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The Class of Noetherian Rings With Finite Valuation Dimension

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Abstract Not a long time ago, Ghorbani and Nazemian [2015] introduced the concept of dimension of valuation which measures how much does the ring differ from the valuation. They've shown that every Artinian ring has a finite valuation dimensions. Further, any comutative ring with a finite valuation dimension is semiperfect. However, there is a semiperfect ring which has an infinite valuation dimension. With those facts, it is of interest to further investigate property of rings that has a finite dimension of valuation. In this article we define conditions that a Noetherian ring requires and suffices to have a finite valuation dimension. In particular we prove that, if and only if it is Artinian or valuation, a Noetherian ring has its finite valuation dimension. In view of the fact that a ring needs a semi perfect dimension in terms of valuation, our investigation is confined on semiperfect Noetherian rings. Furthermore, as a finite product of local rings is a semi perfect ring, the inquiry into our outcome is divided into two cases, the case of the examined ring being local and the case where the investigated ring is a product of at least two local rings. This is, first of all, that every local Noetherian ring possesses a finite valuation dimension, if and only if it is Artinian or valuation. Secondly, any Notherian Ring generated by two or more local rings is shown to have a finite valuation dimension, if and only if it is an Artinian.

Keywords Uniserial Dimension, Dimension of Valuation, Local Rings, Semiperfect Rings

1 Introduction

A valuation ring can be recognized by collecting any ideals fully arranged by the inclusion relationship. The concept of the valuation ring has been widely studied, for examples, in [7], [8], [12], and [13]. The measure that states the dimension of the ring is from the valuation condition is called dimension of valuation, and we can see this notion the first time in [9]. By definition, the dimension of valuation of a ring is the uniserial dimension from its group over itself. We can see the notion of uniserial dimension the first time in [15].

The properties of rings that have a dimension of valuation and groups with uniserial dimension have been studied (see [15]). In another literature, Ghorbani and Nazemian [9] show that any Artinian ring has a finite dimension of valuation. Artinian rings, on the other hand, are not all comutative rings with a finite valuation dimension. Moreover, any rings with a finite valuation dimension is considered semiperfect (see [9]). But the converse is not always true, since there exists a semiperfect Noetherian ring with infinite dimension of valuation.

The foregoing facts make one wonder what conditions commutative rings must meet in order to have a finite valuation dimension. As part of an effort to answer that question, in this note we prove that the class containing Noetherian rings that have finite dimension of valuation consists of all Artinian rings and valuation rings. In Section 2, we start with some review of the concept of the dimension of valuation. Our main result is the characterization of Noetherian rings that has a finite dimension of valuation in Section 3.

2 Review of Finite-Valuation-Dimension Rings

We examine the idea of the dimension of valuation and the properties of rings with a finite dimension of valuation. We assume that any rings be a commutative with a unity in this note, denoted by R. The definition of a uniserial dimension cannot be separated from the presence of the idea of a valuation dimension (see [15]). The concept of the uniserial dimension is related to the concept of ordinal numbers (see Stoll [16]). If $\alpha \geq 1$ be any ordinal number, then the collection of groups ζ_α over R is a group that generated by a transfinite induction, using a collection of groups ζ_β over R, where $\beta < \alpha$ and β, α is any ordinal number.

Definition 2.1. Nazemian *et al.* [15]. Let $\alpha \geq 1$ be any ordinal number and M is a group over R. Then we have:

1. $\zeta_1 = \{M | M \text{ is uniserial}\}$,
2. $\zeta_\alpha = \left\{ M | \left(\forall N < M\right) \left(M/N \ncong M \Longrightarrow M/N \in \bigcup_{\beta<\alpha} \zeta_\beta \right) \right\}$

Based on these definitions, the collection ζ_2 is a class that includes all groups of M over R which not uniserial but for every subgroup N of M where $M/N \ncong M$ then we have M/N is uniserial. Also, the group M over R shown to have a uniserial dimension if $M \in \zeta_\alpha$ for some ordinal number α.

