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Derivation of Some Entries in the Tables of David Bierens De Haan and Anatolii Prudnikov: An Exercise in Integration Theory

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Abstract It is always useful to improve the catalogue of definite integrals available in tables. In this paper we use our previous work on Lobachevsky integrals to derive entries in the tables by Bierens De Haan and Anatolli Prudnikov featuring errata results and new integral formula for interested readers. In this work we derive a definite integral given by

$$\int_0^\infty \frac{e^{-mx}(\log(a)-x)^k + e^{mx}(\log(a)+x)^k}{\cosh^2(cx)+\sinh^2(t)}dx \tag{1}$$

in terms of the Lerch function. The importance of this work lies in the derivation of known and new results not presently found in current literature. We used our contour integral method and applied it to an integral in Prudnikov and used it to derive a closed form solution in terms of a special function. The advantage of using a special function is the added benefit of analytic continuation which widens the range of computation of the parameters. Special functions have significance in mathematical analysis, functional analysis, geometry, physics, and other applications. Special functions are used in the solutions of differential equations or integrals of elementary functions. Special functions are linked to the theory of Lie groups and Lie algebras, as well as certain topics in mathematical physics.

Keywords Entries in Bierens De Haan, Lerch Function, Definite Integral

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1 Introduction

In 1867 David Bierens De Haan [3] and 1990 Anatolii Prudnikov [5] both produced famous books on definite integrals. In this work, the authors expand on their work in [7] and their contour integral method and applied it to an interesting integral in the book of Prudnikov et al. [5] and expressed its closed form in terms of the Lerch function. This derived integral formula was then used to provide formal derivations for some definite integrals in Table 356 in [3] and 2.4.7 in [5], along with some new formula and errata in certain cases. The Lerch function being a special function has the fundamental property of analytic continuation, which enables us to widen the range of evaluation for the parameters involved in our definite integral.

The definite integral derived in this manuscript is given by

$$\int_0^\infty \frac{e^{-mx}(\log(a)-x)^k + e^{mx}(\log(a)+x)^k}{\cosh^2(cx)+\sinh^2(t)}dx \tag{2}$$

where the parameters k, a, m, c and t are general complex numbers. This work is important because the authors were unable to find similar derivations in current literature. The derivation of the definite integral follows the method used by us in [6] which involves Cauchy's integral formula. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{3}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. This method involves using a form of equation (3) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying equation (3) by a function and performing some substitutions so that the contour integrals are the same.

2 Definite integral of the contour integral

We use the method in [6]. The variable of integration in the contour integral is $z = m + w$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we will form two equations and add them. For the first equation replace y by $x + \log(a)$ then multiply by e^{mx}. To form the second equation we replace x by $-x$ in the first and add both equations followed by multiplying both sides by $\frac{1}{2(\cosh^2(cx)+\sinh^2(t))}$. Next we take the infinite integral over $x \in [0, \infty)$ to get

$$\begin{aligned}\frac{1}{k!}\int_0^\infty \frac{e^{-mx}(\log(a)-x)^k + e^{mx}(\log(a)+x)^k}{\cosh^2(cx)+\sinh^2(t)}dx &= \frac{1}{2\pi i}\int_0^\infty\int_C \frac{a^w w^{-k-1}\cosh(x(m+w))}{\cosh^2(cx)+\sinh^2(t)}dwdx \\ &= \frac{1}{2\pi i}\int_C\int_0^\infty \frac{a^w w^{-k-1}\cosh(x(m+w))}{\cosh^2(cx)+\sinh^2(t)}dxdw \\ &= \frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}\text{csch}(2t)\csc\left(\frac{\pi(m+w)}{2c}\right)\sinh\left(\frac{t(m+w)}{c}\right)}{c}dw\end{aligned} \tag{4}$$

from equation (2.4.6.27) in [5] where $-1 < Re(w+m) < 0$ and $2Re(c) > |Re(w+m)|$. The logarithmic function is given for example in section (4.1) in [2]. We are able to switch the order of integration over $w + m$ and x using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

3 The Lerch function

We use (9.550) and (9.556) in [4] where $\Phi(z, s, v)$ is the Lerch function which is a generalization of the Hurwitz zeta $\zeta(s, v)$ and Polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^\infty (v+n)^{-s} z^n \tag{5}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{6}$$

where $Re(v) > 0$, or $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4 Infinite sum of the contour integral

In this section we will again use Cauchy's integral formula (3) and taking the infinite sum to derive equivalent sum representations for the contour integrals. For the first equation we replace y by $y + t$ and multiply both sides by e^{mt}. Next we replace t by $-t$ and take the difference of these two equations followed by replacing t by t/c simplifying to get

$$\frac{e^{-\frac{mt}{c}}\left(y-\frac{t}{c}\right)^k - e^{\frac{mt}{c}}\left(\frac{t}{c}+y\right)^k}{k!} = -\frac{1}{2\pi i}\int_C 2w^{-k-1}e^{wy}\sinh\left(\frac{t(m+w)}{c}\right)dw \tag{7}$$

Next we replace y by $\log(a)+\frac{i\pi(2y+1)}{2c}$ and multiply both sides by $\frac{i}{c}\pi\text{csch}(2t)e^{\frac{i\pi my}{c}+\frac{i\pi m}{2c}}$ and take the infinite over $y\in[0,\infty)$ and simplify using the Lerch function to get

$$\begin{aligned}&\frac{\pi^{k+1}}{c^2k!}\left(\frac{i}{c}\right)^{k-1}\text{csch}(2t)e^{\frac{im(\pi+2it)}{2c}}\left(e^{\frac{2mt}{c}}\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{-2it-2ic\log(a)+\pi}{2\pi}\right)-\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{2it-2ic\log(a)+\pi}{2\pi}\right)\right)\\&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{2i\pi a^w w^{-k-1}\text{csch}(2t)e^{\frac{i\pi(2y+1)(m+w)}{2c}}\sinh\left(\frac{t(m+w)}{c}\right)}{c}dw\\&=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{2i\pi a^w w^{-k-1}\text{csch}(2t)e^{\frac{i\pi(2y+1)(m+w)}{2c}}\sinh\left(\frac{t(m+w)}{c}\right)}{c}\\&=\frac{1}{2\pi i}\int_C\frac{\pi a^w w^{-k-1}\text{csch}(2t)\csc\left(\frac{\pi(m+w)}{2c}\right)\sinh\left(\frac{t(m+w)}{c}\right)}{c}dw\end{aligned}\tag{8}$$

from (1.232.3) in [4] and $Im(m+w)>0$ for convergence of the sum.

