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Category of Submodules of a Uniserial Module

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Abstract Let R be a ring, K, M be R-modules, L a uniserial R-module, and X a submodule of L. The triple (K, L, M) is said to be X-sub-exact at L if the sequence $K \rightarrow X \rightarrow M$ is exact. Let $\sigma(K, L, M)$ is a set of all submodules Y of L such that (K, L, M) is Y-sub-exact. The sub-exact sequence is a generalization of an exact sequence. We collect all triple (K, L, M) such that (K, L, M) is an X-sub exact sequence, where X is a maximal element of $\sigma(K, L, M)$. In a uniserial module, all submodules can be compared under inclusion. So, we can find the maximal element of $\sigma(K, L, M)$. In this paper, we prove that the set $\sigma(K, L, M)$ form a category, and we denoted it by $\mathcal{C}_L$. Furthermore, we prove that $\mathcal{C}_Y$ is a full subcategory of $\mathcal{C}_L$, for every submodule Y of L. Next, we show that if L is a uniserial module, then $\mathcal{C}_L$ is a pre-additive category. Every morphism in $\mathcal{C}_L$ has kernel under some conditions. Since a module factor of L is not a submodule of L, every morphism in a category $\mathcal{C}_L$ does not have a cokernel. So, $\mathcal{C}_L$ is not an abelian category. Moreover, we investigate a monic X-sub-exact and an epic X-sub-exact sequence. We prove that the triple (K, L, M) is a monic X-sub-exact if and only if the triple $\mathbb{Z}$-modules $(Hom_R(N, K), Hom_R(N, L), Hom_R(N, M))$ is a monic $Hom_R(N, X)$-sub-exact sequence, for all R-modules N. Furthermore, the triple (K, L, M) is an epic X-sub-exact if and only if the triple $\mathbb{Z}$-modules $(Hom_R(M, N), Hom_R(L, N), Hom_R(K, N))$ is a monic $Hom_R(X, N)$-sub-exact, for all R-module N.

Keywords Sub-exact Sequences, Pre-additive Category, Uniserial Module

1 Introduction

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules, i.e. $Im\ f = Ker\ g (= g^{-1}(0))$. Davvaz and Parnian-Garamaleky [5] generalize this concept to be a quasi-exact sequence. A sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ is quasi-exact in B or U-exact in B if there exists a submodule U in C such that $Im\ f = g^{-1}(U)$.

Davvaz and ShabaniSolt give new basic properties of the U-homological algebra [4]. In 2002, Anvariyeh and Davvaz introduced U-split sequences and provided several connections between U-split sequences, and projective modules [3]. Then, Anvariyeh dan Davvaz give a generalization of Schanuel Lemma and proved further results about quasi-exact sequences [2].

In 2016, Fitriani et al. [7] introduce an X-sub-exact sequence as a generalization of an exact sequence. Let K, L, M be R-modules and X a submodule of L. The triple (K, L, M) is said to be X-sub-exact at L if $K \rightarrow X \rightarrow M$ is exact. The exact sequence is a special case of X-sub-exact sequence [7]. As an application of a sub-exact sequence, Fitriani et al. introduce an X-sub-linearly independent module [8]. Then, by using the concept of coexact sequence, Fitriani et al. establish a $\mathcal{U}_V$-generated module [10]. This concept is a generalization of the $\mathcal{U}$-generated module [13]. Furthermore, they introduce $\mathcal{U}$-basis and $\mathcal{U}$-free modules [9]. Besides that, the sub-exact sequences can be applied in determining the Noetherian property of the submodule of the generalized power series module [6].

Let $\sigma(K, L, M)$ is a set of all submodules Y of L such that (K, L, M) is Y-sub-exact. In general, if Y_1 and Y_2 are in $\sigma(K, L, M)$, we can not compare Y_1 and Y_2 by inclusion. However, if L is a uniserial module, then any two submodules are comparable concerning inclusion. So, we can find a maximal element of $\sigma(K, L, M)$.