Definition 2.2. Nazemian *et al.* [15]. Let R stand for any ring, M be a group over R, and α be any ordinal number. Then we have:

1. The universal dimension of M is the least ordinal number of α, denote by $u.s.dim\,(M) = \alpha$, if $M \in \zeta_\alpha$.
2. The universal dimension of M is 0 if M is zero.
3. "M has no uniserial dimension" if $M \neq 0$, $M \notin \zeta_\alpha$, and $\alpha \geq 1$ be any ordinal number.

The valuation ring R formed by grid of ideals of R forms a network. In 2015, Ghorbani and Nazemmian [9] proposed the concept of a ring's dimensions of valuation which calculates the dimension of the ring is from valuation conditions.

Definition 2.3. Ghorbani and Nazemian [9]. The dimension of a group R over R is uniserial, denoted by $v.dim\,(R)$, which is called dimension of valuation of R, if it exists.

The studies on rings that have a dimension of valuation have been done (see [15] and [9]). Characteristics of Noetherian and Artinian rings that have a dimension of valuation can be seen in [9].

Rescinding that R is local ring if R has a distinctive maximum ideal. Moreover, a semiperfect ring R with finite calculation of local rings. This means the class of semiperfect rings can be divided into two subclasses, the first subclass is the class of local ring, and the second subclass containing rings that are a product of finite but at least two local rings.

Rescinding also that an Artinian ring is a ring which fulfills the below chain conditions of ideals, and a Noetherian ring is a ring which fulfills the above chain conditions of ideals. Concerning finite dimension of valuation notion, following results were proposed by Ghorbani and Nazemian in [9].

Theorem 2.4. ***Ghorbani and Nazemian [9].*** *Consider R to be a ring, and I to be ideal of R.*

1. *Suppose I be a nonzero with a finite valuation dimension and R/I is not a valuation. Then I as group over R is Artinian.*
2. *Any Artinian ring R has a finite dimension of valuation.*
3. *Any ring R with a finite dimension of valuation is semiperfect. Moreover, R can be viewed as a finite calculation of a maximum of n local rings, such that $v.dim\,(R) = n$.*

In conclusion, the class including all rings with finite dimension of valuation is located between Artinian and semiperfect classes. In this note, when we restrict R being Noether, we obtain all rings that have finite dimension of valuation.

The theorem 2.5 below is very interesting, which says that the valuation dimension of a PID is 1 or ω.

Theorem 2.5. ***Arifin [2].*** *Let R be a PID. Thus, the valuation dimension of R is 1 or ω.*

The integer domain $\mathbb{Z}$ is a PID that has more than one prime element. By referring to the proof of Theorem 2.5 above, the following results are obtained:

Corollary 2.6. ***Arifin [2].*** *For a ring $\mathbb{Z}$, $v.dim\,(\mathbb{Z}) = \omega$ where ω is an ordinal number.*

The following theorem now exists, i.e. a technique to verify the uniserial dimension of a finished primary module across a DVD created with the element k in all order of p as well as the primary module in a uniserial dimension over a DVD which produced using ks element in the order p^{li} in which $i = 1, ..., k$ is created.

Theorem 2.7. ***Arifin [3].*** *Supposse M become a finitely generated primary R-module along a cyclic decomposition*

$$M = \langle x_1 \rangle \oplus \cdots \oplus \langle x_k \rangle \tag{2.1}$$

where the order of x_i $o(x_i) = p^{e_i}, i = 1, \ldots, k$ for some descending finite sequence of integers

$$e = e_1 \geq \cdots \geq e_k \geq 1$$

and k=2,3,.... Thus, the uniserial dimension of M is $e_1 + ... + e_k$.

From Theorema 3.1, we now can get Example 2.8 below, which can give a better understandings of the finit valuation dimensions of ring $\mathbb{Z}_n$. For any integer n such that $n = p^k q^l$, $k, l \in \mathbb{Z}^+$, $p \neq q$, and p, q are primas, the followwing examples is nice to see.