5 Definite integral in terms of the Lerch function

Theorem 1. *For all* $a,k,c,t,m\in\mathbb{C}$,

$$\begin{aligned}\int_0^\infty\frac{e^{-mx}(\log(a)-x)^k+e^{mx}(\log(a)+x)^k}{\cosh^2(cx)+\sinh^2(t)}dx=\frac{2\pi^{k+1}}{c^2}\left(\frac{i}{c}\right)^{k-1}csch(2t)e^{\frac{im(\pi+2it)}{2c}}\left(e^{\frac{2mt}{c}}\Phi\left(e^{\frac{im\pi}{c}},\right.\right.\\ \left.\left.-k,\frac{-2it-2ic\log(a)+\pi}{2\pi}\right)-\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{2it-2ic\log(a)+\pi}{2\pi}\right)\right)\end{aligned}\tag{9}$$

Proof. Since the right-hand side of equation (4) is equal to the right-hand side of (8) we can equate the left-hand sides and simplify the factorials to achieve the stated formula. □

Main results

6 The Fourier Cosine transform in terms of the Lerch function

Theorem 2. *For all* $a,k,c,t,m\in\mathbb{C}$,

$$\begin{aligned}\int_0^\infty\frac{\cosh(mx)\left((\log(a)-x)^k+(\log(a)+x)^k\right)}{\cosh(2cx)+\cosh(2t)}dx=-\frac{1}{2}\pi^{k+1}\left(\frac{i}{c}\right)^{k+1}\text{csch}(2t)e^{-\frac{m(2t+i\pi)}{2c}}\left(\Phi\left(e^{-\frac{im\pi}{c}},\right.\right.\\ \left.-k,\frac{-2it-2ic\log(a)+\pi}{2\pi}\right)\\ -e^{\frac{2mt}{c}}\Phi\left(e^{-\frac{im\pi}{c}},-k,\frac{2it-2ic\log(a)+\pi}{2\pi}\right)\\ +e^{\frac{i\pi m}{c}}\left(e^{\frac{2mt}{c}}\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{-2it-2ic\log(a)+\pi}{2\pi}\right)\right.\\ \left.\left.\left.-\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{2it-2ic\log(a)+\pi}{2\pi}\right)\right)\right)\end{aligned}\tag{10}$$

Proof. Use equation (9) to form a second equation and replace m by $-m$ and add to these two equations and simplify. □

7 Mellin transforms in terms of the Lerch function

Theorem 3. *For all* $k, c, t, m \in \mathbb{C}$,

$$\int_0^\infty \frac{x^k \cosh(mx)}{\cosh(2cx)+\cosh(2t)}dx = -\frac{1}{4}i\pi^{k+1}c^{-k-1}\sec\left(\frac{\pi k}{2}\right)\text{csch}(2t)e^{-\frac{m(2t+i\pi)}{2c}}\left(\Phi\left(e^{-\frac{im\pi}{c}},-k,\frac{\pi-2it}{2\pi}\right)\right. - e^{\frac{2mt}{c}}\Phi\left(e^{-\frac{im\pi}{c}},-k,\frac{2it+\pi}{2\pi}\right) \left. + e^{\frac{i\pi m}{c}}\left(e^{\frac{2mt}{c}}\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{\pi-2it}{2\pi}\right)-\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{2it+\pi}{2\pi}\right)\right)\right) \tag{11}$$

Proof. Use equation (10) and set $a = 1$ and simplify the left-hand side. □

Theorem 4. *For all* $k, c, t, m \in \mathbb{C}$,

$$\int_0^\infty \frac{x^k \cosh(mx)}{(\cosh(2cx)+\cosh(2t))^2}dx = \frac{1}{8}\pi^k c^{-k-2}\sec\left(\frac{\pi k}{2}\right)\text{csch}^2(2t)e^{-\frac{m(2t+i\pi)}{2c}}\left(ck\Phi\left(e^{-\frac{im\pi}{c}},1-k,\frac{1}{2}-\frac{it}{\pi}\right)\right. + cke^{\frac{2mt}{c}}\Phi\left(e^{-\frac{im\pi}{c}},1-k,\frac{it}{\pi}+\frac{1}{2}\right)+cke^{\frac{m(2t+i\pi)}{c}}\Phi\left(e^{\frac{im\pi}{c}},1-k,\frac{1}{2}-\frac{it}{\pi}\right) + cke^{\frac{i\pi m}{c}}\Phi\left(e^{\frac{im\pi}{c}},1-k,\frac{it}{\pi}+\frac{1}{2}\right)-i\pi(2c\coth(2t)+m)\Phi\left(e^{-\frac{im\pi}{c}},-k,\frac{1}{2}-\frac{it}{\pi}\right) - i\pi e^{\frac{2mt}{c}}(m-2c\coth(2t))\Phi\left(e^{-\frac{im\pi}{c}},-k,\frac{it}{\pi}+\frac{1}{2}\right) + i\pi e^{\frac{m(2t+i\pi)}{c}}(m-2c\coth(2t))\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{1}{2}-\frac{it}{\pi}\right) \left. + i\pi e^{\frac{i\pi m}{c}}(2c\coth(2t)+m)\Phi\left(e^{\frac{im\pi}{c}},-k,\frac{it}{\pi}+\frac{1}{2}\right)\right) \tag{12}$$

Proof. Use equation (10) take the first partial derivative with respect to t and multiply both sides by $-\frac{1}{2}\text{csch}(2t)$ and simplify the left-hand side. □

8 Derivation of entry BI(356)(16) in [3]

Theorem 5. *For* $t \in \mathbb{C}$,

$$\int_0^\infty \frac{x^2\log(x)(x\sinh(x)-3(\cos(t)+\cosh(x)))}{(\cos(t)+\cosh(x))^2}dx = \frac{1}{3}(\pi-t)t(t+\pi)\csc(t) \tag{13}$$

Proof. We will form three equations using equation (12) and add to yield the stated result. For the first equation we use equation (12) and set $m = 0, c = 1/2$, then take the first partial derivative with respect to k and set $k = 2$, multiply by $-3\cos(p)$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$-\int_0^\infty \frac{3x^2\cos(p)\log(x)}{(\cosh(2t)+\cosh(x))^2}dx = \frac{1}{2}\cos(p)\text{csch}^2(2t)\left(24i\pi^2\left(i\left(\zeta'\left(-1,\frac{1}{2}-\frac{it}{\pi}\right)+\zeta'\left(-1,\frac{1}{2}+\frac{it}{\pi}\right)\right)\right.\right. + \pi\coth(2t)\left.\left(\zeta'\left(-2,\frac{1}{2}-\frac{it}{\pi}\right)-\zeta'\left(-2,\frac{1}{2}+\frac{it}{\pi}\right)\right)\right) \left. + \left(12t^2+\pi^2\right)(1+\log(4)+2\log(\pi))-4t\left(4t^2+\pi^2\right)\log(2\pi)\coth(2t)\right) \tag{14}$$

Next we derive the second equation by using (12) and setting $m = 1, c = 1/2$ then take the first partial derivative with respect to k and set $k = 2$ and multiply by -3 and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$
\begin{aligned}
-\int_0^\infty \frac{3x^2\log(x)\cosh(x)}{(\cosh(2t)+\cosh(x))^2}dx &= \frac{1}{4}e^{-2t}(\coth(2t)+1)\text{csch}^2(2t)\left(24\pi^2\left(\sinh(4t)\left(\zeta'\left(-1,\frac{1}{2}-\frac{it}{\pi}\right)+\zeta'\left(-1,\frac{1}{2}+\frac{it}{\pi}\right)\right)\right.\right.\\
&\left.\left.-2i\pi\left(\zeta'\left(-2,\frac{1}{2}-\frac{it}{\pi}\right)-\zeta'\left(-2,\frac{1}{2}+\frac{it}{\pi}\right)\right)\right)+8t\left(4t^2+\pi^2\right)\log(2\pi)\\
&\left.-\left(12t^2+\pi^2\right)(1+\log(4)+2\log(\pi))\sinh(4t)\right)
\end{aligned}
\tag{15}
$$

