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Analytical Solutions of ARL for $SAR(p)_L$ Model on a Modified EWMA Chart

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Abstract A modified exponentially weighted moving average (EWMA) scheme expanded from an EWMA chart is an instrument for immediate detection on a small shifted size. The objective of this research is to suggest the average run length (ARL) with the explicit formula on a modified EWMA control chart for observations of a seasonal autoregressive model of order p^{th} ($SAR(p)_L$) with exponential residual. A numerical integral equation method is brought to approximate ARL for checking an accuracy of explicit formulas. The results of two methods show that their ARL solutions are close and the percentage of the absolute relative change (ARC) is obtained to less than 0.002. Furthermore, the modified EWMA chart with the $SAR(p)_L$ model is tested to shift detection when the parameters c and λ are changed. The ARL and the relative mean index (RMI) results are found to be better when c and λ are increased. In addition, the modified EWMA control chart is compared to performance with the EWMA scheme and such that their results encourage the modified EWMA chart for a small shift. Finally, this explicit formula can be applied to various real-world data. For example, two data about information and communication technology are used for the validation and the capability of our techniques.

Keywords Explicit Formula, Seasonal Autoregressive, Average Run Length

1. Introduction

A control chart is one of the instruments for Statistical Process Control (SPC). The ordinary control charts are well known such as a Shewhart control chart [1], an exponentially weighted moving average (EWMA) chart [2], a cumulative sum (CUSUM) scheme [3]. A Shewhart chart is the basis of a control chart that detected a large shift on 3-sigma control limits speedily. Next, EWMA and CUSUM control charts are developed to a small shift detection appropriately. For the EWMA control chart, many recent works of literature were found to this chart usage, for example, Li et al. [4] introduced the EWMA control chart to invent the transient patterns of motivation and potential of specifying seasonal faster motivation. Moreover, Nawaz et al. [5] presented the EWMA control chart by integrating multiscale principal component analysis for improving the monitoring efficiently and detecting an online multiscale mistake. Recently, Hu and Liu [6] detected the positive shifts of zero-inflated poisson models by using a weighted score test statistic on an upper-sided exponentially weighted moving average control chart.

Meanwhile, the EWMA control chart is improved to a performance by many researchers. One of them is the modified EWMA control chart which was originally presented by Patel and Divecha [7] and developed by Khan et al. [8]. The modified EWMA statistic is expanded by adding a multiple of a previous shift term and a constant c for an abrupt detection of autocorrelated observations. The modified EWMA control chart was continuously studied in various literature [9-11].

For the ability comparison of control charts, one of the well-known measurements is the average run length (ARL) [12] presented by using numerous calculations such as a

Markov chain [13], a Monte Carlo simulation [14], a numerical integral equation (NIE) method [15], and an explicit formula such that the later one can be found to exact solutions. For example, Petcharat [16] suggested the explicit formula of the ARL for the SAR(p)$_L$ process in the case of an exponential error on the EWMA chart. Next, Sukparungsee and Areepong [17] presented ARL solutions by using the explicit formula on an EWMA control chart of the AR(p) model for white noise with exponential distribution. Recently, Phanthuna et al. [18] introduced the modified EWMA control chart on the AR(1) process in the case of exponential white noise by using the explicit formula for solving the ARL.

In this research, the modified EWMA chart of the SAR(p)$_L$ model with exponential white noise is proposed. A SAR(p)$_L$ model is an autoregressive model added to consider a seasonal component in time series analysis. A seasonal autoregressive model can be applied to many fields such as engineering [19], environment [20] and communication [21].

Nowadays, global communication has rapidly and continuously entered into the digital age. Therefore, information and communication technology plays a crucial role as a tool for accessing information in a digital platform such that the use of the internet is essential for supporting many areas such as educations, businesses and commerces, entertainment, etc. For this case study, the ARL explicit formula is applied to the percentages of internet users by news and business website categories for preparing to support network applications in the future.

In section 2, the modified EWMA chart of a seasonal autoregressive model is displayed. Moreover, the explicit formula is derived and the NIE method is presented for ARL evaluations. In section 3, the ARL results of two techniques are compared for checking preciseness. Next, the capability of the EWMA and modified EWMA control chart is compared. Afterward, the explicit formula of ARL is applied to real data of time series. Discussion and Conclusion are described in sections 4 and 5, respectively. Finally, the future research is suggested in section 6.

2. Materials and Methods

This section suggests designing a modified EWMA statistic together with observations of a SAR(p)$_L$ model. Next, explicit formulas of the ARL are derived and compared with the NIE method.

2.1. The Modified EWMA Scheme with a SAR(p)$_L$ Model

Patel and Divecha [7] and Khan et al. [8] expanded a modified exponentially weighted moving average (EWMA) control chart originated from the classical EWMA scheme. For a random variable sequence, X_t is an observation of a SAR(p)$_L$ model with average μ and variance σ^2, where t is a positive integer.

SAR(p)$_L$ model:

$$X_t = \eta + \phi_1 X_{t-L} + \phi_2 X_{t-2L} + \dots + \phi_p X_{t-pL} + \varepsilon_t$$

$$\text{or} \quad X_t = \eta + \sum_{i=1}^{p} \phi_i X_{t-iL} + \varepsilon_t, \tag{1}$$

where η is a reasonable constant, ϕ_i is an autoregressive coefficient $(|\phi_p| < 1)$, at i = 1, 2,…, p, L is a seasonal period and ε_t defined white noise sequences with the exponential distribution as $\varepsilon_t \sim Exp(\beta)$.

Modified EWMA statistic:

$$Y_t = (1-\lambda)Y_{t-1} + \lambda X_t + c(X_t - X_{t-1}), \tag{2}$$

where λ is an exponential smoothing parameter with $0 < \lambda < 1$, c is a constant, the mean of Y_t is μ such that $Y_0 = \mu = X_0$ is an initial value and the asymptotic variance of Y_t is $(\lambda + 2\lambda c + 2c^2)\sigma^2 / (2-\lambda)$. In addition, an EWMA statistics is found in (2), when $c = 0$.

