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On the Representation of the Weight Enumerator of d_n^+

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Abstract The weight enumerator of a code is a homogeneous polynomial that provides a lot of information about the code. In this case, for the development of a code, research on the weight enumerator is very important. In this study, we focus on the code d_n^+. Let $W_{d_n^+}(x,y)$ be the weight enumerator of the code d_n^+. Fujii and Oura showed that $W_{d_n^+}(x,y)$ is generated by $W_{d_8^+}(x,y)$ and $W_{d_{24}^+}(x,y)$. Indeed, we show that $W_{d_n^+}(x,y)$ is an element of the polynomial ring $\mathbb{Z}[\frac{1}{24}][W_{d_8^+}(x,y), W_{d_{24}^+}(x,y)]$. We know that the weight enumerator of all self-dual double-even (Type II) code is generated by $W_{d_8^+}(x,y)$ and $\varphi_{24}(x,y) = x^4y^4(x^4-y^4)^4$. Recall d_n^+ is a type II code. Thus, $W_{d_n^+}(x,y)$ is an element of the polynomial ring $\mathbb{Z}[W_{d_8^+}(x,y), \varphi_{24}(x,y)]$ and $\mathbb{Z}[\frac{1}{24}][W_{d_8^+}(x,y), W_{d_{24}^+}(x,y)]$. One of the motivations of this research is to investigate the connection between these two polynomial rings in representing $W_{d_n^+}(x,y)$. Let a_i and b_i be the coefficients of polynomial that represent $W_{d_n^+}(x,y)$ as an element of $\mathbb{Z}[W_{d_8^+}(x,y), \varphi_{24}(x,y)]$ and $\mathbb{Z}[\frac{1}{24}][W_{d_8^+}(x,y), W_{d_{24}^+}(x,y)]$, respectively. We find that b_i is an element of the polynomial $\mathbb{Z}\left[\frac{1}{24}\right][a_i]$. In addition, we also show that there are no weight enumerators of Type II code generated by $W_{d_8^+}(x,y)$ and $\varphi_{24}(x,y)$ that can be written uniquely as isobaric polynomials in five homogeneous polynomial elements of degrees 8, 24, 24, 24, 24.

Keywords Weight Enumator, Polynomial Ring, Isobaric Enumator

1. Introduction

Let C be a linear code of length n. The weight enumerator of a linear code C is

$$W_C(x,y) = \sum_{u\in C} x^{n-wt(u)}\, y^{wt(u)}.$$

The weight enumerator plays an important role in the development of coding theory. In this paper we only focused on binary self-dual doubly-even (Type II, for short) codes in genus 1. It is known that the weight enumerator of all Type II codes can be written uniquely as isobaric polynomials in $W_{d_8^+}(x,y) = x^8 + 14x^4y^4 + y^8$ and $\varphi_{24}(x,y) = x^4y^4(x^4-y^4)^4$ with integer coefficients [1]-[4]. In this case, it is clear that all weight enumerators of type II codes are elements of the ring $\mathbb{Z}\left[W_{d_8^+}(x,y), \varphi_{24}(x,y)\right]$. It should be noted that $W_{d_8^+}(x,y)$ is the weight enumerator of the code d_8^+ and $\varphi_{24}(x,y)$ is not the weight enumerator of a code [5]-[10].

We recall the definition of the code d_n^+. This code is characterized as:

$$d_n^+ = \{(\alpha_1+\beta,\alpha_1,\alpha_2+\beta,\alpha_2,\ldots,\alpha_{n/2}+\beta,\alpha_{n/2}) : \alpha_1,\ldots,\alpha_{n/2},\beta \in F_2, \alpha_1+\cdots+\alpha_{n/2}=0\}.$$

We know that the code d_n^+ is a type II code [11]-[15]. Thus, it is clear that the weight enumerator of the code d_n^+, $W_{d_n^+}(x,y)$, is an element of the polynomial ring $\mathbb{Z}\left[W_{d_8^+}(x,y),\varphi_{24}(x,y)\right]$. In particular, we know that $W_{d_{24}^+}(x,y)\in\mathbb{Z}\left[W_{d_8^+}(x,y),\varphi_{24}(x,y)\right]$. We notice the coefficient of x^{24} is zero at $\varphi_{24}(x,y)$, so it can be written as follows