Finally we derive the third equation using (12) by setting $m=1, c=1/2$ followed by taking the first partial derivative with respect to k and setting $k=2$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$
\begin{aligned}
\int_0^\infty \frac{x^3\log(x)\sinh(x)}{(\cosh(2t)+\cosh(x))^2}dx &= \frac{2}{3}e^{-2t}(\coth(2t)+1)\left(t\left(4t^2+\pi^2\right)(1+\log(8)+3\log(\pi))\right.\\
&\left.-18i\pi^3\left(\zeta'\left(-2,\frac{1}{2}-\frac{it}{\pi}\right)-\zeta'\left(-2,\frac{1}{2}+\frac{it}{\pi}\right)\right)\right)
\end{aligned}
\tag{16}
$$

Add equation (14), (15) and (16) and replace t by $ti/2$, p by t and simplify. Note the result in equation (4.376.10) in [4] is in error. □

9 Derivation of entry BI(356)(17) in [3]

Theorem 6. *For all $k, t \in \mathbb{C}$,*

$$
\begin{aligned}
\int_0^\infty \frac{kx^{k-1}\log(x)(\cos(t)-x\sinh(x)+\cosh(x))}{2(\cos(t)+\cosh(x))^2}dx &= 2^{k-5}k\pi^k\csc\left(\frac{\pi k}{2}\right)\csc(t)\left(8(k-1)\left(\zeta'\left(1-k,\frac{\pi-t}{2\pi}\right)\right.\right.\\
&\left.-\zeta'\left(1-k,\frac{t+\pi}{2\pi}\right)\right)\\
&+\zeta\left(1-k,\frac{t+\pi}{2\pi}\right)\left(8(k-1)\log(2\pi)-4\pi(k-1)\cot\left(\frac{\pi k}{2}\right)+8\right)\\
&+\zeta\left(1-k,\frac{\pi-t}{2\pi}\right)\left(-8k\log(2\pi)+4\pi(k-1)\cot\left(\frac{\pi k}{2}\right)-8\right.\\
&\left.\left.+8\log(\pi)+\log(256)\right)\right)
\end{aligned}
\tag{17}
$$

Proof. We will derive three equations using (12). To derive first equation we set $m=0, c=1/2, t=it/2$, followed by taking the first partial derivative with respect to k and setting $k=k-1$ and multiplying by $\frac{k\cos(t)}{2}$ and using entry (4) in Table below (64:12:7) in [1] and simplify to get

$$
\begin{aligned}
\int_0^\infty \frac{kx^{k-1}\cos(t)\log(x)}{2(\cos(t)+\cosh(x))^2}dx &= 2^{k-4}k\pi^{k-1}\csc\left(\frac{\pi k}{2}\right)\cot(t)\csc(t)\left(2(k-1)\zeta'\left(2-k,\frac{\pi-t}{2\pi}\right)+2(k-1)\zeta'\left(2-k,\frac{t+\pi}{2\pi}\right)\right.\\
&+4\pi\cot(t)\zeta'\left(1-k,\frac{\pi-t}{2\pi}\right)-4\pi\cot(t)\zeta'\left(1-k,\frac{t+\pi}{2\pi}\right)\\
&+\pi\cot(t)\zeta\left(1-k,\frac{t+\pi}{2\pi}\right)\left(-2\pi\cot\left(\frac{\pi k}{2}\right)+4\log(\pi)+\log(16)\right)\\
&+2\pi\cot(t)\zeta\left(1-k,\frac{\pi-t}{2\pi}\right)\left(\pi\cot\left(\frac{\pi k}{2}\right)-2\log(2\pi)\right)\\
&+\zeta\left(2-k,\frac{\pi-t}{2\pi}\right)\left(k\left(-\log\left(4\pi^2\right)\right)+\pi(k-1)\cot\left(\frac{\pi k}{2}\right)-2+2\log(\pi)+\log(4)\right)\\
&\left.+\zeta\left(2-k,\frac{t+\pi}{2\pi}\right)\left(k\left(-\log\left(4\pi^2\right)\right)+\pi(k-1)\cot\left(\frac{\pi k}{2}\right)-2+2\log(\pi)+\log(4)\right)\right)
\end{aligned}
\tag{18}
$$

To derive the second equation we set $m = 1, c = 1/2, t = it/2$, followed by taking the first partial derivative with respect to k and setting $k = k-1$ and multiplying by $\frac{k}{2}$ and using entry (4) in Table below (64:12:7) in [1] and simplify to get

$$\begin{aligned}\int_0^\infty \frac{kx^{k-1}\log(x)\cosh(x)}{2(\cos(t)+\cosh(x))^2}dx &= -i2^{k-5}k\pi^k e^{-it}\csc\left(\frac{\pi k}{2}\right)(1-i\cot(t))\csc^2(t)\left(8\zeta'\left(1-k,\frac{\pi-t}{2\pi}\right)-8\zeta'\left(1-k,\frac{t+\pi}{2\pi}\right)\right.\\
&+\frac{\sin(2t)}{\pi}\left(2(k-1)\left(\zeta'\left(2-k,\frac{\pi-t}{2\pi}\right)+\zeta'\left(2-k,\frac{t+\pi}{2\pi}\right)\right)\right.\\
&+\zeta\left(2-k,\frac{\pi-t}{2\pi}\right)\left(k\left(-\log\left(4\pi^2\right)\right)+\pi(k-1)\cot\left(\frac{\pi k}{2}\right)-2+2\log(\pi)+\log(4)\right)\\
&\left.+\zeta\left(2-k,\frac{t+\pi}{2\pi}\right)\left(k\left(-\log\left(4\pi^2\right)\right)+\pi(k-1)\cot\left(\frac{\pi k}{2}\right)-2+2\log(\pi)+\log(4)\right)\right)\\
&+4\zeta\left(1-k,\frac{\pi-t}{2\pi}\right)\left(\pi\cot\left(\frac{\pi k}{2}\right)+\log\left(\frac{1}{4\pi^2}\right)\right)\\
&\left.+4\zeta\left(1-k,\frac{t+\pi}{2\pi}\right)\left(\log\left(4\pi^2\right)-\pi\cot\left(\frac{\pi k}{2}\right)\right)\right)
\end{aligned} \tag{19}$$

To derive the third equation we set $m = 1, c = 1/2t = it/, k = k-1$, followed by taking the first partial derivative with respect to m and multiplying by $-\frac{k}{2}$ and using entry (4) in Table below (64:12:7) in [1] and simplify to get

$$\begin{aligned}-\int_0^\infty \frac{kx^k\log(x)\sinh(x)}{2(\cos(t)+\cosh(x))^2}dx &= 2^{k-3}k\pi^k\csc\left(\frac{\pi k}{2}\right)\csc(t)\left(2k\left(\zeta'\left(1-k,\frac{\pi-t}{2\pi}\right)-\zeta'\left(1-k,\frac{t+\pi}{2\pi}\right)\right)\right.\\
&+\zeta\left(1-k,\frac{\pi-t}{2\pi}\right)\left(-2k\log(2\pi)+\pi k\cot\left(\frac{\pi k}{2}\right)-2\right)\\
&\left.+\zeta\left(1-k,\frac{t+\pi}{2\pi}\right)\left(2k\log(2\pi)-\pi k\cot\left(\frac{\pi k}{2}\right)+2\right)\right)
\end{aligned} \tag{20}$$

Add equations (18), (19) and (20) and simplify. □

10 Derivation of entry BI(356)(18) in [3]

In this derivation the authors recognized a typo in the denominator of the integrand listed in [3], corrected it and proceeded with the following evaluation.

