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The Relative Rank of Transformation Semigroups with Restricted Range on a Finite Chain

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Abstract Let S be a semigroup and let G be a subset of S. A set G is a generating set G of S which is denoted by $\langle G \rangle = S$. The rank of S is the minimal size or the minimal cardinality of a generating set of S, i.e. $rankS := \min\{|G| : G \subseteq S, \langle G \rangle = S\}$. In last twenty years, the rank of semigroups is worldwide studied by many researchers. Then it lead to a new definition o f r ank that is called the relative rank of S modulo U is the minimal size of a subset $G \subseteq S$ such that $G \cup U$ generates S, i.e. $rank(S : U) := \min\{|G| : G \subseteq S, \langle G \cup U \rangle = S\}$. A set $G \subseteq S$ with $\langle G \cup U \rangle = S$ is called generating set of S modulo U. The idea of the relative rank was generalized from the concept of the rank of a semigroup and it was firstly i ntroduced b y H owie, R uškuc a nd H iggins i n 1998. Let X be a finite c hain a nd l et Y b e a s ubchain o f X . We denote $\mathcal{T}(X)$ the semigroup of full transformations on X under the composition of functions. Let $\mathcal{T}(X,Y)$ be the set of all transformations from X to Y which is so-called the transformation semigroup with restricted range Y. It was firstly i ntroduced a nd s tudied b y S ymons i n 1 975. Many results in $\mathcal{T}(X)$ were extended to results in $\mathcal{T}(X,Y)$. In this paper, we focus on the relative rank of semigroup $\mathcal{T}(X,Y)$ and the semigroup $\mathcal{OP}(X,Y)$ of all orientation-preserving transformations in $\mathcal{T}(X,Y)$. In Section 2.1, we determine the relative rank of $\mathcal{T}(X,Y)$ modulo the semigroup $\mathcal{OD}(X,Y)$ of all order-preserving or order-reversing transformations. In Section 2.2, we describe the results of the relative rank of $\mathcal{T}(X,Y)$ modulo the semigroup $\mathcal{OP}(X,Y)$. In Section 2.3, we determine the relative rank of $\mathcal{T}(X,Y)$ modulo the semigroup $\mathcal{OPR}(X,Y)$ of all orientation-preserving or orientation-reversing transformations. Moreover, we obtain that the relative rank $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ and modulo $\mathcal{OPR}(X,Y)$ are equal.

Keywords Generating Set, Transformations, Relative Rank, Orientation-preserving, Orientation-reversing

1 Introduction and Preliminaries

Let X be a finite chain, i.e. $X = \{1 < \cdots < n\}$ where $n \in \mathbb{N}$. Denote $\mathcal{T}(X)$ by a semigroup of all full transformations on X under the composition of functions. In this paper, we will write functions from the right and compose from the left to the right, i.e. $x(\alpha\beta) = (x\alpha)\beta$. Let $\alpha \in \mathcal{T}(X)$. Define $im\alpha$, $rank\alpha$ and $\ker\alpha$ by $im\alpha := \{x\alpha : x \in X\}$, $rank\alpha := |im\alpha|$ and $\ker\alpha := \{(x,y) \in X \times X : x\alpha = y\alpha\}$, respectively. Then $\ker\alpha$ is an equivalence relation on X is called $\ker\alpha$-classes. Let $T \subseteq X$ with $|C \cap T| = 1$ for all $\ker\alpha$ -classes C. Then T is a transversal of $\ker\alpha$. For sets $A_1, A_2 \subseteq X$, we write $A_1 < A_2$ if $x_1 < x_2$ for all $x_1 \in A_1$ and for all $x_2 \in A_2$. Let $A \subseteq X$. Define $\alpha|_A$ by a mapping $\alpha|_A : A \to X$ with $x(\alpha|_A) := x\alpha$ for all $x \in A$, i.e. $\alpha|_A$ is the mapping α restricted to A.

A set G is a generating set G of a semigroup S which is denoted by $\langle G \rangle = S$. Define the rank of S by a minimal size of a generating set of S, i.e. $rankS := \min\{|G| : G \subseteq S, \langle G \rangle = S\}$. The relative rank of S modulo U is the minimal size of a subset $G \subseteq S$ such that $G \cup U$ generates S, i.e. $rank(S : U) := \min\{|G| : G \subseteq S, \langle G \cup U \rangle = S\}$. A set $G \subseteq S$ with $\langle G \cup U \rangle = S$ is called generating set of S modulo U. The concept of a relative rank generalizes the concept rank of a semigroup and was introduced by Howie, Ruškuc and Higgins [13].