Let L be a uniserial module. The collection of all triple (K, L, M) such that (K, L, M) is an X-sub exact sequence, where X is a maximal element of $\sigma(K, L, M)$ form a category, and we denoted it by $\mathcal{C}_L$. In this paper, we will prove that $\mathcal{C}_Y$ is a full subcategory of $\mathcal{C}_L$, for every submodule Y of L. Furthermore, we will show that $\mathcal{C}_L$ is a pre-additive category, and every morphism in $\mathcal{C}_L$ has kernel under some conditions. We investigate about a monic X-sub-exact and an epic X-sub-exact sequences.

Let K, L, M be R-modules and $\sigma(K, L, M) = \{X \leq L | (K, L, M)\ X\text{-sub-exact at } L\}$. Since $0 \in \sigma(K, L, M)$, $\sigma(K, L, M) \neq \emptyset$. The set $\sigma(K, L, M)$ is not closed under submodules. If a submodule N of L is a direct summand of any element of $\sigma(K, L, M)$, N is contained in $\sigma(K, L, M)$.

Let K, L, M be R-modules and X_1, X_2 submodules of L, where $X_2 \subset X_1$. If $X_1 \in \sigma(K, L, M)$ and X_2 is a direct summand of X_1, then $X_2 \in \sigma(K, L, M)$ [7]. Therefore, if L is semisimple and $L \in \sigma(K, L, M)$, then any submodule of L is contained in $\sigma(K, L, M)$. Moreover, $\sigma(K, L, M)$ is not closed under extensions.

If there are R-module homomorphisms f and g such that the sequence

$$K \xrightarrow{f} L \xrightarrow{g} M$$

is exact, then $\sigma(K, L, M)$ has a maximal element. If not, the set $\sigma(K, L, M)$ has a maximal element if L is Noetherian. Furthermore, $\sigma(K, L, M)$ may has more than one maximal element. But, any two elements of $\sigma(K, L, M)$ are not necesarrily unique up to isomorphism [7].

We recall definition of an additive category and an uniserial module as follow: A category $\mathcal{A}$ is called an additive category if the following conditions hold:

(A1) For every pair of objects X, Y the set of morphisms $Hom_{\mathcal{A}}(X, Y)$ is an abelian group and the composition of morphisms

$$Hom_{\mathcal{A}}(Y, Z) \times Hom_{\mathcal{A}}(X, Y) \to Hom_{\mathcal{A}}(X, Z)$$

is bilinear over the integers.

(A2) $\mathcal{A}$ contains a zero object 0(i.e. for every object $X \in \mathcal{A}$ each morphism set $Hom_{\mathcal{A}}(X, 0)$ and $Hom_{\mathcal{A}}(0, X)$ has precisely one element).

(A3) For every pair of objects X, Y in $\mathcal{A}$ there exists a coproduct $X \oplus Y$ in $\mathcal{A}$.

A category satisfying $(A1)$ and $(A2)$ is called a preadditive category [11].

A module M over any ring R is uniserial if $M \neq 0$ and the submodules of M form a chain (that is, any two of them are comparable under inclusion) [12].

2 Main Result

Let K, L, M be R-modules, where L be a uniserial module. We collect all triples (K, L, M) such that (K, L, M) is X-sub-exact, for some submodule X of L. We define:

$$\sigma(K, L, M) = \{X \leq L | K \to X \to M \text{ exact}\}$$

Let $X_1, X_2 \in \sigma(K, L, M)$. Since L is a uniserial module, we have $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. So, we have a maximal element in $\sigma(K, L, M)$.

We will show that all triples (K, L, M) such that (K, L, M) is X-sub-exact at L, where X is a maximal element of $\sigma(K, L, M)$, form a category, we denote it by $\mathcal{C}_L$. A maximal element of $\sigma(K, L, M)$ will represent (K, L, M) to be an object in category $\mathcal{C}_L$.