Theorem 7.

$$\begin{aligned}\int_0^\infty \frac{2x\log(x)(x\sinh(x)-2\cosh(x)+2)}{(1-2\cosh(x))^2}dx &= \frac{1}{27}\left(12\sqrt{3}\pi\left(\zeta''\left(-1,\frac{5}{6}\right)-\zeta''\left(-1,\frac{1}{6}\right)\right)+18\left(\zeta''\left(0,\frac{1}{6}\right)+\zeta''\left(0,\frac{5}{6}\right)\right.\right.\\
&\left.-2(1+\log(2\pi))\zeta'\left(0,\frac{1}{6}\right)-2(1+\log(2\pi))\zeta'\left(0,\frac{5}{6}\right)\right)\\
&\left.-2\pi^2(3+\log(4)+2\log(\pi))+3(3+\log(4)+2\log(\pi))\psi^{(1)}\left(\frac{1}{3}\right)\right)
\end{aligned} \tag{21}$$

Proof. First equation, use (12) set $c = -1/2, m = 0, t = -\pi i/3$ then take the first partial derivative with respect to k and apply L'Hopital's rule to the right-hand side as $k \to 1$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\begin{aligned}\int_0^\infty \frac{4x\log(x)}{(1-2\cosh(x))^2}dx &= \frac{1}{27}\left(24\sqrt{3}\pi\left(\zeta''\left(-1,\frac{5}{6}\right)-\zeta''\left(-1,\frac{1}{6}\right)\right)\right.\\
&+36\left(\zeta''\left(0,\frac{1}{6}\right)+\zeta''\left(0,\frac{5}{6}\right)-2(1+\log(2\pi))\zeta'\left(0,\frac{1}{6}\right)-2(1+\log(2\pi))\zeta'\left(0,\frac{5}{6}\right)\right)\\
&\left.-4\pi^2\log\left(4\pi^2\right)+6\log\left(4\pi^2\right)\psi^{(1)}\left(\frac{1}{3}\right)\right)
\end{aligned} \tag{22}$$

Second equation, use (12) set $c = -1/2, m = 1, t = -\pi i/3$ then take the first partial derivative with respect to k and apply L'Hopital's rule to the right-hand side as $k \to 1$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\int_0^\infty -\frac{4x\log(x)\cosh(x)}{(1-2\cosh(x))^2}dx = \frac{2}{27}\left(24\sqrt{3}\pi\left(\zeta''\left(-1,\frac{1}{6}\right)-\zeta''\left(-1,\frac{5}{6}\right)\right)-9\left(\zeta''\left(0,\frac{1}{6}\right)+\zeta''\left(0,\frac{5}{6}\right)\right)\right.$$
$$+18(1+\log(2\pi))\zeta'\left(0,\frac{1}{6}\right)+18(1+\log(2\pi))\zeta'\left(0,\frac{5}{6}\right)+4\pi^2\log\left(4\pi^2\right)$$
$$\left.-6\log\left(4\pi^2\right)\psi^{(1)}\left(\frac{1}{3}\right)\right) \tag{23}$$

Third equation, use (12) set $c=-1/2, m=1, t=-\pi i/3$ then take the first partial derivative with respect to m and then take the first partial derivative with respect to k and apply L'Hopital's rule to the right-hand side as $k\to 1$ and simplify using entry (4) in Table below (64:12:7) and equations (64:10:1) and (64:10:3) in [1] to get

$$\int_0^\infty \frac{2x^2\log(x)\sinh(x)}{(1-2\cosh(x))^2}dx = \frac{4\pi\left(\zeta''\left(-1,\frac{5}{6}\right)-\zeta''\left(-1,\frac{1}{6}\right)\right)}{\sqrt{3}}-\frac{2}{9}\pi^2(1+\log(4)+2\log(\pi))$$
$$+\frac{1}{3}(1+\log(4)+2\log(\pi))\psi^{(1)}\left(\frac{1}{3}\right) \tag{24}$$

Add equations (22), (23) and (24) to achieve the stated result. □

11 Derivation of entry BI(356)(21) in [3]

Theorem 8. *For all* $q, p, t \in \mathbb{C}$,

$$\int_0^\infty x^{-q}\log(x)\operatorname{sech}(\pi x)\cosh(px)dx = e^{-\frac{3ip}{2}}2^{-q-2}\sec\left(\frac{\pi q}{2}\right)\left(-2e^{ip}\Phi'\left(e^{-2ip},q,\frac{1}{4}\right)+2\Phi'\left(e^{-2ip},q,\frac{3}{4}\right)\right.$$
$$-2e^{2ip}\Phi'\left(e^{2ip},q,\frac{1}{4}\right)+2e^{3ip}\Phi'\left(e^{2ip},q,\frac{3}{4}\right)$$
$$+e^{ip}\Phi\left(e^{-2ip},q,\frac{1}{4}\right)\left(\log(4)-\pi\tan\left(\frac{\pi q}{2}\right)\right)$$
$$+e^{2ip}\Phi\left(e^{2ip},q,\frac{1}{4}\right)\left(\log(4)-\pi\tan\left(\frac{\pi q}{2}\right)\right)$$
$$-e^{3ip}\Phi\left(e^{2ip},q,\frac{3}{4}\right)\left(\log(4)-\pi\tan\left(\frac{\pi q}{2}\right)\right)$$
$$\left.+\Phi\left(e^{-2ip},q,\frac{3}{4}\right)\left(\pi\tan\left(\frac{\pi q}{2}\right)-\log(4)\right)\right) \tag{25}$$

Proof. Use equation (11) set $m=p, t=ti/2, c=\pi/2, k=-q$ and take the first partial derivative with respect to k finally set $t=\pi/2$ and simplify. □

12 Derivation of entry BI(356)(11) in [4]

Theorem 9. *For all* $a\in\mathbb{C}$

$$\int_0^\infty \frac{1}{2}x^{2a}\log(x)\operatorname{sech}(qx)(2a-qx\tanh(qx)+1)dx = -2^{2a-1}\pi^{2a+1}\left(\zeta\left(-2a,\frac{1}{4}\right)-\zeta\left(-2a,\frac{3}{4}\right)\right)q^{-2a-1}\sec(\pi a) \tag{26}$$

Proof. Derive the first equation. Use equation (11) and set $m=0, t=ti/2, c=q/2$ then set $t=\pi/2$ followed by multiplying both sides by $\frac{2a+1}{2}$ then taking the first partial derivative with respect to k and setting $k=2a$ simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\int_0^\infty \frac{1}{2}(2a+1)x^{2a}\log(x)\operatorname{sech}(qx)dx = 4^{a-1}(2a+1)\pi^{2a+1}q^{-2a-1}\sec(\pi a)\left(-2\zeta'\left(-2a,\frac{1}{4}\right)+2\zeta'\left(-2a,\frac{3}{4}\right)\right.$$
$$\left.+\left(\zeta\left(-2a,\frac{1}{4}\right)-\zeta\left(-2a,\frac{3}{4}\right)\right)\left(\pi\tan(\pi a)-2\log(q)+\log\left(4\pi^2\right)\right)\right) \tag{27}$$