In this paper, we also consider an orientation-preserving transformations with a linear order on X. Let $\alpha \in \mathcal{T}(X)$. Then is called an orientation-preserving (orientation-reversing, respectively) if there is a decomposition $X = A_1 \cup A_2$ with $A_1 < A_2$, $y_1\alpha \geq y_2\alpha$ ($y_1\alpha \leq y_2\alpha$, respectively) for all $y_1 \in A_1$ and $y_2 \in A_2$, and $x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$, respectively) for all $x \leq y \in A_1$ or $x \leq y \in A_2$. If $A_2 = \emptyset$ then α is an order-preserving transformation. Moreover, if $A_1 = \emptyset$ with $x\alpha \geq y\alpha$ for all $x \leq y \in A_2$ then α is an order-reversing transformation. The notation of an orientation-preserving transformation was first studied by McAlister [14] and generalized by Catarino and Higgins [2]. The definition of an orientation-preserving transformation which is given by them is equivalent to that one given in this paper. Notice that the product of two orientation-preserving transformations is an orientation-preserving. Let $\mathcal{O}(X)$ be the semigroup of all order-preserving transformations on X, let $\mathcal{OD}(X)$ be the semigroup of all order-preserving or order-reversing transformations on X, let $\mathcal{OP}(X)$ be the semigroup of all orientation-preserving transformations on X, let $\mathcal{OR}(X)$ be the set of all orientation-reversing transformations on X and let $\mathcal{OPR}(X) := \mathcal{OP}(X) \cup \mathcal{OR}(X)$ be the semigroup of all orientation-preserving or orientation-reversing transformations on X, respectively. Clearly, $\mathcal{O}(X) \subseteq \mathcal{OP}(X)$ and $\mathcal{O}(X) \subseteq \mathcal{OPR}(X)$. It is a proper subsemigroup of $\mathcal{OP}(X)$ and $\mathcal{OPR}(X)$ where $n \geq 2$. The semigroup $\mathcal{OP}(X)$ has been widely investigated (see [1], [2], [4], [6], [9], [16]). In particular, the rank of $\mathcal{OP}(X)$ is 2 [1], the rank of $\mathcal{O}(X)$ is n [6] and the rank of $\mathcal{T}(X)$ is 3 [13]. On the other hand, we have $rank(\mathcal{T}(X) : \mathcal{O}(X)) = 2$ [13], $rank(\mathcal{T}(X) : \mathcal{OP}(X)) = 1$ [13] and $rank(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$ [2].

Let Y be a subchain of X, i.e. $Y = \{l_1 < \cdots < l_m\}$ where $m \in \{1, \ldots, n\}$. Define the following sets by

$$\begin{aligned}
\mathcal{T}(X,Y) &:= \{\alpha \in \mathcal{T}(X) : \mathrm{im}\alpha \subseteq Y\}, \\
\mathcal{O}(X,Y) &:= \{\alpha \in \mathcal{O}(X) : \mathrm{im}\alpha \subseteq Y\}, \\
\mathcal{OD}(X,Y) &:= \{\alpha \in \mathcal{OD}(X) : \mathrm{im}\alpha \subseteq Y\}, \\
\mathcal{OP}(X,Y) &:= \{\alpha \in \mathcal{OP}(X) : \mathrm{im}\alpha \subseteq Y\}, \\
\mathcal{OPR}(X,Y) &:= \{\alpha \in \mathcal{OPR}(X) : \mathrm{im}\alpha \subseteq Y\}.
\end{aligned}$$

All of them form subsemigroups of $\mathcal{T}(X)$. A semigroup $\mathcal{T}(X,Y)$ is called the full transformation semigroup with restricted range and it is defined by Symons [15]. The other semigroups were introduced by Fernandes et al in [8] and [9], respectively. Transformation semigroups with restricted range have been widely investigated (see [7], [8], [10], [14]). The stirling number of second kind $S(n,m)$ is the rank of $\mathcal{T}(X,Y)$ [12]. In [9], it was shown that $rank(\mathcal{OP}(X,Y)) = \binom{n}{m}$. In [8], the authors proved that $rank(\mathcal{O}(X,Y)) = \binom{n-1}{m-1} + |R^\sharp|$, where $R^\sharp$ is the set of captive elements. In [16], Tinpun and Koppitz have already shown $rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y)$ is equal to $S(n,m) - \binom{n-1}{m-1}$ or $S(n,m) - \binom{n-1}{m-1} + 1$ which depends on the given set Y.

The purpose of this paper is to determined the relative rank of $\mathcal{T}(X,Y)$ modulo various subsemigroups. Firstly, the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OD}(X,Y)$ will be determined. Moreover, we also study and describe the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$. Finally, we determine the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OPR}(X,Y)$ which is equal to modulo $\mathcal{OP}(X,Y)$.