Example 2.8. Arifin [5]. Consider the following examples with their respective lattices. Let $\mathbb{Z}_n$ be a ring of integer modulo n:

1. If $n = p^n$, p is prime, $n \in \mathbb{Z}^+$, thus $v.dim\,(\mathbb{Z}_n) = 1$.

$$\begin{array}{c} M = \mathbb{Z}_n \\ \uparrow \\ \langle p \rangle \\ \uparrow \\ \langle p^2 \rangle \\ \uparrow \\ \cdots \\ \uparrow \\ \langle 0 \rangle = \langle p^n \rangle \end{array}$$

2. If $n = pq$ where p, q are primes and $p \neq q$ thus $v.dim\,(\mathbb{Z}_n) = 2$.

$$\begin{array}{ccccc} & & M = \mathbb{Z}_{pq} & & \\ & \nearrow & & \nwarrow & \\ \langle p \rangle & & \uparrow & & \langle q \rangle \\ & \nwarrow & & \nearrow & \\ & & \langle 0 \rangle = \langle pq \rangle & & \end{array}$$

3. If $n = pqr$ where p, q, r are primes, and $p \neq q \neq r$, thus $v.dim\,(\mathbb{Z}_n) = 3$.

$$\begin{array}{ccccc} & & \mathbb{Z}_{pqr} & & \\ & \nearrow & \uparrow & \nwarrow & \\ \langle p \rangle & & \langle q \rangle & & \langle r \rangle \\ \uparrow & \times & & \times & \\ \langle pq \rangle & & \langle pr \rangle & & \langle rq \rangle \\ & \nwarrow & \uparrow & \nearrow & \\ & & \langle 0 \rangle & & \end{array}$$

A Python program can be used to determine the uniserial dimensions of the $(\mathbb{Z}_n)_{\mathbb{Z}}$ [5] module. By using the program, the prime factorization of integers can be written using Python 2.7.14 which can be used to determine the uniserial dimensions of the $(\mathbb{Z}_n)_{\mathbb{Z}}$ module. Readers can directly copy-paste the program code below inside the Python compiler/editor, then we can press the F5 key on the keyboard to run the program.

```
print "===================="
print "Calculate u.s.dim(Zn)"
print "--------------------"
x = input ("Input n:")
def faktorPrima(x) :
a = []
b = []
hasil = 0
bil = x
prime =True

for i in range(2,x):
prima = True
for u in range(2, i) :

if i % u == 0 :
prime = False

if prime :
a.append(i)
idx = 0
while bil > 1 :
try:

if (bil%a[idx]) == 0 :
hasil = bil/a[idx]
bil = hasil
b.append(a[idx])
else :
idx = idx + 1
except IndexError :
break
return b

print "Prime factorization of ",x,":",
faktorPrima(x)

h = faktorPrima(x)
ph = set(h)

if len(ph) == 1:
vdimku = len(ph)
else:
vdimku = len(h)

print "Therefore, u.s.dim(Z_",x, ") =",
vdimku
```

For example, using the Python program above, entering the inputs $n = 1001$ also $n = 729$, i.e. the uniserial dimensions of the module $(\mathbb{Z}_{1001})_{\mathbb{Z}}$ and $(\mathbb{Z}_{729})_{\mathbb{Z}}$. The output results are as follows:

```
====================
Calculate u.s.dim(Zn)
--------------------
Input n:1001
Prime factorization of  1001 : [7, 11, 13]
Therefore, u.s.dim(Z_ 1001 ) = 3
>>>

====================
Calculate u.s.dim(Zn)
```

```
---------------------
Input n:729
Prime factorization of  729 :
[3, 3, 3, 3, 3, 3]
Therefore, u.s.dim(Z_ 729 ) = 1
>>>
```

which means that $u.s.dim\left(\mathbb{Z}_{1001}\right)=3$ and $u.s.dim\left(\mathbb{Z}_{729}\right)=1$. Note again that in Python programs, the maximum value of n is 1.7×10^{308}.