Derive the second equation. Use equation (12) and set $c = q/2, t = ti/2, m = q$ then set $t = \pi/2$ followed by multiplying both sides by $-q/2$ then taking the first partial derivative with respect to k and setting $k = 2a$ simplify using entry (4) in Table below (64:12:7) in [1] to get

$$-\int_0^\infty \frac{1}{2}qx^{2a+1}\log(x)\tanh(qx)\text{sech}(qx)dx = 4^{a-1}\pi^{2a+1}q^{-2a-1}\sec(\pi a)\left(2(2a+1)\left(\zeta'\left(-2a,\frac{1}{4}\right)-\zeta'\left(-2a,\frac{3}{4}\right)\right)\right.$$
$$-\zeta\left(-2a,\frac{1}{4}\right)\left(-2(2a+1)\log(q)+a\log\left(16\pi^4\right)+\pi(2a+1)\tan(\pi a)+2\right.$$
$$\left.+2\log(\pi)+\log(4)\right)+\zeta\left(-2a,\frac{3}{4}\right)\left(-2(2a+1)\log(q)+a\log\left(16\pi^4\right)\right.$$
$$\left.\left.+\pi(2a+1)\tan(\pi a)+2+2\log(\pi)+\log(4)\right)\right) \tag{28}$$

Add equations (27) and (28) and simplify for the stated result. Note: Use L'Hopital's rule to evaluate the right-hand for $q \in \frac{1}{2}\mathbb{Z}$. □

13 Derivation of entry BI(356)(10) in [3]

Theorem 10. *For all $q \in \mathbb{C}$,*

$$\int_0^\infty \frac{1}{2}x^q\log(x)\,\text{sech}(x)(q-x\tanh(x)+1)dx = -2^{q-1}\pi^{q+1}\left(\zeta\left(-q,\frac{1}{4}\right)-\zeta\left(-q,\frac{3}{4}\right)\right)\sec\left(\frac{\pi q}{2}\right) \tag{29}$$

Proof. Derive the first equation. Use equation (11) and set $m = 0, t = ti/2, c = 1/2$ then set $t = \pi/2$ followed by multiplying both sides by $\frac{1+q}{2}$ then taking the first partial derivative with respect to k and setting $k = q, t = \pi/2$ simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\int_0^\infty \frac{1}{2}(q+1)x^q\log(x)\text{sech}(x)dx = 2^{q-2}\pi^{q+1}(q+1)\sec\left(\frac{\pi q}{2}\right)\left(-2\zeta'\left(-q,\frac{1}{4}\right)+2\zeta'\left(-q,\frac{3}{4}\right)\right.$$
$$\left.+\left(\zeta\left(-q,\frac{1}{4}\right)-\zeta\left(-q,\frac{3}{4}\right)\right)\left(\pi\tan\left(\frac{\pi q}{2}\right)+\log\left(4\pi^2\right)\right)\right) \tag{30}$$

Derive the second equation. Use equation (12) take the first partial derivative with respect to m followed by setting $c = 1/2, q = 1, t = ti/2, m = 1$, next we multiply both sides by $-1/2$ and take the first partial derivative with respect to k and set $t = \pi/2, k = q$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\int_0^\infty -\frac{1}{2}x^{q+1}\log(x)\tanh(x)\text{sech}(x)dx = 2^{q-2}\pi^{q+1}\sec\left(\frac{\pi q}{2}\right)\left(2(q+1)\left(\zeta'\left(-q,\frac{1}{4}\right)-\zeta'\left(-q,\frac{3}{4}\right)\right)\right.$$
$$-\zeta\left(-q,\frac{1}{4}\right)\left(q\log(4)+2(q+1)\log(\pi)+\pi(q+1)\tan\left(\frac{\pi q}{2}\right)+2+\log(4)\right)$$
$$\left.+\zeta\left(-q,\frac{3}{4}\right)\left(2q\log(2\pi)+\pi(q+1)\tan\left(\frac{\pi q}{2}\right)+2+2\log(\pi)+\log(4)\right)\right) \tag{31}$$

Add equations (30) and (31) and simplify for the stated result. Note: Use L'Hopital's rule to evaluate the right-hand for $q \in \mathbb{Z}$. □

14 Derivation of entry BI(356)(2) in [3]

Theorem 11. *For all $a, q \in \mathbb{C}$,*

$$-\int_0^\infty \frac{1}{2}x^{2a}\log(x)\operatorname{sech}(qx)(2a-2qx\tanh(qx)+1)dx = 4^{a-1}\pi^{2a+1}q^{-2a-1}\sec(\pi a)\left(-2(2a+1)\left(\zeta'\left(-2a,\frac{1}{4}\right)\right.\right.$$
$$\left.\left.-\zeta'\left(-2a,\frac{3}{4}\right)\right)+\zeta\left(-2a,\frac{1}{4}\right)\left(-2(2a+1)\log(q)\right.\right.$$
$$\left.+a\log\left(16\pi^4\right)+\pi(2a+1)\tan(\pi a)+4+2\log(\pi)+\log(4)\right)$$
$$-\zeta\left(-2a,\frac{3}{4}\right)\left(-2(2a+1)\log(q)+a\log\left(16\pi^4\right)\right.$$
$$\left.\left.+\pi(2a+1)\tan(\pi a)+4+2\log(\pi)+\log(4)\right)\right) \tag{32}$$

Proof. Derive the first equation. Use equation (11) and set $m=0, t=ti/2, c=q/2$ followed by multiplying both sides by $-\frac{1+2a}{2}$ then taking the first partial derivative with respect to k and setting $k=2a, t=\pi/2$ simplify using entry (4) in Table below (64:12:7) in [1] to get

$$-\int_0^\infty \frac{1}{2}(2a+1)x^{2a}\log(x)\operatorname{sech}(qx)dx = -4^{a-1}(2a+1)\pi^{2a+1}q^{-2a-1}\sec(\pi a)\left(-2\zeta'\left(-2a,\frac{1}{4}\right)+2\zeta'\left(-2a,\frac{3}{4}\right)\right.$$
$$\left.+\left(\zeta\left(-2a,\frac{1}{4}\right)-\zeta\left(-2a,\frac{3}{4}\right)\right)\left(\pi\tan(\pi a)-2\log(q)+\log\left(4\pi^2\right)\right)\right) \tag{33}$$

Derive the second equation. Use equation (12) take the first partial derivative with respect to m followed by setting $c=q/2, t=ti/2, m=q$, next we multiply both sides by q and take the first partial derivative with respect to k and set $t=\pi/2, k=2a$ and simplify using entry (4) in Table below (64:12:7) in [1] to get

$$\int_0^\infty qx^{2a+1}\log(x)\tanh(qx)\operatorname{sech}(qx)dx = 2^{2a-1}\pi^{2a+1}q^{-2a-1}\sec(\pi a)\left(-2(2a+1)\left(\zeta'\left(-2a,\frac{1}{4}\right)-\zeta'\left(-2a,\frac{3}{4}\right)\right)\right.$$
$$+\zeta\left(-2a,\frac{1}{4}\right)\left(-2(2a+1)\log(q)+a\log\left(16\pi^4\right)+\pi(2a+1)\tan(\pi a)+2\right.$$
$$\left.+2\log(\pi)+\log(4)\right)-\zeta\left(-2a,\frac{3}{4}\right)\left(-2(2a+1)\log(q)+a\log\left(16\pi^4\right)\right.$$
$$\left.\left.+\pi(2a+1)\tan(\pi a)+2+2\log(\pi)+\log(4)\right)\right) \tag{34}$$