2 Main Results

2.1 Rank($\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)$)

In this section, we compute the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OD}(X,Y)$. Define the set $\mathcal{K}$ by

$$\mathcal{K} := \{\ker\phi : \phi \in \mathcal{T}(X,Y), rank\phi = m\} \setminus \{\ker\phi : \phi \in \mathcal{O}(X,Y), rank\phi = m\}.$$

Notice that $\{\ker\phi : \phi \in \mathcal{O}(X,Y)\} = \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$. Then we have the cardinality of $\mathcal{K}$ as the following lemma.

Lemma 2.1. *[16]* $|\mathcal{K}| = S(n,m) - \binom{n-1}{m-1}$.

For each $K \in \mathcal{K}$, we choose $\alpha_K \in \mathcal{T}(X,Y)$ with $im\alpha_K = Y$ and $\ker\alpha_K = K$. Hence, $|\{\alpha_K : K \in \mathcal{K}\}| = S(n,m) - \binom{n-1}{m-1}$. Then for each $s \in \mathcal{S}(Y)$ there exists $\mu_s \in \mathcal{T}(X,Y) \setminus \mathcal{O}(X,Y)$ with $\mu_s|_Y = s$ and s is not an identity mapping on Y.

Lemma 2.2. *[16] If $\mathcal{S} \subseteq \mathcal{S}(Y)$ and $\langle\mathcal{S}\rangle = \mathcal{S}(Y)$ then*

$$\mathcal{T}(X,Y) = \langle\mathcal{O}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle.$$

In [16], the authors defined subsets $P^*(X)$ of a power set $P(X)$ of X as follows:

1. If $|X| \geq 5$ then $P^*(X) := P(X) \setminus (\{\emptyset, X\} \cup \{\{x\} : x \in X\})$,
2. if $|X| = 4$ then $P^*(X) := \{Y \subseteq X : |Y| \geq 2, |X \setminus Y| = 2$ or $\{x_2, x_3\} \subseteq Y\}$ and
3. if $|X| = 3$ then $P^*(X) := \{Y \subseteq X : |Y| = 2, x_2 \in Y\}$.

Then we obtain the results as shown in the following theorems.

Theorem 2.3. *[16] If $Y \in P^*(X)$ then $rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y)) = S(n,m) - \binom{n-1}{m-1}$.*

Theorem 2.4. *[16] If $Y \notin P^*(X)$ then $rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y)) = S(n,m) - \binom{n-1}{m-1} + 1$.*

Lemma 2.5. *Let $B \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OD}(X,Y)$ and $\langle\mathcal{OD}(X,Y), B\rangle = \mathcal{T}(X,Y)$. Then $\mathcal{K} \subseteq \{\ker\phi : \phi \in B\}$.*

Proof. Suppose that $B \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OD}(X,Y)$ with $\langle\mathcal{OD}(X,Y), B\rangle = \mathcal{T}(X,Y)$. Assume that there exists $K \in \mathcal{K}$ with $K \notin \{\ker\phi : \phi \in B\}$. Since $\alpha_K \in \mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), B\rangle$, there are $\theta_1 \in \mathcal{OD}(X,Y) \cup B$ and $\theta_2 \in$

$\mathcal{T}(X,Y)$ such that $\alpha_K = \theta_1\theta_2$. Because $rank\alpha_K = m$, we obtain $\ker\alpha_K = \ker\theta_1$, i.e. $\ker\theta_1 = K$. Hence, $\theta_1 \notin B$ and $\theta_1 \notin \mathcal{OD}(X,Y)$ because $K \notin \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$ which is a contradiction. □

Theorem 2.6. *Let $\mathcal{S} \subseteq \mathcal{S}(Y)$ and $\langle\mathcal{S}\rangle = \mathcal{S}(Y)$. Then*

$$\mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle.$$

Proof. By Lemma 2.2 and $\mathcal{O}(X,Y)$ is a proper subsemigroup of $\mathcal{OD}(X,Y)$, $\mathcal{T}(X,Y) = \langle\mathcal{O}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle \subseteq \langle\mathcal{OD}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle$. It is clear that $\langle\mathcal{OD}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle \subseteq \mathcal{T}(X,Y)$. Altogether, we obtain that $\mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), \{\mu_s : s \in \mathcal{S}\}, \{\alpha_K : K \in \mathcal{K}\}\rangle$. □

From Theorem 2.6, we obtain the following corollaries.