$$\varphi_{24}(x,y)=\frac{W_{d_{24}^+}(x,y)-\left(W_{d_8^+}(x,y)\right)^3}{b}$$

for nonzero b. As a result, we claim that all weight enumerators of Type II codes can also be generated by $W_{d_8^+}$ and $W_{d_{24}^+}$. Since d_n^+ is an element of type II codes, all weight enumerators of the code d_n^+ in genus 1 are elements of the polynomial ring $C\left[W_{d_8^+},W_{d_{24}^+}\right]$. Indeed, we show that $C=\mathbb{Z}[\frac{1}{24}]$. Here we have a claim that the weight enumerator of the code d_n^+ can be represented as an element of the polynomial ring $\mathbb{Z}\left[W_{d_8^+},\varphi_{24}\right]$ and $\mathbb{Z}[\frac{1}{24}]\left[W_{d_8^+},W_{d_{24}^+}\right]$. In theorem 2.1, we will show how the two rings are related in expressing $W_{d_n^+}(x,y)$ [16]-[19].

Previously, in genus 2 showed that there exist no weight enumerators of Type II code could be written uniquely as an isobaric polynomial in homogeneous polynomials of degrees 8, 24, 24, 32, 40. Theorem 2.1 in this paper shows that, in genus 1, there is also no weight enumerator of Type II code that can be written uniquely as isobaric polynomials in five homogeneous polynomial elements of degrees 8, 24, 24, 24, 24.

2. Preliminaries

Before proving Lemma 3.1, we first recall a definition of ring of polynomials. Ring is an algebraic structure equipped by two binary operations. A polynomial ring $\boldsymbol{R}[\boldsymbol{x}]$ is a set of polynomials with coefficients in the ring $\boldsymbol{R}$.

Definition 2.1 Let $\boldsymbol{R}$ be a ring. A polynomial $\boldsymbol{p}(\boldsymbol{x})$ with coefficients in $\boldsymbol{R}$ is an infinite formal sum [20]

$$\sum_{i=0}^{\infty}\boldsymbol{a_i x^i}=\boldsymbol{a_0}+\boldsymbol{a_1 x}+\cdots+\boldsymbol{a_n x^n}+\cdots$$

where $\boldsymbol{a_i}\in\boldsymbol{R}$ and $\boldsymbol{a_i}=\boldsymbol{0}$ for all but a finite number of values of $\boldsymbol{i}$. If for some $\boldsymbol{i}\geq\boldsymbol{0}$ it is true that $\boldsymbol{a_i}\neq\boldsymbol{0}$, the largest such value of $\boldsymbol{i}$ is the degree of $\boldsymbol{p}(\boldsymbol{x})$. If all $\boldsymbol{a_i}=\boldsymbol{0}$, then the degree of $\boldsymbol{p}(\boldsymbol{x})$ is undefined.

Theorem 2.1. The set $\boldsymbol{R}[\boldsymbol{x}]$ of all polynomials in an indeterminate $\boldsymbol{x}$ with coefficients in a ring $\boldsymbol{R}$ is a ring under polynomial addition and multiplication. If $\boldsymbol{R}$ is commutative, then so is $\boldsymbol{R}[\boldsymbol{x}]$, and if $\boldsymbol{R}$ has unity 1, then 1 also unity for $\boldsymbol{R}[\boldsymbol{x}]$ [20].

3. Result and Discussion

Lemma 3.1 $W_{d_n^+}(x,y)$ is an element of the polynomial ring $\mathbb{Z}[\frac{1}{24}]\left[W_{d_8^+}(x,y),W_{d_{24}^+}(x,y)\right]$. In particular, we have

$$W_{d_n^+}(x,y)=b_0\left(W_{d_8^+}\right)^{\frac{n}{8}}+\sum_{i=1}^{k}b_i\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}\left(W_{d_{24}^+}\right)^{i}$$

Where

$$b_0=\left(1+\sum_{i=1}^{k}(-1)^i\frac{a_i}{24^i}\right),$$

$$b_i=\left(\sum_{j=i}^{k}(-1)^{j-i}\binom{j}{i}\frac{a_j}{24^j}\right)\text{ for } i=1,2,\ldots,k$$

And $a_i\in\mathbb{Z}$ is the coefficient of $\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}(\varphi_{24})^i$ in the representation of $W_{d_n^+}(x,y)$ as an element of ring $\mathbb{Z}\left[W_{d_8^+},\varphi_{24}\right]$.