Add equations (33) and (34) and simplify for the stated result. Note: Use L'Hopital's rule to evaluate the right-hand for $q \in \frac{1}{2}\mathbb{Z}$. □

15 Derivation of entry 2.4.6.27 in [5]

Theorem 12. *For all $m, c, t \in \mathbb{C}$,*

$$\int_0^\infty \frac{\cosh(mx)}{\cosh(cx)+\cosh(t)}dx = \frac{\pi\operatorname{csch}(t)\csc\left(\frac{\pi m}{c}\right)\sinh\left(\frac{mt}{c}\right)}{c} \tag{35}$$

Proof. Use equation (9) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [1]. □

16 Derivation of an integral in terms of the loggamma function

Theorem 13. *For all $a, c, t \in \mathbb{C}$,*

$$\int_0^\infty \frac{\log\left(\log^2(a)-x^2\right)}{\cosh(cx)+\cosh(t)}dx = -\frac{2i\operatorname{csch}(t)}{c}\left(\pi log\Gamma\left(-\frac{-it+ic\log(a)+\pi}{2\pi}\right)-\pi log\Gamma\left(-\frac{it+ic\log(a)+\pi}{2\pi}\right)\right.$$
$$\left.+\pi\log(-ic\log(a)+it-\pi)-\pi\log(-i(c\log(a)+t-i\pi))+it\log\left(\frac{2i\pi}{c}\right)\right) \tag{36}$$

Proof. Use equation (9) and set $m = 0$ then take the first partial derivative with respect to k and set $k = 0, t = t/2, c = c/2$ and simplify using entry (4) in Table below (64:12:7), equations (64:12:1) and (64:10:2) in [1]. □

Derivation of entries in Prudnikov et. al. [5]

17 Derivation of entry 2.4.7.8 in [5]

Theorem 14. *For* $Re(c) \geq Re(b), |t| < \pi, Re(z) > 0$

$$\int_0^\infty \frac{\cosh(bx)}{(x^2+z^2)(\cosh(cx)+\cosh(t))}dx = -\frac{i}{4z}\operatorname{csch}(t)e^{-\frac{b(t+i\pi)}{c}}\left(\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{-it+cz+\pi}{2\pi}\right) - e^{\frac{2bt}{c}}\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{it+cz+\pi}{2\pi}\right) + e^{\frac{2i\pi b}{c}}\left(e^{\frac{2bt}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{-it+cz+\pi}{2\pi}\right) - \Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{it+cz+\pi}{2\pi}\right)\right)\right) \tag{37}$$

Proof. Use (10) first set $k = -1, a = e^{zi}$ multiply both sides by $\frac{1}{2zi}$ and simplify. Next set $t = t/2, c = c/2, m = b$ and simplify. □

18 Derivation of new entry Table 2.4.7 in [5]

Theorem 15. *For* $Re(c) \geq Re(b), |t| < \pi, Re(z) > 0,$

$$\int_0^\infty \frac{x\sinh(bx)}{(x^2+z^2)(\cosh(cx)+\cosh(t))}dx = \frac{1}{4}\operatorname{csch}(t)e^{-\frac{b(t+i\pi)}{c}}\left(\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{-it+cz+\pi}{2\pi}\right) - e^{\frac{2bt}{c}}\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{it+cz+\pi}{2\pi}\right) + e^{\frac{2i\pi b}{c}}\left(\Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{it+cz+\pi}{2\pi}\right) - e^{\frac{2bt}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{-it+cz+\pi}{2\pi}\right)\right)\right) \tag{38}$$

Proof. Use (37) and take the first partial derivative with respect to b and simplify. □

19 Derivation of entry 2.4.7.2 in [5]

Theorem 16. *For* $Re(\alpha) > 0, |t| < \pi$

$$\int_0^\infty \frac{x^{\alpha-1}}{\cos(t)+\cosh(x)}dx = 2^{\alpha-1}\pi^\alpha \csc\left(\frac{\pi\alpha}{2}\right)\csc(t)\left(\zeta\left(1-\alpha,\frac{\pi-t}{2\pi}\right) - \zeta\left(1-\alpha,\frac{t+\pi}{2\pi}\right)\right) \tag{39}$$

Proof. Use (10) and set $m = 0, k = \alpha - 1, t = ti/2, c = 1/2$ and simplify using entry (4) in Table below (64:12:7) in [1]. □

20 Derivation of entry 2.4.7.3 in [5]

Theorem 17. *For* $Re(\alpha) > 0, |t| < \pi$

$$\int_0^\infty \frac{x^{\alpha-1}\cosh\left(\frac{x}{2}\right)}{\cosh(t)+\cosh(x)}dx = \frac{4^{\alpha-1}}{e^t+1}\pi^\alpha e^{t/2}\csc\left(\frac{\pi\alpha}{2}\right)\left(\zeta\left(1-\alpha,\frac{\pi-it}{4\pi}\right) + \zeta\left(1-\alpha,\frac{it+\pi}{4\pi}\right) - \zeta\left(1-\alpha,\frac{3}{4}-\frac{it}{4\pi}\right) - \zeta\left(1-\alpha,\frac{it}{4\pi}+\frac{3}{4}\right)\right) \tag{40}$$

Proof. Use (10) and set $k = \alpha - 1, t = t/2, c = 1/2, m = 1/2$ and simplify using entry (4) in Table below (64:12:7) in [1] □

Theorem 18. *For $Re(\alpha) > 0, |t| < \pi$*

$$\int_0^\infty \frac{x^{\alpha-1}\sinh\left(\frac{x}{2}\right)}{\cosh(t)+\cosh(x)}dx = i2^{2\alpha-3}\pi^\alpha e^{-t/2}\left(e^t+1\right)\sec\left(\frac{\pi\alpha}{2}\right) csch(t)\left(-\zeta\left(1-\alpha,\frac{\pi-it}{4\pi}\right)+\zeta\left(1-\alpha,\frac{it+\pi}{4\pi}\right)\right.$$
$$\left.+\zeta\left(1-\alpha,\frac{3}{4}-\frac{it}{4\pi}\right)-\zeta\left(1-\alpha,\frac{it}{4\pi}+\frac{3}{4}\right)\right) \quad (41)$$

Proof. Use (10) and set $k = \alpha - 2, t = t/2, c = 1/2, m = 1/2$ and simplify using entry (4) in Table below (64:12:7) in [1]. □

21 Derivation of entry 2.4.7.4 in [5]

Theorem 19. *For $Re(c) > |Re(b)|$,*

$$\int_0^\infty \frac{x\sinh(bx)}{\cosh(cx)+1}dx = \frac{\pi\left(c-\pi b\cot\left(\frac{\pi b}{c}\right)\right)\csc\left(\frac{\pi b}{c}\right)}{c^3} \quad (42)$$

Proof. Use (10) take the first partial derivative with respect to m and set $k = 0, t = to/2, c = c/2, m = b$ and simplify using entry (2) in Table below (64:12:7) in [1]. Next apply L'Hopitals rule to the right-hand side as $t \to 0$ and simplify. □