Category of $\mathcal{C}_L$ is given by:

1. Objects: Class of all triples (K, L, M) such that (K, L, M) is X-sub-exact, where X is a maximal element of $\sigma(K, L, M)$.

2. Morphisms:
Let $(K_1, L, M_1), (K_2, L, M_2) \in Obj(\mathcal{C}_L)$. Then, there exist submodules X_1, X_2 of L and R-homomorphisms f_1, g_1, f_2, g_2 such that the sequences:

$$K_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} M_1$$

and

$$K_2 \xrightarrow{f_2} X_2 \xrightarrow{g_2} M_2$$

are exact, where X_1 and X_2 are maximal element of $\sigma(K_1, L, M_1)$ and $\sigma(K_2, L, M_2)$, respectively.
A morphism $\theta = (\alpha, \beta, \gamma)$ from (K_1, L, M_1) to (K_2, L, M_2), where $\alpha : K_1 \to K_2$, $\beta : X_1 \to X_2$ and $\gamma : M_1 \to M_2$ are R-module homomorphisms such that the following diagram with exact rows:

$$\begin{array}{ccccc} K_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{g_1} & M_1 \\ \downarrow{\alpha_1} & & \downarrow{\beta_1} & & \downarrow{\gamma_1} \\ K_2 & \xrightarrow{f_2} & X_2 & \xrightarrow{g_2} & M_2 \end{array}$$

is commutative.

3. Composition of morphisms:
Let $\bar{K}_1 = (K_1, L, M_1)$, $\bar{K}_2 = (K_2, L, M_2)$, $\bar{K}_3 = (K_3, L, M_3) \in Obj(\mathcal{C}_L)$,
$\theta_1 = (\alpha_1, \beta_1, \gamma_1) \in Mor_{\mathcal{C}_L}(\bar{K}_1, \bar{K}_2)$, and
$\theta_2 = (\alpha_2, \beta_2, \gamma_2) \in Mor_{\mathcal{C}_L}(\bar{K}_2, \bar{K}_3)$.
Hence, we have the following commutative diagrams:

$$\begin{array}{ccccc} K_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{g_1} & M_1 \\ \downarrow{\alpha_1} & & \downarrow{\beta_1} & & \downarrow{\gamma_1} \\ K_2 & \xrightarrow{f_2} & X_2 & \xrightarrow{g_2} & M_2 \end{array}$$

and

$$\begin{array}{ccccc} K_2 & \xrightarrow{f_2} & X_2 & \xrightarrow{g_2} & M_2 \\ \downarrow{\alpha_2} & & \downarrow{\beta_2} & & \downarrow{\gamma_2} \\ K_3 & \xrightarrow{f_3} & X_3 & \xrightarrow{g_3} & M_3 \end{array}$$

Then $\theta_3 = (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1, \gamma_2 \circ \gamma_1)$ is a morphism from $\bar{K}_2$ to $\bar{K}_3$. We can see this in the following commutative diagram with exact rows:

$$\begin{array}{ccccc} K_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{g_1} & M_1 \\ \downarrow{\scriptstyle \alpha_2\circ\alpha_1} & & \downarrow{\scriptstyle \beta2\circ\beta_1} & & \downarrow{\scriptstyle \gamma_2\circ\gamma_1} \\ K_3 & \xrightarrow{f_3} & X_3 & \xrightarrow{g_3} & M_3 \end{array}$$