Proof. We know that $W_{d_8^+}$ and φ_{24} generate all weight enumerators of Type II codes. It is clear that $W_{d_n^+}(x,y)\in\mathbb{Z}\left[W_{d_8^+},\varphi_{24}\right]$, since d_n^+ is a type II code. Consequently, we have

$$W_{d_n^+}(x,y)=\left(W_{d_8^+}\right)^{\frac{n}{8}}+\textstyle\sum_{j=1}^{k}a_j\left(W_{d_8^+}\right)^{\frac{n}{8}-3j}(\varphi_{24})^j \tag{1}$$

Where $a_i\in\mathbb{Z}$ and $k=\left\lfloor\frac{n}{24}\right\rfloor$. It is easy to check that

$$W_{d_{24}^+}=W_{d_8^+}{}^3+24\varphi_{24}$$

Which implies

$$W_{d_n^+}(x,y)=\left(W_{d_8^+}\right)^{\frac{n}{8}}+\textstyle\sum_{j=1}^{k}\frac{a_j}{24^j}\left(W_{d_8^+}\right)^{\frac{n}{8}-3j}\left(W_{d_{24}^+}-W_{d_8^+}{}^3\right)^j \tag{2}$$

Furthermore, for $j\geq i\geq 0$, we will investigate the coefficient of $\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}\left(W_{d_{24}^+}\right)^i$ on the right-hand side of (2). We consider that the coefficient of $\left(W_{d_8^+}\right)^{3(j-i)}\left(W_{d_{24}^+}\right)^i$ on expansion $\left(W_{d_{24}^+}-W_{d_8^+}{}^3\right)^j$ is $(-1)^{j-i}\binom{j}{i}$. As a result, we get the coefficient of $\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}W_{d_{24}^+}{}^i$ on the right-hand side of (2) is $\sum_{j=i}^{k}(-1)^{j-i}\binom{j}{i}\frac{a_j}{24^j}$. Finally, we obtain

$$W_{d_n^+}(x,y) = \left(W_{d_8^+}\right)^{\frac{n}{8}} + \sum_{i=0}^{k}\left(\sum_{j=i}^{k}(-1)^{j-i}\binom{j}{i}\frac{a_j}{24^j}\right)\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}\left(W_{d_{24}^+}\right)^{i}$$

Or

$$W_{d_n^+}(x,y) = \left(1+\sum_{i=1}^{k}(-1)^{i}\frac{a_i}{24^i}\right)\left(W_{d_8^+}\right)^{\frac{n}{8}} + \sum_{i=1}^{k}\left(\sum_{j=i}^{k}(-1)^{j-i}\binom{j}{i}\frac{a_j}{24^j}\right)\left(W_{d_8^+}\right)^{\frac{n}{8}-3i}\left(W_{d_{24}^+}\right)^{i}$$

It is obvious that

$$b_0 = 1+\sum_{j=1}^{k}(-1)^{j}\frac{a_j}{24^j} \text{ dan } b_i = \sum_{j=i}^{k}(-1)^{j-i}\binom{j}{i}\frac{a_j}{24^j} \text{ for } i = 1,2,\dots,k$$

Are elements of $\mathbb{Z}[\frac{1}{24}]$ since $a_i \in \mathbb{Z}$. This completes the proof.

Generally, based on Theorem 2.1, in terms of finding the value of b_i for the same n, we need to construct as many as k + 1 equations. Meanwhile, to find the value of a_i, we only need as many as k equations. Thus, to find the value of b_i will be more efficient when using the value of a_i, since b_i depends on a_i.

Example 3.1 To find coefficients of polynomial ring $\mathbb{Z}[\frac{1}{24}]\left[W_{d_8^+}, W_{d_{24}^+}\right]$ as a representation of $W_{d_{56}^+}$, we need to construct and solve a system of equations with three variables based on (3)

$$W_{d_{56}^+} = b_0\left(W_{d_8^+}\right)^7 + b_1\left(W_{d_8^+}\right)^4 W_{d_{24}^+} + b_2 W_{d_8^+}\left(W_{d_{24}^+}\right)^2 \quad (3)$$

$$b_0 = 1-\frac{a_1}{24}+\frac{a_2}{24^2};\ b_1 = \frac{a_1}{24}-\frac{2a_2}{24^2};\ b_2 = \frac{a_2}{24^2}$$

It is clear that we can only use a_1 and a_2 to find b_0, b_1, dan b_2. We can simply construct and solve a system of equations with only two variables based on (4)

$$W_{d_{56}^+} = \left(W_{d_8^+}\right)^7 + a_1\left(W_{d_8^+}\right)^4\varphi_{24} + a_2 W_{d_8^+}(\varphi_{24})^2 \quad (4)$$

And we obtain $a_1 = 280$ and $a_2 = 1792$. As a result, we get

$$b_0 = -\frac{68}{9};\ b_1 = \frac{49}{9};\text{dan } b_2 = \frac{28}{9}$$

Corollary 3.1 Let a_i and b_i as defined in theorem 2.1. For all i, we have $b_i \in \mathbb{Z}$ if and only if $24^i | a_i$.