22 Derivation of entry 2.4.7.6 in [5]

Theorem 20. *For $t < \pi$,*

$$\int_0^\infty \frac{x\cosh\left(\frac{x}{2}\right)}{\cos(t)+\cosh(x)}dx = 4\pi\sec\left(\frac{t}{2}\right)\left(\zeta'\left(-1,\frac{\pi-t}{4\pi}\right)+\zeta'\left(-1,\frac{t+\pi}{4\pi}\right)-\zeta'\left(-1,\frac{3}{4}-\frac{t}{4\pi}\right)-\zeta'\left(-1,\frac{1}{4}\left(\frac{t}{\pi}+3\right)\right)\right) \quad (43)$$

Proof. Use (10) and set $t = ti/2, c = 1/2, m = 1/2$ and simplify using entry (4) in Table below (64:12:7) in [1], followed by applying L'Hopital's rule to the right-hand side as $k \to 1$ □

23 Derivation of entry 2.4.7.7 in [5]

Theorem 21. *For $Re(c) > 0, t < \pi$,*

$$\int_0^\infty \frac{x^{2n+1}\sinh(cx)}{(\cosh(cx)+\cos(t))^2}dx = 4^n(2n+1)\pi^{2n+1}c^{-2(n+1)}\sec(\pi n)\csc(t)\left(\zeta\left(-2n,\frac{\pi-t}{2\pi}\right)-\zeta\left(-2n,\frac{t+\pi}{2\pi}\right)\right) \quad (44)$$

Proof. Use (10) and set $t = ti/2, c = c/2$, next take the first partial derivative with respect to m and then with respect to t and set $k = 2n, m = c$ and simplify using entry (4) in Table below (64:12:7) in [1]. □

24 Derivation of entry 2.4.7.9 in [5]

Theorem 22. *For $2Re(c) > Re(b)$,*

$$\int_0^\infty \frac{\cosh(bx)}{\sqrt{x}(\cosh(2cx)+2)}dx = -\frac{i}{2\sqrt{c}}\sqrt{\frac{\pi}{6}}e^{-\frac{ib\left(\pi+\cos^{-1}(2)\right)}{2c}}\left(e^{\frac{i\pi b}{c}}\Phi\left(e^{\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+2\sin^{-1}(2)}{4\pi}\right)\right.$$
$$-\Phi\left(e^{-\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+\cos^{-1}(2)}{2\pi}\right)-e^{\frac{ib\left(\pi+\cos^{-1}(2)\right)}{c}}\Phi\left(e^{\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+\cos^{-1}(2)}{2\pi}\right) \quad (45)$$
$$\left.+e^{-\frac{b\cosh^{-1}(2)}{c}}\Phi\left(e^{-\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+2\sin^{-1}(2)}{4\pi}\right)\right)$$

Proof. Use (10) set $t = ti/2, k = -1/2$, next set $t = \cos^{-1}(2), m = b$ and simplify. □

Theorem 23. *For* $2Re(c) > Re(b)$*,*

$$\int_0^\infty \frac{\sinh(bx)}{\sqrt{x}(\cosh(2cx)+2)}dx = \frac{1}{2\sqrt{c}}\sqrt{\frac{\pi}{6}}e^{-\frac{ib\left(\pi+\cos^{-1}(2)\right)}{2c}}\left(e^{-\frac{b\cosh^{-1}(2)}{c}}\left(\Phi\left(e^{-\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+2\sin^{-1}(2)}{4\pi}\right)\right.\right.$$
$$\left.+e^{\frac{i\pi b}{c}}\left(\Phi\left(e^{\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+\cos^{-1}(2)}{2\pi}\right)-e^{\frac{b\cosh^{-1}(2)}{c}}\Phi\left(e^{\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+2\sin^{-1}(2)}{4\pi}\right)\right)\right) \quad (46)$$
$$\left.-\Phi\left(e^{-\frac{ib\pi}{c}},\frac{1}{2},\frac{\pi+\cos^{-1}(2)}{2\pi}\right)\right)$$

Proof. Use (10) take the first partial derivative with respect to m and set $t = ti/2, k = -3/2$ and simplify. Next set $t = \cos^{-1}(2), m = b$ and simplify. □

25 Derivation of entry 2.4.5.24 in [5]

Theorem 24. *For* $Re(c) > Re(b)$*,*

$$\int_0^\infty \frac{\cosh(bx)\operatorname{sech}(cx)}{x^2+z^2}dx = -\frac{1}{4z}e^{-\frac{3i\pi b}{2c}}\left(-e^{\frac{i\pi b}{c}}\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{2cz+\pi}{4\pi}\right)+\Phi\left(e^{-\frac{2ib\pi}{c}},1,\frac{cz}{2\pi}+\frac{3}{4}\right)\right.$$
$$\left.-e^{\frac{2i\pi b}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{2cz+\pi}{4\pi}\right)+e^{\frac{3i\pi b}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1,\frac{cz}{2\pi}+\frac{3}{4}\right)\right) \quad (47)$$

Proof. Use equation (37) set $t = ti$ then set $t = \pi/2$ and simplify. □

26 Derivation of errata for entry 2.4.5.20 in [5]

Theorem 25. *For* $|Re(b)| < \pi, z \in \mathbb{C}$*,*

$$\int_0^\infty \frac{x\sinh(bx)\operatorname{sech}\left(\frac{\pi x}{z}\right)}{x^2+z^2}dx = \frac{1}{2}\left(-4\sin\left(\frac{bz}{2}\right)+\pi\sin(bz)+\cos(bz)\log\left(\frac{1-\sin\left(\frac{bz}{2}\right)}{\sin\left(\frac{bz}{2}\right)+1}\right)\right) \quad (48)$$

Proof. Use equation (37) set $t = ti$ then set $t = \pi/2, c = \pi/4$ and simplify using equation (9.559) in [4] and entry (1) in Table below (64:12:7) in [1]. □

27 Derivation of entry 2.4.5.10 in [5]

Theorem 26. *For* $Re(c) > 0, Re(\alpha) > 1$

$$\int_0^\infty x^{\alpha+1}\tanh(cx)\operatorname{sech}(cx)dx = 2^\alpha\pi^{\alpha+1}(\alpha+1)\sec\left(\frac{\pi\alpha}{2}\right)c^{-\alpha-2}\left(\zeta\left(-\alpha,\frac{1}{4}\right)-\zeta\left(-\alpha,\frac{3}{4}\right)\right) \quad (49)$$

Proof. Use equation (12) take the first partial derivative with respect to m and set $t = ti/2, c = c/2, m = c$ and simplify using entry (4) in Table below (64:12:7) in [1]. □

28 Derivation of entry 2.4.5.8 in [5]

Theorem 27. *For* $Re(c) > Re(b)$*,*

$$\int_0^\infty x\sinh(bx)\operatorname{sech}^2(cx)dx = -\frac{\pi\left(\pi b\cot\left(\frac{\pi b}{2c}\right)-2c\right)\csc\left(\frac{\pi b}{2c}\right)}{4c^3} \quad (50)$$

Proof. Use equation (12) take the first partial derivative with respect to m and set $t = ti/2, c = c/2$ and simplify. Next set $t = \pi/2, k = 0$ and simplify using entry (2) in Table below (64:12:7) in [1]. □