Then, we will check whether the morphisms hold associative law.
Let $\bar{K_1} = (K_1, L, M_1), \bar{K_2} = (K_2, L, M_2), \bar{K_3} = (K_3, L, M_3)$ and $\bar{K_4} = (K_4, L, M_4)$ are objects in $\mathcal{C}_L$, $\theta_1 \in Mor_{\mathcal{C}_L}(\bar{K_1}, \bar{K_2})$, $\theta_2 \in Mor_{\mathcal{C}_L}(\bar{K_2}, \bar{K_3})$, $\theta_3 \in Mor_{\mathcal{C}_L}(\bar{K_3}, \bar{K_4})$. Then,

$$\begin{aligned} \theta_3 \circ_c (\theta_2 \circ_c \theta_1) &= (\alpha_3, \beta_3, \gamma_3) \circ_c ((\alpha_2, \beta_2, \gamma_2) \circ_c (\alpha_1, \beta_1, \gamma_1) \\ &= (\alpha_3, \beta_3, \gamma_3) \circ_c (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1, \gamma_2 \circ \gamma_1) \\ &= (\alpha_3 \circ \alpha_2 \circ \alpha_1, \beta_3 \circ \beta_2 \circ \beta_1, \gamma_3 \circ \gamma_2 \circ \gamma_1) \\ &= (\alpha_3 \circ \alpha_2, \beta_3 \circ \beta_2, \gamma_3 \circ \gamma_2) \circ_c (\alpha_1, \beta_1, \gamma_1) \\ &= ((\alpha_3, \beta_3, \gamma_3) \circ_c (\alpha_2, \beta_2, \gamma_2)) \circ_c (\alpha_1, \beta_1, \gamma_1) \\ &= (\theta_3 \circ_c \theta_2) \circ_c \theta_1. \end{aligned}$$

Hence, morphisms of category of $\mathcal{C}_L$ hold associative law, i.e

$$\theta_3 \circ_c (\theta_2 \circ_c \theta_1) = (\theta_3 \circ_c \theta_2) \circ_c \theta_1,$$

for every $\theta_1 \in Mor_{\mathcal{C}_X}(\bar{K_1}, \bar{K_2})$, $\theta_2 \in Mor_{\mathcal{C}_X}(\bar{K_2}, \bar{K_3})$, $\theta_3 \in Mor_{\mathcal{C}_X}(\bar{K_3}, \bar{K_4})$.

For every $\bar{K} = (K, L, M) \in Obj(\mathcal{C}_L)$, there is a morphism $id_{\bar{K}} = (id_K, id_X, id_M)$ in $Mor_{C_L}(\bar{K}, \bar{K})$, the identity of $\bar{K}$, with

$$\theta \circ_c id_{\bar{K}} = id_{\bar{K_1}} \circ_c \theta = \theta,$$

for every $\theta \in Mor_{C_X}(\bar{K}, \bar{K_1})$, $\bar{K_1} = (K_1, L, M_1) \in Obj(\mathcal{C}_L)$.

$$\begin{array}{ccccc} K & \xrightarrow{f} & X & \xrightarrow{g} & M \\ \downarrow{\scriptstyle id_K} & & \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_M} \\ K & \xrightarrow{f} & X & \xrightarrow{g} & M \end{array}$$

So, we can conclude that $\mathcal{C}_L$ is a category.

In the following proposition, we will show that if L is a uniserial R-module, then a category $\mathcal{C}_L$ is pre-additive.

Proposition 1 *Let L be a uniserial module. The category $\mathcal{C}_L$ is a pre-additive category.*

Proof.

1. Let the triples $\bar{K_1} = (K_1, L, M_1)$ and $\bar{K_2} = (K_2, L, M_2)$ are objects in $\mathcal{C}_L$.
   Then, there are submodules X_1 and X_2 of L, where X_1 and X_2 are maximal element of $\sigma(K_1, L, M_1)$ and $\sigma(K_2, L, M_2)$, respectively, such that the sequences

   $$K_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} M_1$$

   and

   $$K_2 \xrightarrow{f_2} X_2 \xrightarrow{g_2} M_2$$

   are exact.
   We define:

   $$(\alpha_1, \beta_1, \gamma_1) +_c (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2),$$

   for all $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in Hom_{\mathcal{C}_X}(\bar{K_1}, \bar{K_2})$.