Control bounds of a modified EWMA chart:

$$\mu \pm W_s \sigma \sqrt{\frac{\lambda + 2\lambda c + 2c^2}{2-\lambda}}, \tag{3}$$

where W_s is suitable to control width limits, and λ is an exponential smoothing parameter. In similar ways, the control bounds of a classical EWMA chart can be looked for (3), as $c = 0$.

If Equation (1) of a SAR(p)$_L$ model is substituted into (2), then the modified EWMA statistic with a SAR(p)$_L$ model can be written as:

$$Y_t = (1-\lambda)Y_{t-1} + \lambda X_t + c(X_t - X_{t-1})$$

$$Y_t = (1-\lambda)Y_{t-1} + \lambda\left(\eta + \sum_{i=1}^{p} \phi_i X_{t-iL} + \varepsilon_t\right) + c\left(\eta + \sum_{i=1}^{p} \phi_i X_{t-iL} + \varepsilon_t - X_{t-1}\right)$$

$$Y_t = (1-\lambda)Y_{t-1} + (c+\lambda)\sum_{i=1}^{p} \phi_i X_{t-iL} - cX_{t-1} + (c+\lambda)\varepsilon_t + (c+\lambda)\eta. \tag{4}$$

For detection of an out-of-control process, the corresponding stopping time on SAR(p)$_L$ model of the modified EWMA control chart, where $Y_t = u$ is the initial value, a and b are lower and upper control limits, is defined as:

$$\tau_{a,b} = \inf\{t > 0;\ Y_t < a \cup Y_t > b\}. \tag{5}$$

On the other hand, Y_t is in an in-control process can be reorganized in the error term ε_t as:

$$a < Y_t < b$$

$$a < \begin{pmatrix} (1-\lambda)u + (c+\lambda)\sum_{i=1}^{p}\phi_i X_{t-iL} \\ -cX_0 + (c+\lambda)\varepsilon_t + (c+\lambda)\eta \end{pmatrix} < b$$

$$\begin{pmatrix} \frac{a-(1-\lambda)u+cX_0}{(c+\lambda)} \\ -\sum_{i=1}^{p}\phi_i X_{t-iL} - \eta \end{pmatrix} < \varepsilon_t < \begin{pmatrix} \frac{b-(1-\lambda)u+cX_0}{(c+\lambda)} \\ -\sum_{i=1}^{p}\phi_i X_{t-iL} - \eta \end{pmatrix}. \quad (6)$$

An initial value of the ARL for the $SAR(p)_L$ model on the modified EWMA control chart is determined as:

$$ARL(u) = E_\infty(\tau_{a,b}). \quad (7)$$

2.2. Analytical Solutions of ARL

For the modified EWMA control chart of the $SAR(p)_L$ model, ARL is solved by using the method of the Fredholm integral equation of the second kind [22]. $ARL(u)$ recalled for an initial ARL can be described as:

$$ARL(u) = 1 + \int_{\frac{a-(1-\lambda)u+cX_0}{(c+\lambda)} - \sum_{i=1}^{p}\phi_i X_{t-iL} - \eta}^{\frac{b-(1-\lambda)u+cX_0}{(c+\lambda)} - \sum_{i=1}^{p}\phi_i X_{t-iL} - \eta} ARL\begin{bmatrix} (1-\lambda)u - cX_0 \\ +(c+\lambda)\sum_{i=1}^{p}\phi_i X_{t-iL} \\ +(c+\lambda)k + (c+\lambda)\eta \end{bmatrix} f(k)dk.$$

After that, the integral variable is changed for an easier way such that

$$z = (1-\lambda)u - cX_0 + (c+\lambda)\sum_{i=1}^{p}\phi_i X_{t-iL} + (c+\lambda)k + (c+\lambda),$$

then $ARL(u)$ is rearranged as:

$$ARL(u) = 1 + \frac{1}{c+\lambda}\int_a^b ARL(z)\, f\begin{pmatrix} \frac{z-(1-\lambda)u+cX_0}{(c+\lambda)} \\ -\sum_{i=1}^{p}\phi_i X_{t-iL} - \eta \end{pmatrix} dz, \quad (8)$$

where $f(z) = \exp(-z/\beta)/\beta$. Therefore,

$$ARL(u) = 1 + \frac{e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}} \cdot e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\beta(c+\lambda)} \int_a^b ARL(z) \cdot e^{\frac{-z}{\beta(c+\lambda)}} dz. \quad (9)$$

In the next step, the integral equation in (9) is proved by using Banach's fixed point theorem [23] for checking the persistence and uniqueness of ARL solutions on the $SAR(p)_L$ model of the modified EWMA scheme such that this procedure is presented in Appendix.

2.2.1. Explicit Formula

After checking a unique solution of ARL, Equation (9) can be converted by setting new variables as:

$$IE = \int_a^b ARL(z) \cdot e^{\frac{-z}{\beta(c+\lambda)}} dz$$

and

$$D(u) = e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}} \cdot e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}},$$

then this equation can be rewritten as:

$$ARL(u) = 1 + \frac{D(u)}{\beta(c+\lambda)} \cdot IE. \quad (10)$$

Next step, IE is discussed and $ARL(z)$ is replaced by (10) as follows:

$$IE = \int_a^b ARL(z) \cdot e^{\frac{-z}{\beta(c+\lambda)}} dz$$

$$IE = \int_a^b \left(1 + \frac{D(u)}{\beta(c+\lambda)} \cdot IE\right) \cdot e^{\frac{-z}{\beta(c+\lambda)}} dz$$