Corollary 2.7. *If $|X| \geq 5$ and $|Y| \geq 3$ then $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = S(n,m) - \binom{n-1}{m-1}$.*

Proof. Since $|Y| \geq 3$, two bijections, say s_1 and s_2, generate the symmetric group $\mathcal{S}(Y)$. Then there are $K_1, K_2 \notin \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$, i.e. $K_1, K_2 \in \mathcal{K}$ and Y is a transversal of K_1, K_2. Without loss of generality, we can assume that $\alpha_{K_1}|_Y = s_1$ and $\alpha_{K_2}|_Y = s_2$, i.e. $\mu_{s_1} = \alpha_{K_1}$ and $\mu_{s_2} = \alpha_{K_2}$. By Theorem 2.6 , we have $\mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}\rangle$. Hence, $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) \leq |\{\alpha_K : K \in \mathcal{K}\}| = S(n,m) - \binom{n-1}{m-1}$. By Lemma 2.5, we obtain that $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) \geq |\mathcal{K}| = S(n,m) - \binom{n-1}{m-1}$. □

Corollary 2.8. *If $|X| \geq 3$ and $|Y| = 2$ then $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = 2^{n-1} - n$.*

Proof. Since $|Y| = 2$, one bijection, say s_1, generates the symmetric group $\mathcal{S}(Y)$. Then there is $K_1 \in \{\ker\beta : \beta \in \mathcal{OD}(X,Y)\}$ such that Y is a transversal of K_1. Without loss of generality, we can assume that $\alpha_{K_1}|_Y = s_1$, i.e. $\mu_{s_1} = \alpha_{K_1}$. By Theorem 2.6, we have $\mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}\rangle$. Then we have $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) \leq |\{\alpha_K : K \in \mathcal{K}\}| = S(n,2) - \binom{n-1}{2-1} = 2^{n-1} - n$. By Lemma 2.5, we obtain $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) \geq |\mathcal{K}| = S(n,2) - \binom{n-1}{2-1} = 2^{n-1} - n$. □

Corollary 2.9. *Let $|X| = 4$ and $|Y| = 3$. If $\{x_2,x_3\} \nsubseteq Y$ and $\{x_2,x_3\} \subseteq Y$ then $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = 4$ and $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = 3$, respectively.*

Proof. Since $|Y| = 3$, two bijections, say s_1 and s_2, generate the symmetric group $\mathcal{S}(Y)$. If $\{x_2,x_3\} \nsubseteq Y$ then there is only one $K_1 \in \mathcal{K}$ such that Y is a transversal of K_1 and $K_1 \notin \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$. Without loss of generality, we can assume that $\alpha_{K_1}|_Y = s_1$, i.e. $\mu_{s_1} = \alpha_{K_1}$. Then $\mu_{s_2} \notin \mathcal{OD}(X,Y) \cup \{\alpha_K : K \in \mathcal{K}\}$. Put $\alpha := \mu_{s_2}$ and $\alpha_{K_1} := \mu_{s_1}$. By Theorem 2.6, we have $\mathcal{T}(X,Y) = \langle\mathcal{OD}(X,Y), \{\alpha_K : K \in \mathcal{K}\}, \alpha\rangle$. So, we obtain that $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) \leq |\{\alpha_K : K \in \mathcal{K}\} \cup \{\alpha\}| = S(4,3) - \binom{4-1}{3-1} + 1 = 4$. Since there exists only one $K_1 \in \mathcal{K}$ such that Y is a transversal of K_1 and $K_1 \notin \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$. Thus, we want one element in B which is not in $\mathcal{OD}(X,Y) \cup \{\alpha_K : K \in \mathcal{K}\}$. Hence, $|B| \geq |\mathcal{K}| + 1 = S(4,3) - \binom{4-1}{3-1} + 1 = 4$.

For the case $\{x_2,x_3\} \subseteq Y$, the proof is similar to Corollary 2.7. Then we have $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = S(4,3) - \binom{4-1}{3-1} = 3$. □

Example 2.10. *Let $X = \{1,2,3\}$ and $Y = \{1,2\}$. Then $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = 1$.*

Solution. So, we have $|X| = 3$, $|Y| = 2$, and $|\mathcal{K}| = 1$. Put $\theta := \begin{pmatrix} 1 & \overline{2,3} \\ 2 & 1 \end{pmatrix}$ and $\alpha := \begin{pmatrix} \overline{1,3} & 2 \\ 1 & 2 \end{pmatrix}$. It is clear that $\theta \in \mathcal{OD}(X,Y)$, $\ker\theta \in \{\ker\phi : \phi \in \mathcal{OD}(X,Y)\}$ with Y is a transversal of $ker\theta$, and $\langle\theta|_Y\rangle = \mathcal{S}(Y)$. By the definition of α, we have $\ker\alpha \in \mathcal{K}$. Since $|\mathcal{K}| = 1$, there is exactly one $K \in \mathcal{K}$. Without loss of generality, we can assume that $\alpha_K = \alpha$. So, it is easy to see that $\alpha_K \in \mathcal{T}(X,Y) \setminus \mathcal{OD}(X,Y)$ and $\langle\mathcal{OD}(X,Y), \alpha_K\rangle = \mathcal{T}(X,Y)$. Hence, $rank(\mathcal{T}(X,Y) : \mathcal{OD}(X,Y)) = 1$ which satisfies a result in Corollary 2.8.