3 Noetherian Rings With Finite Valuation Dimension

Now, research around Noetherian rings that have finite dimension of valuation. According to Theorem 2.4 above, this class is included into semiperfect rings class. Since there are two types of semiperfect rings; local rings or finite calculation of local rings; our investigation is divided into instances. For the instance of the ring R is local, the following is the result.

Theorem 3.1. *Assume (R,m) is a Noetherian ring and local. Then R has a finite dimension of valuation if and only if R is valuation or Artinian.*

Proof. We already know that valuation rings and Artinian rings have finite dimension of valuation. Hence we only need to prove the one direction of the theorem from left to right.

Let R be a finite valuation dimension local Noetherian ring. Suppose R is neither valuation nor Artinian. Then for any natural number n, we have $m^n\neq 0$ and it is not Artinian [6]. Since R is Noetherian, according to Iversen and Nielsen [11], we also have

$$\cap_{n=1}^{\infty} m^n = 0. \quad (3.1)$$

Meanwhile R is not valuation implies there exists $x,y\in R$ such that $Rx \nsubseteq Ry$ and $Ry \nsubseteq Rx$. From (3.1) we obtain some natural number k such that $x,y\notin m^k$. As a result we have R/m^k is not valuation and m^k is not Artinian. This last statement is a contradiction statement to Theorem 2.4 point (1) above. Thus R is valuation or Artinian. □

Before we proceed with the second case, here is an example of local Artinian ring from [9, Remark 3.2], which is not valuation. that means that the class Artinian rings in Theorem 3.1 can not be deleted.

Example 3.2. Consider $R=\mathbb{Z}_2\left[x,y\right]/\left\langle x^2,y^2,xy\right\rangle$ be a ring. We can see that the ring R is local Artinian but not valuation. Moreover $v.dim\left(R\right)=2$ since for any N submodules of $M=R_R$, $M/N\in\zeta_1$, and $M\notin\zeta_1$ since R is not valuation.

Next we consider semiperfect rings which are product of finite but at least two local rings. The result of this case is a direct consequence of this theorem which is a generalization of [9, Theorem 3.3].

Theorem 3.3. *Let R,S are commutative rings. Then R,S are both Artinian if and only if $R\times S$ has a finite valuation dimension.*

Proof. If R,S are Artinian then $R\times S$ is also Artinian, since finite product of Artinian rings are also Artinian. Hence, by Theorem 2.4 part (2), $R\times S$ has a finite dimension of valuation.

Conversely, we shall prove, without loss of generality, if R is not Artinian then $v.dim\left(R\times S\right)$ is not finite. So, suppose R is not Artinian. Let

$$I_1 > I_2 > \cdots > I_n > \ldots$$

be an unstable chain of proper ideals. Define an ideal

$$I = I_2\times 0 \subseteq R\times S.$$

We obtain I is a proper ideal of $R\times S$ which is not Artinian since I_2 is not Artinian. Let

$$J_1=(I_1\times 0)/I, J_2=(I_2\times S)/I.$$

We have J_1,J_2 are proper nonzero ideals in $(R\times S)/I$ such that $J_1\cap J_2=\{0\}$ which implies $(R\times S)/I$ is not valuation. Thus, according to Theorem 2.4 point (1), $v.dim\left(R\times S\right)$ is not finite. □

The following corollary is a direct consequence of Theorem 3.3.

Corollary 3.4. *Assume R is a commutative ring with $R\cong R_1\times ...\times R_k$, $k>1$ and R_i are local rings for all $i=1,2,\ldots,k$. Then R has a finite dimension of valuation if and only if R_i is an Artinian for every $i=1,\ldots,k$.*

Summarizing our result in Theorem 3.1 and Corollary 3.4 we obtain the following theorem which is the main result of this paper.