$$IE = \int_a^b e^{\frac{-z}{\beta(c+\lambda)}} dz + \int_a^b \frac{D(z)}{\beta(c+\lambda)} \cdot IE \cdot e^{\frac{-z}{\beta(c+\lambda)}} dz$$

$$IE = -\beta(c+\lambda)\left[e^{\frac{-b}{\beta(c+\lambda)}} - e^{\frac{-a}{\beta(c+\lambda)}}\right] - \frac{IE \cdot e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\lambda}\left[e^{\frac{-\lambda b}{\beta(c+\lambda)}} - e^{\frac{-\lambda a}{\beta(c+\lambda)}}\right]$$

$$IE = \frac{-\beta(c+\lambda)\left[e^{\frac{-b}{\beta(c+\lambda)}} - e^{\frac{-a}{\beta(c+\lambda)}}\right]}{1 + \frac{e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\lambda}\left[e^{\frac{-\lambda b}{\beta(c+\lambda)}} - e^{\frac{-\lambda a}{\beta(c+\lambda)}}\right]}. \quad (11)$$

Thus, the IE of (11) is substituted into (10), then $ARL(u)$ can be rearranged as:

$$ARL(u) = 1 + \frac{e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}} \cdot e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\beta(c+\lambda)} \cdot \left(\frac{-\beta(c+\lambda)\left[e^{\frac{-b}{\beta(c+\lambda)}} - e^{\frac{-a}{\beta(c+\lambda)}}\right]}{1 + \frac{e^{\frac{-cX_0}{\beta(c+\lambda)}} \cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\lambda}\left[e^{\frac{-\lambda b}{\beta(c+\lambda)}} - e^{\frac{-\lambda a}{\beta(c+\lambda)}}\right]}\right)$$

$$ARL(u)=1-\frac{\lambda\cdot e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}}\cdot e^{\frac{-cX_0}{\beta(c+\lambda)}}\cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}\left(e^{\frac{-b}{\beta(c+\lambda)}}-e^{\frac{-a}{\beta(c+\lambda)}}\right)}{\lambda+e^{\frac{-cX_0}{\beta(c+\lambda)}}\cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}\left(e^{\frac{-\lambda b}{\beta(c+\lambda)}}-e^{\frac{-\lambda a}{\beta(c+\lambda)}}\right)}$$

$$ARL(u)=1-\frac{\lambda\cdot e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}}\left(e^{\frac{-b}{\beta(c+\lambda)}}-e^{\frac{-a}{\beta(c+\lambda)}}\right)}{\lambda\cdot e^{\frac{cX_0}{\beta(c+\lambda)}}\cdot e^{-\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}+e^{\frac{-\lambda b}{\beta(c+\lambda)}}-e^{\frac{-\lambda a}{\beta(c+\lambda)}}}. \quad (12)$$

Finally, Equation (12) is the explicit formula to solve the ARL on the modified EWMA control chart of the SAR(p)$_L$ model.

2.2.2. NIE Method

Otherwise, the NIE method can be used to calculate ARL solutions to the SAR(p)$_L$ model on the modified EWMA control chart. From Equation (8), the ARL of the NIE method or $ARL_N(u)$ is estimated by using the n linear equation systems with the composite midpoint quadrature rule [18] on the interval [a, b] such that given the distance of n equal separated intervals to be $d_j = (b - a)/n$ and the intermediate value of the j^{th} interval to be $z_j = (j - 0.5)d_j + a$. Finally, the NIE method can be solved to the ARL as follows:

$$ARL_N(u)\approx 1+\frac{1}{c+\lambda}\sum_{j=1}^{n} d_j ARL\left(z_j\right) f\left(\frac{z_j-(1-\lambda)u+cX_0}{(c+\lambda)}-\sum_{i=1}^{p}\phi_i X_{t-iL}-\eta\right). \quad (13)$$

3. Results

For ARL results of an in-control process, ARL_0 is defined when the exponential parameter of the ARL is set to $\beta=\beta_0$. Otherwise, the β_1 is assigned as: $\beta_1=(1+\delta)\beta_0$, where $\beta_1>\beta_0$ and δ is the size of mean shift for an out-of-control process such that this ARL situation is called to be ARL_1.

ARL solutions of the NIE method and the explicit formula are compared on the modified EWMA chart for observations of the SAR(p)$_L$ model with the absolute relative change (ARC) [24] computed as:

$$\text{ARC}(\%) = \left|\frac{ARL(u)-ARL_N(u)}{ARL(u)}\right|\times 100\%. \quad (14)$$

Moreover, each control chart can be compared to perform by measuring the relative mean index (RMI) [25] defined as:

$$\text{RMI}=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{ARL_i(c)-ARL_i(s)}{ARL_i(s)}\right], \quad (15)$$

where $ARL_i(c)$ is the ARL of row i on the tested control chart, $ARL_i(s)$ is the lowest ARL of row i from all the control charts such that a control chart is more effective if the RMI value is lower.

3.1. Experimental Results

For this section, a simulation of the in-control process is typically given ARL_0 = 370 such that the initial parameters are set u = 1, $X_0, X_{t-L},\ldots, X_{t-pL}$ = 1, β_0 =1. For the out-of-control process, β_1 is computed by determining mean shift sizes (δ) to be 0.01, 0.02, 0.03, 0.05, 0.10, 0.15, 0.2, 0.3, 0.5, 1.0, 1.5 and 2.0. Since the lower bound a is studied on the exponential distribution of ε_t which is in the interval $[0,\infty)$., the upper bound b is found by using the least a to be 0.

In Tables 1 and 2, the ARL_0 of the modified EWMA control chart at c = 1 is computed by using two techniques to be the explicit formula and the NIE method with various λ and ϕ for the SAR(1)$_{12}$ model and the SAR(2)$_{12}$ model, respectively. The results of two methods are compared with the ARC for checking the precision of solutions such that all of them are of little value, less than 0.002, correspondingly.