2.2 Rank($\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)$)

In this section, we study and describe the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$. Assume that Y is a proper subset of X. Define the set $\mathcal{P}$ by

$$\mathcal{P} := \{\ker\alpha : \alpha \in \mathcal{OP}(X,Y), rank\alpha = m\} \setminus \{\ker\alpha : \alpha \in \mathcal{O}(X,Y), rank\alpha = m\}.$$

Then $\mathcal{P}$ is the set of all partitions of X into $m-1$ intervals and one block which is the union of two intervals B_1 and B_n such that $1 \in B_1$ and $n \in B_n$. For each $P \in \mathcal{P}$, we fix any $\alpha_P \in \mathcal{OP}(X,Y)$ with $im\alpha_P = Y$ and $\ker\alpha_P = P$. The following lemma gives the cardinality of $\mathcal{P}$.

Lemma 2.11. *[4] $|\mathcal{P}| = |\{\alpha_P : P \in \mathcal{P}\}| = \binom{n-1}{m}$.*

Next, we define a mapping $\alpha^* : X \to Y$ by

$$x\alpha^* := \begin{cases} l_{i+1} & \text{if } l_i \leq x < l_{i+1} \text{ and } 1 \leq i < m \\ l_1 & \text{if } l_m \leq x \text{ or } x < l_1. \end{cases}$$

Then $\alpha^*|_Y$ is a permutation on Y and $(l_1\alpha^*, \ldots, l_m\alpha^*)$ is cyclic, i.e. $\alpha^* \in \mathcal{OP}(X,Y)$. Since $\mathcal{O}(X,Y) \subseteq \mathcal{OP}(X,Y)$, we have $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) \leq rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y))$, i.e. $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) \leq S(n,m) - \binom{n-1}{m-1} + 1$. Define the set $\mathcal{M}$ by

$$\mathcal{M} := \{\ker\alpha : \alpha \in \mathcal{T}(X,Y), rank\alpha = m\} \setminus \{\ker\alpha : \alpha \in \mathcal{OP}(X,Y), rank\alpha = m\}.$$

Then the following lemma gives the cardinality of $\mathcal{M}$.

Lemma 2.12. *[3]* $|\mathcal{M}| = S(n,m) - \binom{n}{m}$.

Notice that for any $\beta \in \mathcal{S}(Y)$, there is $\beta' \in \mathcal{T}(X,Y)$ such that $\beta'|_Y = \beta$, i.e. $\mathcal{S}(Y) \subseteq \{\alpha|_Y : \alpha \in \mathcal{T}(X,Y)\}$. The following lemmas give necessary and sufficient conditions for a set $A \subseteq \mathcal{T}(X,Y)$ to be a generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$

Lemma 2.13. *[3] For any generating set A of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$, there is a set $C \subseteq A$ with $\mathcal{M} \subseteq \{\ker\alpha : \alpha \in C\}$ such that there is a set $B \subseteq C$ with $\langle\{\alpha|_Y : \alpha \in B\}, \eta\rangle = \mathcal{S}(Y)$ where $\eta = \alpha^*|_Y$.*

Lemma 2.14. *[3] Let $A \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\mathcal{M} \subseteq \{\ker\alpha : \alpha \in A\}$ and $\mathcal{S}(Y) = \langle\{\alpha|_Y : \alpha \in B\}, \eta\rangle$ for some $B \subseteq A$. Then A is a generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$.*

Then we obtain the main result of this section as shown in the following theorem.

Theorem 2.15. *[3] $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) = |\mathcal{M}|$.*

Proof. By Lemma 2.13, we get that $\mathrm{rank}(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) \geq |\mathcal{M}|$. For all $M \in \mathcal{M}$, we fix $\alpha_M \in \mathcal{T}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\ker\alpha_M = M$. Without loss of generality, we can assume that there is $\beta \in \{\alpha_M : M \in \mathcal{M}\}$ with $\beta|_Y := \begin{pmatrix} l_1 & l_2 & l_3 & \cdots & l_m \\ l_2 & l_1 & l_3 & \cdots & l_m \end{pmatrix}$. It is well known that $\langle\beta|_Y, \eta\rangle = \mathcal{S}(Y)$ where $\eta = \alpha^*|_Y$. By Lemma 2.14, we have $\langle\mathcal{OP}(X,Y), \{\alpha_M : M \in \mathcal{M}\}\rangle = \mathcal{T}(X,Y)$. Then $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) \leq |\{\alpha_M : M \in \mathcal{M}\}| = |\mathcal{M}|$. Altogether, $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) = |\mathcal{M}|$. □

Example 2.16. *Let $X = \{1,2,3,4\}$ and $Y = \{1,2,3\}$. Then $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) = 2$.*