Proof. If $24^i | a_i$, then it is clear that $b_i \in \mathbb{Z}$ for all i. Conversely, Let $b_i \in \mathbb{Z}$ for all i. We consider the sequence

$$x_m = \left(\frac{a_{k+1-n}}{24^{k+1-n}} : n = 1,2,\dots m\right)$$

We need to show that $x_m \in \mathbb{Z}^m$ for all $m \in \mathbb{N}$. We next use induction on m. Based on theorem 1, for $i = k$, we have $x_1 = \frac{a_k}{24^k} = b_k \in \mathbb{Z}$. So, the statement is true for $m = 1$. Next, suppose it is true for $m = p$. Then, we have

$$x_p = \left(\frac{a_k}{24^k}, \frac{a_{k-1}}{24^{k-1}}, \frac{a_{k-2}}{24^{k-2}}, \dots, \frac{a_{k+1-p}}{24^{k+1-p}}\right) \in \mathbb{Z}^p$$

Moreover, based on Theorem 2.1, we know that

$$b_{k-p} = \sum_{j=k-p}^{k}(-1)^{j-i}\binom{j}{k-p}\frac{a_j}{24^j} = \sum_{j=k+1-p}^{k}(-1)^{j-k+p}\binom{j}{k-p}\frac{a_j}{24^j} + (-1)^{j-k+p}\frac{a_{k-p}}{24^{k-p}}$$

It is clear that

$$\sum_{j=k+1-p}^{k}(-1)^{j-k+p}\binom{j}{k-p}\frac{a_j}{24^j} \in \mathbb{Z}$$

Since $x_p \in \mathbb{Z}^p$. Because of $b_{k-p} \in \mathbb{Z}$, we also have $\frac{a_{k-p}}{24^{k-p}} \in \mathbb{Z}$. Finally, for case $m = p+1$, we get

$$x_{p+1} = \left(\frac{a_k}{24^k}, \frac{a_{k-1}}{24^{k-1}}, \frac{a_{k-2}}{24^{k-2}}, \dots, \frac{a_{k+1-p}}{24^{k+1-p}}, \frac{a_{k-p}}{24^{k-p}}\right) \in \mathbb{Z}^{p+1}$$

Hence, we have a claim that

$$x_m = \left(\frac{a_{k+1-n}}{24^{k+1-n}} : n = 1,2,\dots m\right) \in \mathbb{Z}^m$$

for all $m \in \mathbb{N}$. If we choose $m = k$, we get

$$x_k = \left(\frac{a_k}{24^k}, \frac{a_{k-1}}{24^{k-1}}, \dots, \frac{a_2}{24^2}, \frac{a_1}{24^1}\right) \in \mathbb{Z}^k$$

which implies $24^i | a_i$ for $i = 1,2,\dots,k$. This completes the proof.

Lemma 3.2 There exist no weight enumerators of Type II code generated by $W_{d_8^+}$ and φ_{24} that can be written uniquely as isobaric polynomial in five homogeneous polynomial elements of degrees 8, 24, 24, 24, 24.

Proof. Suppose there exists such a weight enumerator of a Type II code (call it code C^*). Then we have a representation of weight enumerator of the code C^* as a polynomial in $W_{d_8^+}$ and φ_{24} and it can be written as

$$W_{C^*}(x,y) = \sum_{i=0}^{k} a_i W_{d_8^+}{}^{\frac{n}{8}-3i}\varphi_{24}{}^{i}, \quad (5)$$

Where $k = \left\lfloor \frac{n}{24} \right\rfloor$. Since $W_{C^*}(x,y)$ is generated by $W_{d_8^+}$ and φ_{24}, it is obvious that there exists at least one nonzero coefficient a_i for an $i \geq 1$. We consider φ_{24} can be written as

$$\varphi_{24} = X_8^3 - X_{24} - 41Y_{24} - 595Z_{24} - 2822W_{24} = X_8^3 - X_{24} + f(Y_{24}, Z_{24}, W_{24}) \quad (6)$$

Where

$$X_8 = W_{d_8^+};\ X_{24} = x^{24} + y^{24};\ Y_{24} = x^4y^4(x^{16} + y^{16});\ Z_{24} = x^8y^8(x^8 + y^8);\ W_{24} = x^{12}y^{12} \quad (7)$$

So, we can write $W_{C^*}(x,y)$ as an isobaric polynomial in five homogeneous polynomial elements of degrees 8, 24, 24, 24, 24. On the other hand, we can also write

$$\varphi_{24} = \frac{-X_8^3 + X_{24} + 66Y_{24} + 495Z_{24} + 2972W_{24}}{24} = \frac{-X_8^3 + X_{24}}{24} + g(Y_{24}, Z_{24}, W_{24}) \quad (8)$$