Table 1. Comparison of ARL_0 = 370 from two techniques for the SAR(1)$_{12}$ model on the modified EWMA control chart at c = 1 and λ = 0.05

λ	ϕ	b	Explicit	NIE	ARC (%)
0.05	0.05	2.47647	370.116233	370.115577	0.0001772
	0.10	2.34842	370.111274	370.110694	0.0001567
	0.20	2.112831	370.020466	370.020012	0.0001227
0.10	0.05	2.63585	370.167256	370.165554	0.0004598
	0.10	2.49127	370.062673	370.061195	0.0003994
	0.20	2.2279	370.321510	370.320389	0.0003027
0.20	0.05	3.01639	370.175279	370.168555	0.0018164
	0.10	2.82791	370.336107	370.330383	0.0015456
	0.20	2.49307	370.002909	369.998734	0.0011284

Table 2. Comparison of $ARL_0 = 370$ from two techniques for the $SAR(2)_{12}$ model on the modified EWMA control chart at $c = 1$ and $\lambda = 0.05$

λ	ϕ_1	ϕ_2	***b***	**Explicit**	**NIE**	**ARC (%)**
0.05	0.1	0.2	1.90196	370.10454	370.10418	0.0000973
	0.2	0.3	1.54352	370.14372	370.14349	0.0000621
	0.3	0.5	1.13179	370.39688	370.39677	0.0000297
0.10	0.1	0.2	1.99495	370.33503	370.33417	0.0002322
	0.2	0.3	1.60479	370.01020	370.00969	0.0001378
	0.3	0.5	1.16523	370.37571	370.37547	0.0000648
0.20	0.1	0.2	2.20547	370.21995	370.21686	0.0008346
	0.2	0.3	1.74013	370.11578	370.11404	0.0004701
	0.3	0.5	1.237881	370.00551	370.00472	0.0002135

The capability of control charts can be compared by calculating the ARL in Table 3 and Fig. 1A for the $SAR(1)_{12}$ model and Table 4 and Fig. 1B for the $SAR(2)_{12}$ model. The results of the modified EWMA control charts for all *c* can be effectively detected to be faster than the EWMA scheme for a small shift size. Correspondingly, the RMI of the modified EWMA control charts is lower than the EWMA chart. Moreover, the modified EWMA chart is more performance when *c* is increased.

Table 3. ARL Comparison among the original and the modified EWMA control charts adjusted various *c* for the $SAR(1)_{12}$ model at $\lambda = 0.05$, $\phi = 0.05$, $ARL_0 = 370$

δ	**EWMA**	**Modified EWMA**			
		***c* = 1**	***c* = 3**	***c* = 10**	***c* = 50**
0.00	370	370	370	370	370
0.01	300.784	108.536	70.895	59.901	56.414
0.02	245.347	63.667	39.624	33.085	31.052
0.03	200.934	45.087	27.696	23.085	21.661
0.05	136.365	28.527	17.515	14.646	13.764
0.10	55.244	14.984	9.472	8.038	7.597
0.15	24.489	10.252	6.708	5.777	5.489
0.2	11.887	7.854	5.310	4.633	4.423
0.3	3.779	5.448	3.899	3.477	3.345
0.5	1.308	3.544	2.761	2.538	2.467
1.0	1.008	2.173	1.897	1.813	1.785
1.5	1.000	1.745	1.605	1.560	1.546
2.0	1.000	1.541	1.457	1.430	1.421
RMI	3.074	0.886	0.383	0.245	0.202

Table 4. ARL Comparison among the original and the modified EWMA control charts adjusted various *c* for the $SAR(2)_{12}$ model at $\lambda = 0.20$, $\phi_1 = 0.3$, $\phi_2 = 0.5$, $ARL_0 = 370$

δ	**EWMA**	**Modified EWMA**			
		***c* = 1**	***c* = 3**	***c* = 10**	***c* = 50**
0.00	370	370	370	370	370
0.01	306.045	101.396	79.656	71.983	69.383
0.02	258.348	58.944	45.053	40.361	38.791
0.03	221.575	41.648	31.611	28.274	27.162
0.05	168.950	26.368	20.031	17.945	17.252
0.10	98.256	13.956	10.811	9.774	9.430
0.15	64.225	9.630	7.627	6.962	6.740
0.2	45.113	7.436	6.013	5.536	5.376
0.3	25.563	5.230	4.382	4.094	3.996
0.5	11.395	3.470	3.064	2.921	2.873
1.0	3.776	2.179	2.059	2.016	2.001
1.5	2.251	1.764	1.718	1.701	1.695
2.0	1.720	1.563	1.545	1.538	1.536
RMI	4.620	0.308	0.096	0.024	0