Theorem 3.5. *Let R be a Noetherian ring with commutative properties. Then the following is the same as:*

1. *R is valuation or Artinian;*
2. *R has finite valuation dimension.*

4 Concluding Remark

Result of Corollary 3.4, in fact, works for general commutative rings without condition of the rings under consideration being Noetherian. In contrast, Theorem 3.1 requires the rings under consideration being Noetherian. As a result, if we can improve Theorem 3.1 by replacing the object rings from being local Noetherian to being local, including local non-Noetherian, we will obtain the class of commutative rings with a finite valuation dimension; additionally, it is an open problem that is currently under investigation.

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REFERENCES

[1] Arifin, S., Garminia, H., ”Uniserial dimension of module Zm x Zn over Z using Python,” Int. J. Sci. Technol. Res., 8, 2019, p 194-9.

[2] Arifin, S., Garminia, H., and Astuti, P., ”Dimensi Valuasi Dari Daerah Ideal Utama,” Prosiding Seminar Nasional Aljabar, Penerapan dan Pembelajarannya, Sanata Dharma University, 2016, p 9-16.

[3] Arifin, S., Garminia, H., and Astuti, P., ”Uniserial Dimensions of Finitely Generated Primary Modules Over A Discrete Valuation Domain”, In Journal of Physics: Conference Series, IOP Publishing, Vol. 1245, No. 1, 2019, p. 012057. DOI: 10.1088/1742-6596/1245/1/012057

[4] Arifin, S., Muktyas, I.B., and Sukmawati, K.I., ”Product of two groups integers modulo m,n and their factor groups using python”, In Journal of Physics: Conference Series, IOP Publishing, Vol. 1778, No. 1, 2021, p. 012026. DOI: 10.1088/1742-6596/1778/1/012026

[5] Arifin S and Garminia H, Valuation Dimension of Ring Zn Using Python, Int. J. Eng. Technol., 7, 2018, p 6351–6

[6] Atiyah, M. F., MacDonald, I. G., Introduction to Commutative Algebra, Addison-Wesley Publication Co., London, 1969.

[7] Faith, C., The structure of valuation rings, J. Pure Appl. Algebra, 31, 7-27, 1984.

[8] Faith, C., The structure of valuation rings II, J. Pure Appl. Algebra, 42, 37-43, 1986.

[9] Ghorbani, A. dan Nazemian, Z., On commutative rings with uniserial dimension, Journal of Algebra and Its Aplications, 14, 1, 1-10, 2015, https://doi.org/10.1142/S0219498815500085.

[10] Halushchak, I., Novosad, Z., Tsizhma, Y., Zagorodnyuk, A., Logistic Map on the Ring of Multisets and Its Application in Economic Models, Mathematics and Statistics, 8(4), 424-429, 2020, DOI: 10.13189/ms.2020.080408.

[11] Iversen, B., Nielsen, H. A., Lecture Notes on Local Rings, World Scientific Pub. Co. Inc., 2014.

[12] Kaplansky, I., Groups over Dedekind rings and valuation rings, Trance. Amer. Math. Soc., 72, 327-340, 1952.

[13] Manis, M., Valuation on a commutative ring, Proc. Amer. Math. Soc, 20, 193-198, 1969.

[14] Mouchili, S.B.L, Discrete Logarithm in Galois Rings, Mathematics and Statistics, 6(3), 34-45, 2018, DOI: 10.13189/ms.2018.060302.

[15] Nazemian, Z., Ghorbani, A., Behboodi, M., Uniserial dimension of groups, Journal of Algebra, 399, 894-903, 2014, https://doi.org/10.1016/j.jalgebra.2013.09.054.

[16] Stoll, R. R., Set Theory and Logic, Dover Publication Inc., New York, 1961.

[17] Warfield, R. B., Serial rings and finitely presented modules, J. Algebra, 37, 187-222, 1975.