   It is easy to see that $(Hom_{\mathcal{C}_X}(\bar{K_1}, \bar{K_2}), +_c)$ is an Abelian group and the composition of morphisms

   $$Hom_{\mathcal{C}_L}(\bar{K_2}, \bar{K_3}) \times Hom_{\mathcal{C}_L}(\bar{K_1}, \bar{K_2}) \to Hom_{\mathcal{C}_L}(\bar{K_1}, \bar{K_3})$$

   is bilinear, i.e.

   $$\begin{aligned} &((\alpha_1, \beta_1, \gamma_1) +_c (\alpha_2, \beta_2, \gamma_2)) \circ_c (f, g, h) \\ &= ((\alpha_1, \beta_1, \gamma_1) \circ_c (f, g, h)) +_c \\ &\quad ((\alpha_2, \beta_2, \gamma_2) \circ_c (f, g, h)) \end{aligned}$$

   and

   $$\begin{aligned} &(f', g', h') \circ_c ((\alpha_1, \beta_1, \gamma_1) +_c (\alpha_2, \beta_2, \gamma_2)) \\ &= ((f', g', h') \circ_c (\alpha_1, \beta_1, \gamma_1)) +_c \\ &\quad ((f', g', h') \circ_c (\alpha_2, \beta_2, \gamma_2)) \end{aligned}$$

2. The zero object in $\mathcal{C}_L$ is triple $(0, 0, 0)$.

Hence, the category $\mathcal{C}_L$ is a pre-additive category.

Let L be a uniserial module, and Y be a submodule of L. Then we can construct the category $\mathcal{C}_L$ and $\mathcal{C}_Y$. Since every object in $\mathcal{C}_Y$ is an object in $\mathcal{C}_L$, we have the following proposition.

Proposition 2 *Let L be a uniserial module, and Y be a submodule of L. Then $\mathcal{C}_Y$ is a full subcategory of $\mathcal{C}_L$.*

We recall that the sequence $0 \to M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2$ is exact if and only if the sequence: $0 \to Hom_R(N, M_1) \xrightarrow{\phi_\star} Hom_R(N, M) \xrightarrow{\psi_\star} Hom_R(N, M_2)$ is an exact sequence of $\mathbb{Z}$-modules for all R-modules N. The sequence $M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \to 0$ is exact if and only if the sequence:

$$0 \to Hom_R(M_2, N) \xrightarrow{\psi^\star} Hom_R(M, N) \xrightarrow{\phi^\star} Hom_R(M_1, N)$$

is an exact sequence of $\mathbb{Z}$-modules for all R-modules N [1]. Next, we will investigate whether the Hom-functor preserves the sub-exactness. Now, we define a *monic X-sub-exact* and *epic X-sub-exact* as follow:

Definition 1 *Let K, L, M be R-modules and X be a submodule of L. Then the triple (K, L, M) is said to be a monic X-sub-exact at L if there exist R-homomorphisms f and g such that the sequence:*

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is an exact sequence and f is a monomorphism.
The triple (K, L, M) is said to be an epic X-sub-exact at L if there exist R-homomorphisms f and g such that the sequence of R-modules and R-homomorphisms:

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is an exact sequence and g is an epimorphism.

Next, we will prove that a monic X-sub-exactness of (K, L, M) implies a monic $Hom_R(N, X)$-sub-exactness of $(Hom_R(N, K), Hom_R(N, L), Hom_R(N, M))$, for any R-module N.

Proposition 3 *Let K, L, M be R-modules and X be a submodule of L. The triple (K, L, M) is a monic X-sub-exact, i. e. the sequence*

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is exact at X and f is a monomorphism, if and only if the triple $\mathbb{Z}$-modules :

$$(Hom_R(N, K), Hom_R(N, L), Hom_R(N, M))$$

is a monic $Hom_R(N, X)$-sub-exact, for all R-modules N.