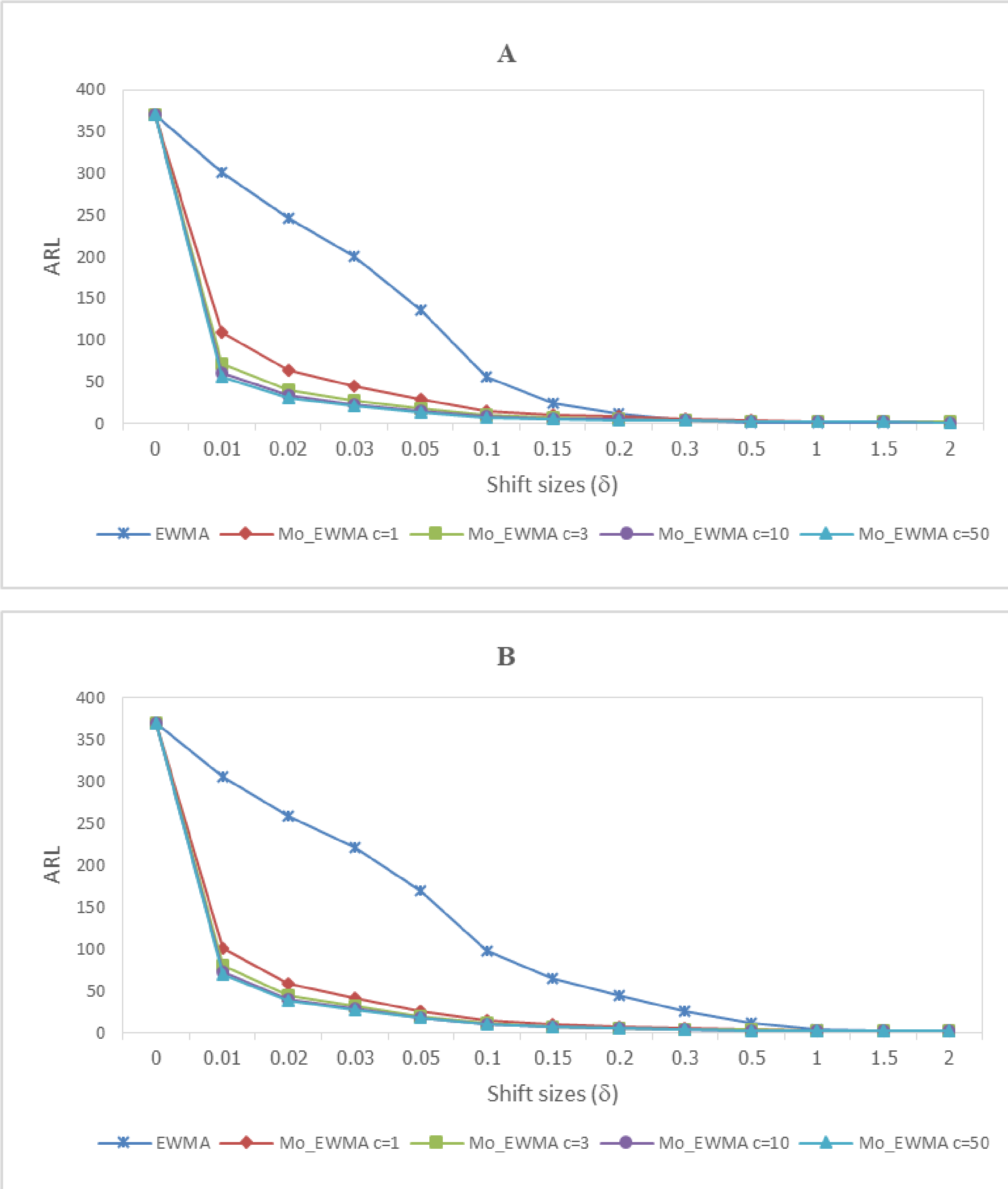

Figure 1. ARL comparison of control charts for the SAR(1)$_{12}$ model in Part A and the SAR(2)$_{12}$ model in Part B.

For Fig. 2A and Fig. 2B, three different λ, 0.05, 0.10 and 0.20, are compared to ARL_1 of the modified EWMA chart with c = 1 such that higher λ can be detected to shift faster than on observations of the SAR(1)$_{12}$ model and the SAR(2)$_{12}$ model, respectively.