29 Derivation of entry 2.4.5.7 in [5]

Theorem 28. *For $Re(\alpha) > 0, Re(c) > Re(b)$,*

$$\int_0^\infty x^{\alpha-1}\cosh(bx)\operatorname{sech}(cx)dx = -2^{\alpha-2}\pi^\alpha\csc\left(\frac{\pi\alpha}{2}\right)e^{-\frac{3i\pi b}{2c}}c^{-\alpha}\left(-e^{\frac{i\pi b}{c}}\Phi\left(e^{-\frac{2ib\pi}{c}},1-\alpha,\frac{1}{4}\right)\right.$$
$$\left.+\Phi\left(e^{-\frac{2ib\pi}{c}},1-\alpha,\frac{3}{4}\right)-e^{\frac{2i\pi b}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1-\alpha,\frac{1}{4}\right)+e^{\frac{3i\pi b}{c}}\Phi\left(e^{\frac{2ib\pi}{c}},1-\alpha,\frac{3}{4}\right)\right) \quad (51)$$

Proof. Use equation (11) set $t = ti/2, c = c/2$ next set $t = \pi/2, k = \alpha - 1, m = b$ and simplify. □

30 Derivation of entry 2.4.5.5 in [5]

Theorem 29. *For $Re(c) > Re(b)$,*

$$\int_0^\infty \frac{\sinh(bx)\operatorname{sech}(cx)}{x}dx = \frac{1}{3}ie^{-\frac{3i\pi b}{2c}}\left(3e^{\frac{i\pi b}{c}}\,{}_2F_1\left(\frac{1}{4},1;\frac{5}{4};e^{-\frac{2ib\pi}{c}}\right)-3e^{\frac{2i\pi b}{c}}\,{}_2F_1\left(\frac{1}{4},1;\frac{5}{4};e^{\frac{2ib\pi}{c}}\right)\right.$$
$$\left.-\,{}_2F_1\left(\frac{3}{4},1;\frac{7}{4};e^{-\frac{2ib\pi}{c}}\right)+e^{\frac{3i\pi b}{c}}\,{}_2F_1\left(\frac{3}{4},1;\frac{7}{4};e^{\frac{2ib\pi}{c}}\right)\right) \quad (52)$$
$$= \log\left(\tan\left(\frac{\pi(b+c)}{4c}\right)\right)$$

Proof. Use equation (51) and take the first partial derivative with respect to b and set $\alpha = -1$ and simplify using equation (9.559) in [4] and Table 18-1 in [1]. □

31 Derivation entry 3.531.1 in [4]

Theorem 30.

$$\int_0^\infty \frac{x}{2\cosh(x)-1}, \frac{1}{9}\left(3\psi^{(1)}\left(\frac{1}{3}\right)-2\pi^2\right) \quad (53)$$

Proof. Use equation (11) and set $c = 1/2, m = 0, t = -\frac{i\pi}{3}$ then apply L'Hopital's rule to the right-hand side as $k \to 1$ and simplify using entry (4) in Table below (64:12:7) and equation (64:4:1) in [1]. □

32 Derivation of entry 3.533.2 in [4]

Theorem 31. *For $m \in \mathbb{R}, t \in \mathbb{C}$*

$$\int_0^\infty \frac{x\sinh(mx)}{(\cos(t)-\cosh(mx))^2}dx = \operatorname{sgn}(m)\frac{(\pi-t)\csc(t)}{m^2} \quad (54)$$

Proof. Use equation (11) take the first partial derivatives with respect to m and t and set $t = \frac{1}{2}i(\pi - t), c = m/2$ and simplify using entry (4) in Table below (64:12:7) in [1]. Next set $k = 0$ and simplify. Where $\operatorname{sgn}(x)$ gives -1, 0, or 1 depending on whether x is negative, zero, or positive. □

33 Derivation of entry 3.533.3 in [4]

Theorem 32. *For all $t \in \mathbb{C}$,*

$$\int_0^\infty \frac{x^3\sinh(x)}{(\cos(t)+\cosh(x))^2}dx = (\pi-t)t(t+\pi)\csc(t) \quad (55)$$

Proof. Use equation (11) take the first partial derivatives with respect to m and t and set $t = ti/2, c = 1/2, m = 1$ and simplify using entry (4) in Table below (64:12:7) in [1]. Next set $k = 2$ and simplify. □

34 Derivation of entry 3.533.4 in [4]

Theorem 33. *For* $m, a \in \mathbb{C}$

$$\int_0^\infty \frac{x^{2m+1}\sinh(x)}{(\cos(2\pi a)-\cosh(x))^2}dx = e^{-2i\pi a}4^m(2m+1)\pi^{2m+1}(\cot(2\pi a)+i)\sec(\pi m)(\zeta(-2m,a)-\zeta(-2m,1-a)) \quad (56)$$

Proof. Use equation (11) take the first partial derivatives with respect to m and t and set $t=\frac{1}{2}i(\pi-2\pi a), c=1/2, m=1, k=2m$ and simplify using entry (4) in Table below (64:12:7) in [1]. Note when $a=1/2$ apply L'Hopital's rule to the right-hand side to evaluate. □

35 Special cases

Theorem 34.

$$\int_0^\infty \frac{1}{(1-2\cosh(x))^2}dx = \frac{1}{27}\left(9+2\sqrt{3}\pi\right) \quad (57)$$

Proof. Use equation (11) take the first partial derivative with respect to t and set $t=-\frac{i\pi}{3}, c=1/2, m=0$ and simplify using entry (4) in Table below (64:12:7) in [1]. Next apply L'Hopital's rule as $k\to 0$. □

Theorem 35.

$$\int_0^\infty \frac{x}{(1-2\cosh(x))^2}dx = \frac{1}{27}\left(3\psi^{(1)}\left(\frac{1}{3}\right)-2\pi^2\right) \quad (58)$$

Proof. Use equation (11) take the first partial derivatives with respect to t and set $t=-\frac{i\pi}{3}, c=1/2, m=0$ and simplify using entry (4) in Table below (64:12:7) and equation (64:4:1) in [1]. Next apply L'Hopital's rule as $k\to 1$. □

Theorem 36.

$$\int_0^\infty \frac{2\cosh(x)-7}{(2\cosh(x)-1)^3}dx = -\frac{2}{27}\left(9+5\sqrt{3}\pi\right) \quad (59)$$

Proof. Use equation (11) take the second partial derivatives with respect to t and set $t=-\frac{i\pi}{3}, c=1/2, m=0$ and simplify using entry (4) in Table below (64:12:7) and equation (64:4:1) in [1]. Next apply L'Hopital's rule as $k\to 0$. □

36 Conclusions

In this work we used our contour integral method and applied it to an integral in Prudnikov et.al. [5] and produced formal derivations for some integrals in the book by Bierens De Haan [3], Prudnikov et. al. [5] and Gradshteyn and Rhyzik [4]. We also derived a few new integral formula for interested readers to use where applicable and more are possible using the derived definite integral in terms of the Lerch function. Formal derivations of integrals are very important especially in cases where they are not present. Formal derivations assist in determining whether previously published formulae are correct, and they also provide an avenue in generating new integral formula. The authors will be using their method to pursue deriving other formula in [3], [4] and [5]. The formula in this work were numerically evaluated using Wolfram's Mathematica software.

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