Proof. The triple (K, L, M) is a monic X-sub-exact, i.e the sequence $K \xrightarrow{f} X \xrightarrow{g} M$ is exact at X and f is a monomorphism, for any R-module N, if and only if the sequence of $\mathbb{Z}$-modules:

$$Hom_R(N, K) \xrightarrow{f_\star} Hom_R(N, X) \xrightarrow{g_\star} Hom_R(N, M)$$

is exact at $Hom_R(N, X)$ and $f_\star$ is a monomorphism.

Furthermore, for any $h \in Hom_R(N, X)$, $h \in Hom_R(N, L)$. Hence,
$Hom_R(N, X) \subseteq Hom_R(N, L)$. So, we can conclude that the triple (K, L, M) is a monic X-sub-exact, i. e. the sequence $K \xrightarrow{f} X \xrightarrow{g} M$ is exact at X and f is a monomorphism, if and only if the triple $\mathbb{Z}$-modules :

$$(Hom_R(N, K), Hom_R(N, L), Hom_R(N, M))$$

is a monic $Hom_R(N, X)$-sub-exact, for all R-modules N. □

On the other hand, we will investigate whether the triple:

$$(Hom_R(M, N), (Hom_R(L, N), (Hom_R(K, N))$$

is also a $(Hom_R(X, N)$-sub-exact, for all R-modules N. If $h \in Hom_R(X, N)$, then h is not necessary an element of $Hom_R(L, N)$. For example, the inclusion $i \in Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$, but $i \notin Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

In the following proposition, we provide a necessary condition to a submodule X of L so that the triple $(Hom_R(M, N), (Hom_R(L, N), (Hom_R(K, N))$ is a $(Hom_R(X, N)$-sub-exact, for all R-module N.

Proposition 4 *Let K, L, M be R-modules and X be a direct summand of L. The triple (K, L, M) is an epic X-sub-exact, i. e. the sequence*

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is exact at X and g is an epimorphism, if and only if the triple $\mathbb{Z}$-modules :

$$(Hom_R(M, N), Hom_R(L, N), Hom_R(K, N))$$

is a monic $Hom_R(X, N)$-sub-exact, for all R-module N.

Proof. The triple (K, L, M) is an epic X-sub-exact, i.e the sequence

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is exact at X and g is an epimorphism, if and only if the sequence of $\mathbb{Z}$-modules:

$$Hom_R(M, N) \xrightarrow{g^\star} Hom_R(X, N) \xrightarrow{f_\star} Hom_R(M, N)$$

is a monic $Hom_R(X, N)$-sub-exact.

Since X is a direct summand of L, there is a submodule Y of L such that $L \simeq X \oplus Y$. Let $h \in Hom_R(X, N)$. We can define a homomorphism

$$h^{'} : L \to N,$$

where:

$$h^{'}(a) = \begin{cases} h(a) & ; \text{if } a \in X, \\ 0 & ; \text{otherwise.} \end{cases}$$

We will show that $h^{'}$ is an R-homomorphism from L to N. Let $a, b \in L$ and $r \in R$. We have $a = x_1 + y_1$ and $b = x_2 + y_2$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Therefore, we get:

$$\begin{aligned} f^{'}(a + b) &= f^{'}((x_1 + y_1) + (x_2 + y_2)) \\ &= f^{'}((x_1 + x_2) + (y_1 + y_2)) \\ &= f(x_1 + x_2) \\ &= f(x_1) + f(x_2) \\ &= f^{'}(x_1 + y_1) + f^{'}(x_2 + y_2) \\ &= f^{'}(a) + f^{'}(b). \end{aligned}$$

and

$$\begin{aligned} f^{'}(ra) &= f^{'}(r(x + y)) \\ &= f^{'}(rx + ry) \\ &= f(rx) = rf(x) \\ &= rf^{'}(a). \end{aligned}$$

We can conclude that $h^{'}$ is an R-homomorphism from L to N.