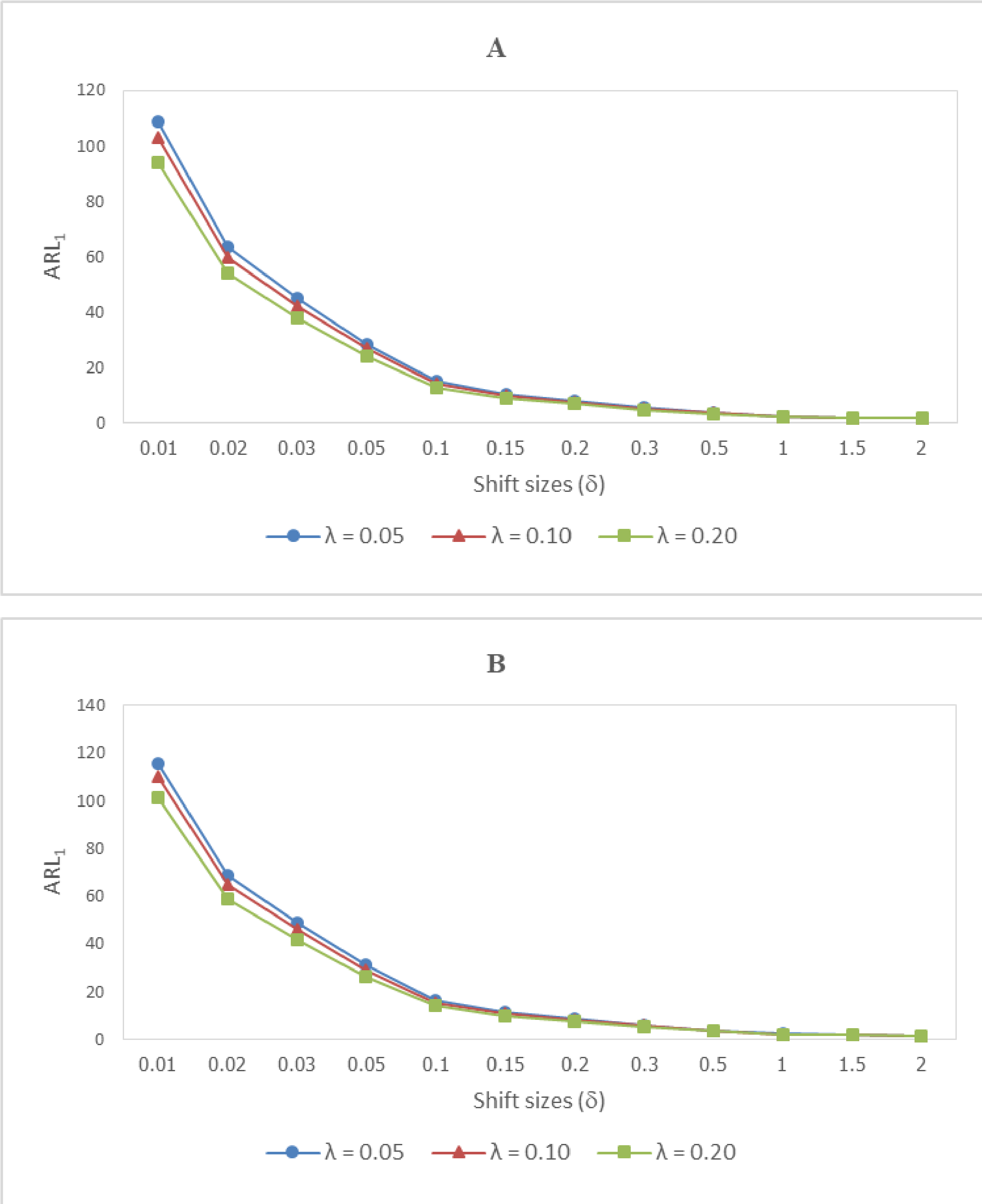

Figure 2. ARL comparison of various λ for the $SAR(1)_{12}$ model in Part A and the $SAR(2)_{12}$ model in Part B

3.2. Application for Real Data

Two practical datasets are used in this case study which are the percentages of internet users by news and business website categories in Thailand [26]. First, the autocorrelation of the observations is assessed by using the Box-Jenkins technique to determine the fitting of forecast time series data models. Next, applying the t-statistic is proved that the datasets are suitable for a $SAR(p)_L$ model. Researchers verified that the white noise is followed by an exponential distribution.

Dataset 1 is the percentage of internet users by news category collected monthly from January 2015 to December 2020. This dataset proves an autocorrelated time series suitable for the $SAR(1)_{12}$ model. This $SAR(1)_{12}$ model can be written as follows:

$$X_t = 22.42 + 0.767 X_{t-L} + \varepsilon_t, \text{ where } \varepsilon_t \sim Exp(4.27).$$

Dataset 2 is the percentage of internet users by business category collected monthly from January 2017 to

December 2020. The $SAR(2)_{12}$ model is suitable for this dataset. This $SAR(2)_{12}$ model can be found as:

$$X_t = 0.619X_{t-L} + 0.358X_{t-2L} + \varepsilon_t, \text{ where } \varepsilon_t \sim Exp(2.19).$$

For Dataset 1, the ARL of control charts is evaluated in Table 5 and Fig. 3A for the $SAR(1)_{12}$ model. Accordingly, Dataset 2 of the $SAR(2)_{12}$ model is presented to the ARL results on control charts in Table 6 and Fig. 3B. The results of both datasets similarly appear to simulated data such that the modified EWMA control charts are found to have higher performance than the EWMA chart when a shift detection is of small size and these charts adjusted *c* enlarge.

Table 5. ARL Comparison of control charts for observations of the percentage of internet users by news category on the $SAR(1)_{12}$ model $\lambda = 0.05$, $ARL_0 = 370$

δ	EWMA	Modified EWMA			
		***c* = 1**	***c* = 3**	***c* = 10**	***c* = 50**
0.00	370	370	370	370	370
0.01	286.443	36.378	21.709	17.934	16.780
0.02	222.705	19.063	11.411	9.485	8.900
0.03	174.030	12.900	7.846	6.582	6.197
0.05	107.867	7.835	4.953	4.231	4.012
0.10	35.537	4.007	2.777	2.465	2.370
0.15	13.290	2.765	2.067	1.887	1.832
0.2	5.758	2.1714	1.724	1.606	1.570
0.3	1.883	1.623	1.400	1.340	1.321
0.5	1.059	1.254	1.174	1.151	1.144
1.0	1.001	1.065	1.049	1.044	1.043
1.5	1.000	1.028	1.022	1.021	1.020
2.0	1.000	1.015	1.013	1.012	1.012
RMI	8.954	0.496	0.133	0.040	0.012

Table 6. ARL Comparison of control charts for observations of the percentage of internet users by business category on the $SAR(2)_{12}$ model $\lambda = 0.20$, $ARL_0 = 370$

δ	EWMA	Modified EWMA			
		***c* = 1**	***c* = 3**	***c* = 10**	***c* = 50**
0.00	370	370	370	370	370
0.01	304.601	97.284	75.617	67.994	65.406
0.02	256.187	56.196	42.534	37.927	36.384
0.03	219.077	39.610	29.791	26.528	25.440
0.05	166.301	25.030	18.859	16.828	16.152
0.10	96.045	13.237	10.187	9.181	8.846
0.15	62.513	9.139	7.199	6.554	6.339
0.2	43.779	7.063	5.685	5.223	5.068
0.3	24.705	4.976	4.158	3.878	3.784
0.5	10.967	3.315	2.923	2.785	2.738
1.0	3.636	2.100	1.984	1.941	1.926
1.5	2.181	1.711	1.665	1.648	1.642
2.0	1.677	1.522	1.504	1.497	1.495
RMI	4.833	0.319	0.099	0.025	0