Solution. So, we have $|X| = 4$, $|Y| = 3$, and $|\mathcal{M}| = 2$. Then we put $\alpha^* := \begin{pmatrix} 1 & 2 & \overline{3,4} \\ 2 & 3 & 1 \end{pmatrix}$, $\beta := \begin{pmatrix} 1 & \overline{2,4} & 3 \\ 2 & 1 & 3 \end{pmatrix}$, and $\gamma := \begin{pmatrix} \overline{1,3} & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$. It is immediately to see that $\alpha^* \in \mathcal{OP}(X,Y)$. By the definition of β and γ, we obtain that $\ker\beta, \ker\gamma \in \mathcal{M}$ and $\ker\beta \neq \ker\gamma$. Since $|\mathcal{M}| = 2$, there are only $M_1, M_2 \in \mathcal{M}$. Without loss of generality, we can assume that $\alpha_{M_1} = \beta$ and $\alpha_{M_2} = \gamma$. Since Y is a transversal of $\ker\alpha^*$ and $\ker\alpha_{M_1}$, we can show that $\langle\alpha^*|_Y, \alpha_{M_1}|_Y\rangle = \mathcal{S}(Y)$. Then it is easy to verify that $\langle\mathcal{OP}(X,Y), \alpha_{M_1}, \alpha_{M_2}\rangle = \mathcal{T}(X,Y)$. So, $rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) = 2$ which satisfies a result in Theorem 2.15.

2.3 Rank($\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)$)

The both semigroups $\mathcal{OPR}(X,Y)$ and $\mathcal{OP}(X,Y)$ coinside when Y has only two elements. So, we will consider Y is a proper subset of X with $|Y| \geq 3$. Recall that $\mathcal{OPR}(X,Y) = \mathcal{OP}(X,Y) \cup \mathcal{OR}(X,Y)$ and notice that $\{\ker\alpha : \alpha \in \mathcal{OP}(X,Y)\} = \{\ker\alpha : \alpha \in \mathcal{OPR}(X,Y)\}$. Define a mapping $\beta^* : X \to Y$ by

$$x\beta^* := \begin{cases} l_m & \text{if } x < l_1 \\ l_{m-i+1} & \text{if } l_i \leq x < l_{i+1} \text{ and } 1 \leq i < m \\ l_1 & \text{if } x \geq l_m \ . \end{cases}$$

We note that β^* is orientation-reversing. Then we obtain that $\mathcal{OP}(X,Y)$ and β^* will be a generating set for $\mathcal{OPR}(X,Y)$ as shown in the following theorem.

Theorem 2.17. *$\mathcal{OPR}(X,Y) = \langle\mathcal{OP}(X,Y), \beta^*\rangle$.*

Proof. Let $\gamma \in \mathcal{OR}(X,Y)$ with $rank\gamma = m$. We put $\theta := \gamma\beta^*$ and observe that $\theta \in \mathcal{OP}(X,Y)$ as a product of two orientation-reversing transformations. Then $\theta\beta^* = \gamma\beta^*\beta^* = \gamma$ since $\beta^*\beta^*|_Y$ is the identity mapping on Y, i.e. $\gamma \in \langle\mathcal{OP}(X,Y), \beta^*\rangle$.

Let $2 \leq p \in \mathbb{N}$ with $p \leq m$ and suppose that $\{\alpha \in \mathcal{OR}(X,Y) : rank\alpha = p\} \subseteq \langle\mathcal{OP}(X,Y), \beta^*\rangle$. Let $\gamma \in \mathcal{OR}(X,Y)$ with $rank\gamma = p-1$. We observe that $\ker\gamma = \{G_1, G_2 < \cdots < G_{p-1}\}$, where $G_1 < G_2$ or $\{1,n\} \subseteq G_1$, where $im\gamma = \{k_1, \ldots, k_{p-1}\}$ with $k_i\gamma^{-1} = G_i$ for $1 \leq i \leq p-1$. Let us put $k_p := k_1$ due to technical reasons. Since γ is orientation-reversing, there is one $r \in \{1, \cdots, p-1\}$ such that $k_r < k_{r+1}$ and $k_i > k_{i+1}$ for all $i \in \{1, \cdots, p-1\} \setminus \{r\}$. Since $p-1 < m$, we have $Y \setminus im\gamma \neq \emptyset$. So, there are $b \in Y \setminus im\gamma$ and $i \in \{1, \ldots, p-1\}$ such that (i) $k_i > b > k_{i+1}$ or (ii) $k_i < b$ and $b > k_{i+1}$ or (iii) $k_i > b$ and $b < k_{i+1}$. In all the three cases, we put

$$m_j := \begin{cases} k_j & \text{if } 1 \leq j \leq i \\ b & \text{if } j = i+1 \\ k_{j-1} & \text{if } i+1 < j \leq p. \end{cases}$$

