	0.3365
					8	2.5488	0.1126	0.0561	0.2219

Continuance Table 4

α	λ	r	n	T_1	T_2	$\sim H_B$	SD	MSE	$RBias$
	16	100	30		4	3.1074	0.4475	0.1750	0.7415
					6	3.1915	0.3374	0.1980	0.7887
					8	3.0570	0.3067	0.1006	0.7133
	14				4	3.1805	0.3155	0.1173	0.6546
					6	3.0944	0.2617	0.0922	0.5327
					8	2.8306	0.1818	0.0813	0.4431
	16	150	50		4	2.9230	0.2279	0.1296	0.6382
					6	3.2988	0.4690	0.2129	0.8488
					8	3.2281	0.3732	0.2081	0.8092
	14				4	2.3538	0.3917	0.0108	0.7732
					6	2.3007	0.3721	0.0072	0.5324
					8	2.1134	0.2287	0.0061	0.4210
	16	100	30		4	2.8875	0.1927	0.1217	0.6183
					6	2.8039	0.1905	0.1039	0.5715
					8	2.9726	0.3780	0.1412	0.6660
	14				4	2.7745	0.2187	0.0100	0.4427
					6	2.6405	0.2298	0.0093	0.4042
					8	2.5921	0.2655	0.0065	0.0929

Table 5. The residual entropy estimates $\sim H_B$, SD, MSE and $RBias$ for selected values of α, λ, r when $t = 20$.

α	λ	r	n	T_1	T_2	$\sim H_B$	SD	MSE	$RBias$
6	16	50	40	2	5	2.3032	0.2561	0.1274	0.9611
					7	2.3226	0.2117	0.1038	0.7760
					9	2.0863	0.1173	0.0836	0.7764
	14				5	2.3356	0.3316	0.0134	0.9920
					7	2.2833	0.2488	0.0111	0.7582
					9	2.2387	0.1904	0.0081	0.6322
	16	100	60		5	2.4443	0.2994	0.1611	1.0811
					7	2.1354	0.2293	0.1094	0.8182
					9	2.1593	0.1146	0.0970	0.7386
	14				5	2.3376	0.2207	0.0997	0.7111
					7	2.2711	0.1807	0.0925	0.6331
					9	2.2207	0.1598	0.0741	0.4230
	16	150	100		5	2.2785	0.5129	0.1190	0.9400
					7	2.0799	0.3398	0.0898	0.7709
					9	2.0994	0.1850	0.0855	0.6876
	14				5	2.1187	0.2318	0.1000	0.8432
					7	2.0735	0.1810	0.0818	0.6349
					9	2.0306	0.1614	0.0611	0.4287
	16	200	120		5	2.0256	0.6214	0.0724	0.7247
					7	1.9223	0.4708	0.0559	0.6368
					9	1.9888	0.1986	0.0362	0.5934
	14				5	2.0111	0.2108	0.0779	0.6332
					7	2.0087	0.1784	0.0588	0.3717
					9	2.0025	0.1103	0.0398	0.1769

Continuance Table 5

α	λ	n	r	T_1	T_2	$\sim H_B$	SD	MSE	$RBias$
7	16	50	40		5	2.3198	0.0776	0.0808	0.6331
					7	2.3349	0.0130	0.0836	0.6437
					9	2.2831	0.0011	0.0744	0.6073
	14				5	2.3377	0.1173	0.0889	0.6637
					7	2.2766	0.0843	0.0633	0.4478
					9	2.2487	0.0662	0.0472	0.2171
	16	100	60		5	2.3639	0.1223	0.0890	0.6642
					7	2.2003	0.0105	0.0620	0.5544
					9	2.2192	0.0026	0.0630	0.5623
	14				5	2.2287	0.1185	0.0866	0.5388
					7	2.1767	0.0944	0.0633	0.3786
					9	2.1270	0.0695	0.0498	0.1099
	16	150	100		5	2.2533	0.2492	0.0693	0.5863
					7	2.1741	0.0825	0.0568	0.5306
					9	2.1092	0.0380	0.0474	0.4851
	14				5	2.2055	0.0991	0.0632	0.4487
					7	2.1654	0.0743	0.0530	0.1776
					9	2.1155	0.0525	0.0317	0.1332
	16	200	120		5	2.1531	0.2672	0.0536	0.5157
					7	2.0790	0.1193	0.0433	0.4636
					9	2.0639	0.0705	0.0414	0.4529
	14				5	2.1195	0.0866	0.0831	0.3779
					7	2.0944	0.0522	0.0639	0.2301
					9	2.0721	0.0427	0.0387	0.1877
9	16	50	40		5	2.6598	0.1199	0.0766	0.4907
					7	2.6375	0.1318	0.9087	0.5342
					9	2.5892	0.1945	0.0647	0.4511
	14				5	2.6635	0.0862	0.0732	0.2267
					7	2.4478	0.0677	0.0411	0.1501
					9	2.3478	0.0589	0.0201	0.0882
	16	100	60		5	2.9007	0.0710	0.1246	0.6257
					7	2.5530	0.1161	0.0591	0.4308
					9	2.3803	0.1715	0.0633	0.4461
	14				5	2.7892	0.0907	0.0448	0.3379
					7	2.6577	0.0644	0.0278	0.1806
					9	2.3771	0.0428	0.0017	0.0866
	16	150	100		5	2.5812	0.1617	0.0635	0.4466
					7	2.4783	0.1352	0.0481	0.3889
					9	2.3242	0.1019	0.0547	0.4147
	14				5	2.6277	0.0860	0.0740	0.3371
					7	2.4307	0.0631	0.0486	0.1189
					9	2.2741	0.0486	0.0168	0.0888
	16	200	120		5	2.4611	0.1601	0.0458	0.3793
					7	2.4044	0.1713	0.0490	0.3924
					9	2.3449	0.1034	0.0436	0.3702
	14				5	2.3799	0.0721	0.0522	0.4476
					7	2.2085	0.0642	0.0276	0.2256
					9	2.1718	0.0379	0.0019	0.1143