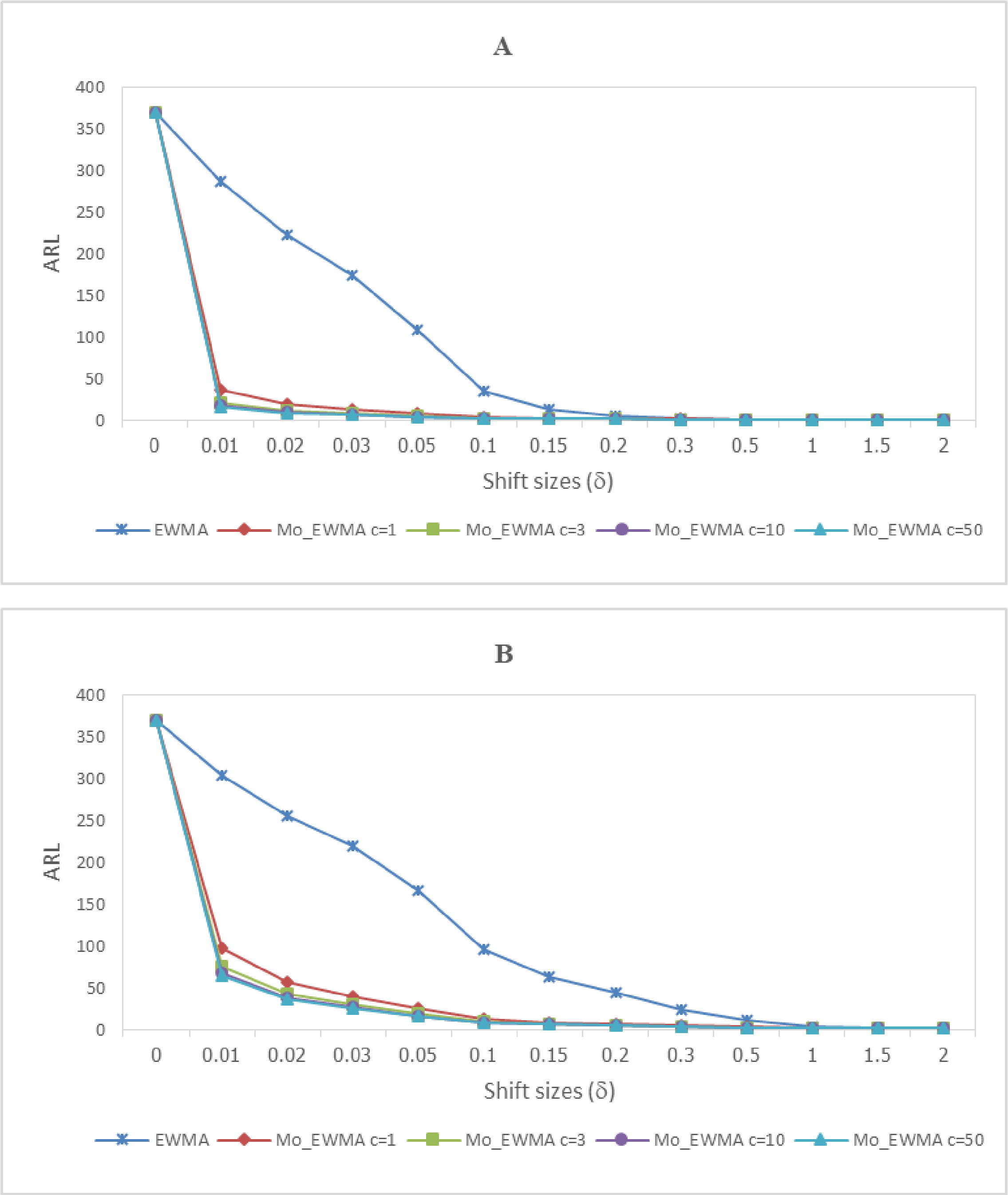

Figure 3. ARL comparison of control charts for real data of the $SAR(1)_{12}$ model in Part A and the $SAR(2)_{12}$ model in Part B

For the modified EWMA control chart with $c = 1$, ARL_1 solutions when λ adapts, 0.05, 0.10 and 0.20, are shown in Fig. 4A and Fig. 4B for observations of Datasets 1 and 2, respectively. They increase λ to make a shift detection quick.