Substitute (6) and (8) to (5) so we get two representations of $W_{C^*}(x,y)$ in the form of homogeneous polynomials $X_8, X_{24}, Y_{24}, Z_{24}$, and W_{24}, call them $W_{C^*}(x,y;f)$ and $W_{C^*}(x,y;g)$. Furthermore, we consider the coefficient of ${X_8}^{\frac{n}{8}-3i}{X_{24}}^{i}$ on $W_{C^*}(x,y;f)$ and $W_{C^*}(x,y;g)$. Because of the uniqueness of $W_{C^*}(x,y)$, then it must be fulfilled by $W_{C^*}(x,y;f) = W_{C^*}(x,y;g)$, so we get

$$\sum_{j=i}^{k}(-1)^i \binom{j}{i} a_j = \sum_{j=i}^{k}\binom{j}{i}\frac{a_j}{24^j} \quad (9)$$

for all $i = 1, 2, \ldots, k$. In the following step, we consider the sequence

$$y_m = (a_{k-n+1}: n = 1,2, \ldots, m)$$

Using mathematical induction on m, we will show that

$$y_m = (0, 0, \ldots, 0) \in \mathbb{Z}^m$$

for all $m \in \mathbb{N}$.See (9), when $i = k$ we have $(-1)^k a_k = \frac{a_k}{24^k}$. This condition occurs if and only if $a_k = 0$, and we get $y_1 = 0 \in \mathbb{Z}$. So, it is true for $m = 1$. Suppose it is also true for $m = p$, then we get

$$y_p = (a_k, a_{k-1}, \ldots, a_{k-p+1}) = (0, 0, \ldots, 0) \in \mathbb{Z}^p$$

When $i = k - p$, base on (8), we have

$$\sum_{j=k-p}^{k}(-1)^{k-p}\binom{j}{k-p}a_j = \sum_{j=k-p}^{k}\binom{j}{k-p}\frac{a_j}{24^j}$$

Because of $a_k = a_{k-1} = \cdots = a_{k-p+1} = 0$, then we get $(-1)^{k-p}a_{k-p} = \frac{a_{k-p}}{24^{k-p}}$, where this condition also occurs if and only if $a_{k-p} = 0$, i.e

$$y_{p+1} = (a_k, a_{k-1}, \ldots, a_{k-p+1}, a_{k-p}) = (0, 0, \ldots, 0) \in \mathbb{Z}^{p+1}$$

In this case we have shown that

$$y_m = (a_{k-n+1}: n = 1,2, \ldots, m) = (0, 0, \ldots, 0) \in \mathbb{Z}^m$$

for all $m \in \mathbb{N}$. Next, if we choose $m = k$, we get

$$y_k = (a_k, a_{k-1}, \ldots, a_2, a_1) = (0, 0, \ldots, 0) \in \mathbb{Z}^k.$$

Which also means $a_1 = a_2 = a_3 = \cdots = a_{k-1} = a_k = 0$. Now, we have a claim that the weight enumerator of code C is not generated by φ_{24}. So, we get a contradiction. This completes the proof of the Theorem.

Example 3.2 If we define X_8, X_{24}, Y_{24}, Z_{24}, and W_{24} based on (7), then we get

$$W_{d_{40}^+}(x,y) = 121{X_8}^5 - 120{X_8}^2X_{24} - 4920{X_8}^2Y_{24} - 71400{X_8}^2Z_{24} - 338640{X_8}^2W_{24}$$

and

$$W_{d_{40}^+}(x,y) = -4{X_8}^5 + 5{X_8}^2X_{24} + 330{X_8}^2Y_{24} + 2475{X_8}^2Z_{24} + 14860{X_8}^2W_{24}$$

Here we have claim that the weight enumerator of the code d_{40}^+ has at least two different representations as isobaric polynomials in five elements of polynomials of degree 8, 24, 24, 24, and 24.

4. Conclusions

In this paper, we have shown that the weight enumerator of the code d_n^+ is an element of ring $\mathbb{Z}[\frac{1}{24}][W_{d_8^+}, W_{d_{24}^+}]$. In addition, the connections between polynomial ring $\mathbb{Z}\left[W_{d_8^+}, \varphi_{24}\right]$ and $\mathbb{Z}[\frac{1}{24}][W_{d_8^+}, W_{d_{24}^+}]$ in representing $W_{d_n^+}$ were also presented in this paper.

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