Let $q \in \{1, \ldots, p\}$ such that m_q is the least element in $B := \{m_1, \ldots, m_p\}$. Next, we define a mapping $\delta : B \to B$ by

$$m_j\delta := \begin{cases} m_{q+1-j} & \text{if } 1 \leq j \leq q \\ m_{p+q+1-j} & \text{if } q+1 \leq j \leq p. \end{cases}$$

Clearly, δ is bijective. Further, we define a mapping $\delta^* : X \to Y$ by

$$x\delta^* := \begin{cases} m_1 & \text{if } x < m_1\delta^{-1} \\ m_j & \text{if } m_j\delta^{-1} \leq x < m_{j+1}\delta^{-1} \text{ and } 1 \leq j \leq p-1 \\ m_p & \text{if } x \geq m_p\delta^{-1}. \end{cases}$$

It is easy to verify that $m_1(\delta^*)^{-1} < \cdots < m_p(\delta^*)^{-1}$, $m_q < m_{q+1}$, and $m_j > m_{j+1}$ for all $j \in \{1, \ldots, p\} \setminus \{q\}$, where $m_{p+1} := m_1$. Thus, δ^* is an orientation-reversing transformation with $\mathrm{rank}\delta^* = p$, i.e. $\delta^* \in \langle\mathcal{OP}(X,Y), \beta^*\rangle$ by the assumption. Moreover, we observe that $\delta = \delta^*|_B$. Let $\theta : X \to B$ be defined by

$$x\theta := \begin{cases} m_j\delta^{-1} & \text{if } x \in G_j \text{ and } 1 \leq j \leq i \\ m_{j+1}\delta^{-1} & \text{if } x \in G_j \text{ and } i < j \leq p-1. \end{cases}$$

Notice that $\ker\theta = \ker\gamma$. Since $m_j\delta^{-1} < m_{j+1}\delta^{-1}$ for all $j \in \{1, \ldots, p-1\}$, it is easy to verify that $\theta \in \mathcal{O}(X,Y) \subseteq \mathcal{OP}(X,Y)$.

Finally, we show that $\gamma = \theta\delta^*$. Let $x \in X$. Then $x \in G_j$ for some $j \in \{1, \ldots, p-1\}$. If $1 \leq j \leq i$ then we get $x\theta\delta^* = x\theta\delta = m_j\delta^{-1}\delta = m_j = k_j = x\gamma$. If $i < j \leq p-1$ then we have $x\theta\delta^* = x\theta\delta = m_{j+1}\delta^{-1}\delta = m_{j+1} = k_j = x\gamma$. Hence, we have $\gamma = \theta\delta^* \in \langle\mathcal{OP}(X,Y), \beta^*\rangle$. □

From Theorem 2.17, we obtain immediately the following corollary.

Corollary 2.18. $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) = 1$.

Proof. By Theorem 2.17, we obtain that $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) \leq 1$. Since $\mathcal{OP}(X,Y)$ is a proper subsemigroup of $\mathcal{OPR}(X,Y)$, we obtain that $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) \geq 1$. Altogether, we can conclude that $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) = 1$. □

Example 2.19. *Let* $X = \{1,2,3,4\}$ *and* $Y = \{2,3,4\}$. *Then* $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) = 1$.

Solution. So, we have $|X| = 4$ and $|Y| = 3$. Then we put $\beta^* := \begin{pmatrix} \overline{1,2} & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix}$. By the definition of β^*, we obtain that β^* is orientation-reversing, i.e. $\beta^* \in \mathcal{OR}(X,Y)$. Hence, it is easy to show that $\langle \mathcal{OP}(X,Y), \beta^* \rangle = \mathcal{OPR}(X,Y)$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{OP}(X,Y)) = 1$ which satisfies a result in Corollary 2.18.

Lemma 2.20. *For any generating set* A *of* $\mathcal{T}(X,Y)$ *modulo* $\mathcal{OPR}(X,Y)$*, there is a set* $A' \subseteq A$ *with* $\mathcal{M} = \{\ker\alpha : \alpha \in A'\}$.