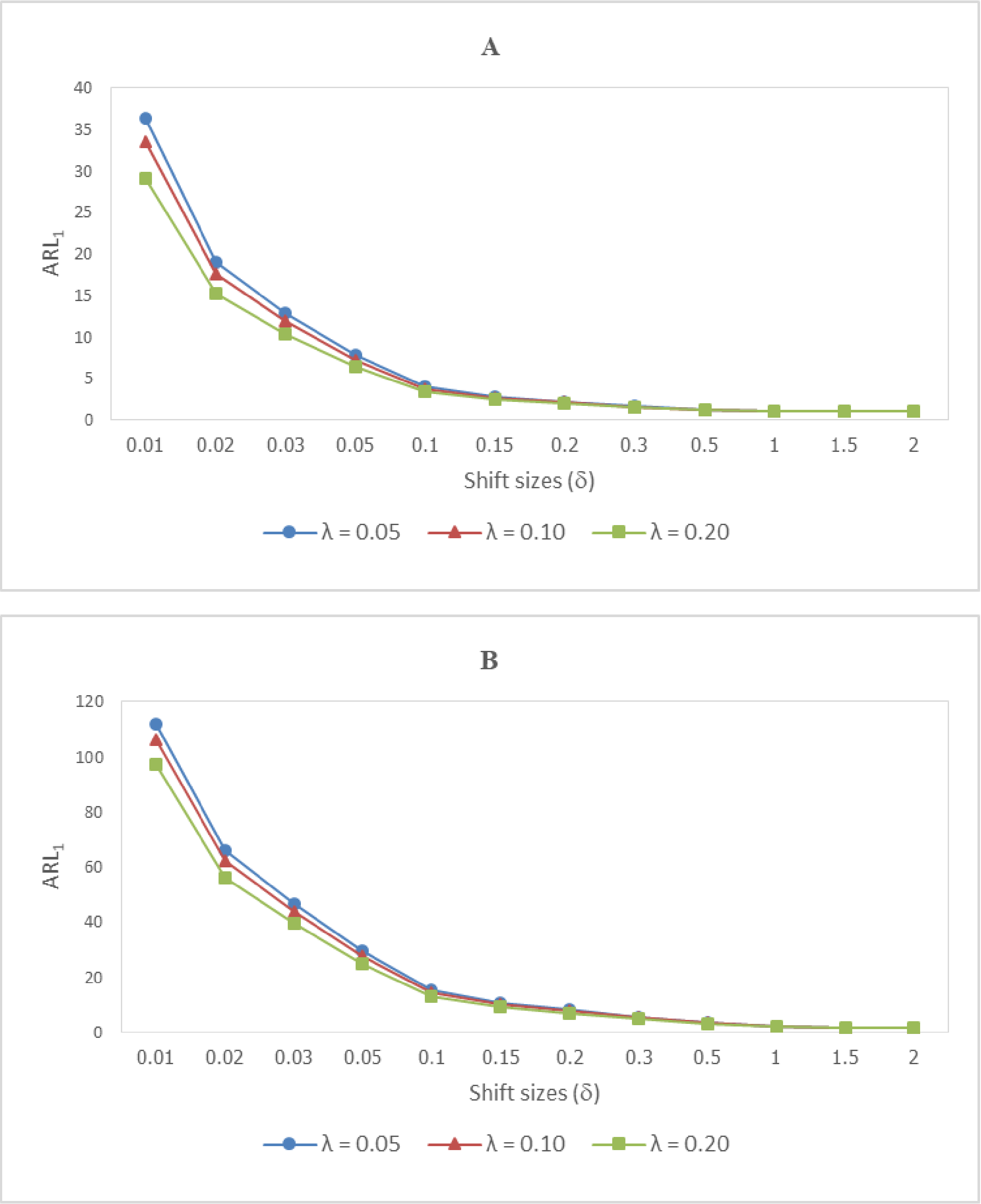

Figure 4. ARL comparison of various λ for real data of the SAR(1)$_{12}$ model in Part A and the SAR(2)$_{12}$ model in Part B

4. Discussion

For the proposed explicit formula, their ARL_0 and ARL_1 results are closed to the NIE method such that the percentage of the absolute relative change (ARC) is obtained to lower than 0.002. Moreover, this explicit formula of ARL is used for comparing the EWMA and the modified EWMA control charts adjusted c such that the modified EWMA control charts get less ARL and RMI value than the EWMA chart for a small shift and a high c, then they summarized that modified EWMA control charts can be detected more quickly. Afterward, the modified EWMA control chart is experimented to vary λ such that the results are presented to better efficacy for higher λ. In addition, this explicit formula can be applied with real data such as the percentages of internet users by news and business website categories in Thailand, then these results are agreeable to simulated situations. For this research, the explicit formula can be used under the conditions of the SAR(p)L model in the case of exponential white noise.

5. Conclusions

In this research, the modified EWMA control chart is presented for observations of the $\mathrm{SAR(p)_L}$ model with exponential residual. The explicit formula of the ARL is derived to measure the efficiency of this control chart and compared to the NIE method for checking the exactness of this explicit formula by using the ARL and ARC solutions such that the results of two methods are not much different. Furthermore, the modified EWMA control charts adjusted *c*are compared to the EWMA by using the ARL and RMI calculations. Finally, this process can be applied for observing real-life situations.

6. Future Research

For future studies, we will develop the explicit formula of the ARL on this control chart for other models and construct explicit formulas for modern control charts. Other techniques will be suggested for calculating the ARL such as the Markov chain approach, Monte Carlo simulation, and Martingale approach.

Appendix

For proving Banach's fixed point theorem, we suppose that $f:[a,b]\rightarrow[a,b]$ is a contraction mapping for all $u\in[a,b]$ on the complete metric space $\left(\mathbb{C}([a,b]),\|\cdot\|_\infty\right)$ such that $\mathbb{C}([a,b])$ is a set of all continuous functions on the compact interval [a, b]. Next step, ARL_1, ARL_2 are given to be a solution to (9) for all

$$ARL_1,\ ARL_2\in\mathbb{C}([a,b])$$

such that $\|f(ARL_1)-f(ARL_2)\|_\infty\le r\|ARL_1-ARL_2\|_\infty$ is proved as follows:

$$\|f(ARL_1)-f(ARL_2)\|_\infty=\sup_{u\in[a,b]}\left|ARL_1(u)-ARL_2(u)\right|$$

$$=\sup_{u\in[a,b]}\left|\frac{e^{\frac{(1-\lambda)u}{\beta(c+\lambda)}}\cdot e^{\frac{-cX_0}{\beta(c+\lambda)}}\cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\beta(c+\lambda)}\cdot\int_a^b\left(ARL_1(z)-ARL_2(z)\right)\cdot e^{\frac{-z}{\beta(c+\lambda)}}dz\right|$$

$$\le\sup_{u\in[a,b]}\left\{\frac{e^{\frac{(1-\lambda)u-cX_0}{\beta(c+\lambda)}}\cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}}{\beta(c+\lambda)}\left|\int_a^b e^{\frac{-z}{\beta(c+\lambda)}}dz\right|\right\}\|ARL_1-ARL_2\|_\infty$$

$$\le\sup_{u\in[a,b]}\left\{e^{\frac{(1-\lambda)u+cX_0}{\beta(c+\lambda)}}\cdot e^{\frac{\sum_{i=1}^{p}\phi_i X_{t-iL}+\eta}{\beta}}\left(e^{\frac{-a}{\beta(c+\lambda)}}-e^{\frac{-b}{\beta(c+\lambda)}}\right)\right\}\cdot\|ARL_1-ARL_2\|_\infty$$

$$\le r\|ARL_1-ARL_2\|_\infty,$$

where a positive constant $r\in[0,1)$. Thus, $f:[a,b]\rightarrow[a,b]$ is the contraction mapping and has a fixed point. Therefore, $ARL(u)$ is summarized to existence and unique solution.

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