So, for every $h \in Hom_R(X, N)$, we can define an R-homomorphism $h^{'} \in Hom_R(L, N)$. Therefore, there exists a monomorphism

$$\theta : Hom_R(X, N) \to Hom_R(L, N),$$

where $\theta(h) = h^{'}$. We have $Hom_R(X, N)$ is isomorphic to a submodule of $Hom_R(L, N)$. Consequently, the triple $\mathbb{Z}$-modules:

$$(Hom_R(M, N), Hom_R(L, N), Hom_R(K, N))$$

is a monic $Hom_R(X, N)$-sub-exact, for all R-modules N.
□

Consider now the family of monic X-sub-exact sequences, where X is a submodule of a uniserial module L, as follow:

$$Obj(\mathcal{C}_L^\star) = \{(K, L, M) | (K, L, M) \text{ is a monic } X\text{- sub-exact}\}$$

It is clear that $Obj(\mathcal{C}_L^\star) \subseteq Obj(\mathcal{C}_L)$.

In Proposition 1, we proved that C_L is a pre-additive category. According to [11], an Abelian category is an additive category in which every morphism has kernel and cokernel, and for every morphism $f : X \to Y$, the natural morphism $coim\ f \to im\ f$ is an isomorphism. We will show that every morphism in $\mathcal{C}_L^\star$ has a kernel.

Proposition 5 *Let L be a uniserial module. Then every morphism in $\mathcal{C}_L^\star$ has a kernel.*

Proof. Let $(\alpha, \beta, \gamma) \in Hom((K_1, L, M_1), (K_2, L, M_2))$. Then, there are submodules X_1, X_2 of L, where X_1 and X_2 are maximal element of $\sigma(K_1, L, M_1)$ and $\sigma(K_2, L, M_2)$, respectively, such that the following sequences:

$$K_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} M_1$$

and

$$K_2 \xrightarrow{f_2} X_2 \xrightarrow{g_2} M_2$$

are exact, where f_1, f_2 are monomorphisms. We have the following diagram:

$$\begin{array}{ccccc}
Ker\ \alpha & & Ker\ \beta & & Ker\ \gamma \\
\downarrow & & \downarrow & & \downarrow \\
K_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{g_1} & M_1 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
K_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{g_2} & M_2
\end{array}$$

Since f_1, f_2 are monomorphisms, then by Snake Lemma, the first row, i.e. $Ker\ \alpha \to Ker\ \beta \to Ker\ \gamma$ is exact. So, $(Ker\ \alpha, L, Ker\ \gamma)$ is in $\mathcal{C}_L^\star$ and it is kernel of (α, β, γ). □

Since a module factor of L is not a submodule of L, every morphism in a category $\mathcal{C}_L$ does not have a cokernel. So, $\mathcal{C}_L$ is not an abelian category.

3 Conclusions

For any uniserial R-module L, we can construct a category $\mathcal{C}_L$. The object of a category $\mathcal{C}_L$ is triple (K, L, M) such that (K, L, M) is an X-sub-exact sequence, where X is the maximal element of the set of all submodules Y of L such that (K, L, M) is a Y-sub-exact. We proved that $\mathcal{C}_L$ is a pre-additive category, a category $\mathcal{C}_Y$ is a full subcategory of $\mathcal{C}_L$, for any submodule Y of L. Every morphism in $\mathcal{C}_L^\star$ has a kernel.

Furthermore, we proved that a monic X-sub-exactness of (K, L, M) implies a monic sub-exactness of $(Hom_R(N, K), Hom_R(N, L), Hom_R(N, M))$. If X is a direct summand of L, then an epic X-sub-exactness of (K, L, M) implies a monic sub-exactness of $(Hom_R(M, N), Hom_R(L, N), Hom_R(K, N))$, for any R-module N.

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