Proof. Suppose that $A \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OPR}(X,Y)$ with $\langle \mathcal{OPR}(X,Y), A \rangle = \mathcal{T}(X,Y)$. Assume that there is $M \in \mathcal{M}$ with $M \notin \{\ker\alpha : \alpha \in A\}$. Let γ be a transformation with $\ker\gamma = M$, i.e. $rank\gamma = m$. Since $\gamma \in \mathcal{T}(X,Y) = \langle \mathcal{OPR}(X,Y), A \rangle$, there are $\theta_1 \in \mathcal{OPR}(X,Y) \cup A$ and $\theta_2 \in \mathcal{T}(X,Y)$ such that $\gamma = \theta_1\theta_2$. Because $rank\gamma = m$, we obtain that $\ker\gamma = \ker\theta_1$, i.e. $\ker\theta_1 = M$. We have $\ker\theta_1 = M \notin \{\ker\alpha : \alpha \in \mathcal{OP}(X,Y)\}$. Moreover, it is clear that $\{\ker\alpha : \alpha \in \mathcal{OPR}(X,Y)\} = \{\ker\alpha : \alpha \in \mathcal{OP}(X,Y)\}$. Thus, $\ker\theta_1 \notin \{\ker\alpha : \alpha \in \mathcal{OPR}(X,Y)\}$, i.e. $\theta_1 \notin \mathcal{OPR}(X,Y)$. Additional, it holds that $\theta_1 \notin A$ since $\mathcal{M} \notin \{\ker\alpha : \alpha \in A\}$, i.e. $\theta_1 \notin \mathcal{OPR}(X,Y) \cup A$, a contradiction. This shows that there is a set $A' \subseteq A$ with $\{\ker\alpha : \alpha \in A'\} = \mathcal{M}$. □

Then we can state the main result as follows:

Theorem 2.21. $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) = rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y))$.

Proof. Since $\mathcal{OP}(X,Y) \subseteq \mathcal{OPR}(X,Y)$, we obtain that $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) \leq rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y)) = |\mathcal{M}|$ by Theorem 2.15. Conversely, let $A \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OPR}(X,Y)$ such that $\langle \mathcal{OPR}(X,Y), A \rangle = \mathcal{T}(X,Y)$. By Lemma 2.20, there is $A' \subseteq A$ with $\mathcal{M} = \{\ker\alpha : \alpha \in A'\}$, i.e. $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) \geq |A'| \geq |\mathcal{M}|$. Therefore, $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) = |\mathcal{M}| = rank(\mathcal{T}(X,Y) : \mathcal{OP}(X,Y))$. □

Example 2.22. *Let* $X = \{1,2,3,4\}$ *and* $Y = \{1,2,3\}$. *Then* $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) = 2$.

Solution. So, we have $|X| = 4$, $|Y| = 3$, and $|\mathcal{M}| = 2$. Then we put $\alpha^* := \begin{pmatrix} 1 & 2 & \overline{3,4} \\ 2 & 3 & 1 \end{pmatrix}$, $\beta := \begin{pmatrix} 1 & \overline{2,4} & 3 \\ 2 & 1 & 3 \end{pmatrix}$, and $\gamma := \begin{pmatrix} \overline{1,3} & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$. It is clear that $\alpha^* \in \mathcal{OP}(X,Y) \subseteq \mathcal{OPR}(X,Y)$. By the definition of β and γ, we obtain that $\ker\beta, \ker\gamma \in \mathcal{M}$ and $\ker\beta \neq \ker\gamma$. Since $|\mathcal{M}| = 2$, there are only $M_1, M_2 \in \mathcal{M}$. Without loss of generality, we can assume that $\alpha_{M_1} = \beta$ and $\alpha_{M_2} = \gamma$. Then we have $\alpha_{M_1}, \alpha_{M_2} \in \mathcal{T}(X,Y) \setminus \mathcal{OPR}(X,Y)$. Since Y is a transversal of $\ker\alpha^*$ and $\ker\alpha_{M_1}$, we can show that $\langle \alpha^*|_Y, \alpha_{M_1}|_Y \rangle = \mathcal{S}(Y)$. Hence, it is easy to calculate that $\langle \mathcal{OPR}(X,Y), \alpha_{M_1}, \alpha_{M_2} \rangle = \mathcal{T}(X,Y)$. Then $rank(\mathcal{T}(X,Y) : \mathcal{OPR}(X,Y)) = 2$ which satisfies a result in Theorem 2.21.

3 Conclusions

In this paper, we study transformation semigroup $\mathcal{T}(X)$ and transformation semigroup with restricted range $\mathcal{T}(X,Y)$. We also determine the relative rank of transformation semigroup $\mathcal{T}(X,Y)$ modulo various subsemigroups in $\mathcal{T}(X,Y)$. In Section 1, we define some notation and introduce some definition about transformation semigroups in order to use through this paper. In section 2.1, we obtain the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OD}(X,Y)$ as shown in Theorem 2.6 and Corollary 2.7-2.9. In section 2.2, we study and describe the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ as shown in Theorem 2.15. In section 2.3, we calculate the relative rank of $\mathcal{OPR}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ as shown in Theorem 2.17 and Corollary 2.18. In addition, we obtain that the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ and modulo $\mathcal{OPR}(X,Y)$ coincide as shown in Theorem 2.21. Finally, we also illustrate examples in each section in order to get a good understandable.

In future work, we can study other kind structure of transformation semigroup with restricted range.

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