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# Approximate Solution of Higher Order Fuzzy Initial Value Problems of Ordinary Differential Equations Using Bezier Curve Representation

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**Abstract** The Bezier curve is a parametric curve used in the graphics of a computer and related areas. This curve, connected to the polynomials of Bernstein, is named after the design curves of Renault's cars by Pierre Bézier in the 1960s. There has recently been considerable focus on finding reliable and more effective approximate methods for solving different mathematical problems with differential equations. Fuzzy differential equations (known as FDEs) make extensive use of various scientific analysis and engineering applications. They appear because of the incomplete information from their mathematical models and their parameters under uncertainty. This article discusses the use of Bezier curves for solving elevated order fuzzy initial value problems (FIVPs) in the form of ordinary differential equation. A Bezier curve approach is analyzed and updated with concepts and properties of the fuzzy set theory for solving fuzzy linear problems. The control points on Bezier curve are obtained by minimizing the residual function based on the least square method. Numerical examples involving the second and third order linear FIVPs are presented and compared with the exact solution to show the capability of the method in the form of tables and two dimensional shapes. Such findings show that the proposed method is exceptionally viable and is straightforward to apply.

**Keywords** Fuzzy Set Theory, Linear Fuzzy Initial Value Problem, Fuzzy Differential Equation, Bezier Curve Method (BCM), Residual Function

## 1. Introduction

In dynamical system modelling, the approximate solution of fuzzy boundary or an initial value problem of differential equations plays a major role in dealing with uncertainty in real-world settings. The model of a fuzzy derivative is firstly initiated by Chang and Zadeh [1], followed by the used of addition principle approach Dubois and Prade [2]. In [3], Kendel and Byatt introduced the model of "Fuzzy differential equation (FDE)" and the study of Fuzzy Initial Value Problem (FIVP) was proposed by Kavela and Seikkalaan others in [4-7]. There are several authors discussing the use of approximate methods for solving linear or non-linear FIVPs of FDEs. For example, A domain Decomposition method is proposed for rectifying the first order and second order FIVPs [8,9].

Furthermore, the solution to a different order of FIVPs using Homotopy Perturbation Method (HPM) is discussed in [10,11]. Recently, Jameel et al. In [12], propose a method, namely Optimum Homotopy Asymptotic for solving first order nonlinear FIVP. The Variational Iteration Method (VIM) is implemented in [13] to solve the FIVP Bratu Equation without decomposing the nonlinear terms of the given equation, in order to solve nonlinear equations quickly and more reliably compared to previous methods. This is the key advantage of VIM relative to ADM and HPM, because it is extraordinarily helpful to solve the problem of elevated order nonlinear and linear initial value straight without converting the nonlinear concept into a first method and prohibitive assumptions.

In contrast with the review of previous methods, the current paper explains a new method based on control point method through Bezier curves representation [14] for solving higher order FIVPs without reducing to system of the first order equations or require ADM polynomials and also without constructing a correction functional by a general Lagrange multiplier as in VIM. The Bezier control points will be determined using least square method by minimizing the residual error function [15].

The paper structure is as folllows. In Section 2 the fundamental models of the FIVPs falsification are described . The Bezier curves in fuzzy form are presented in Section 3. The least square method falsification analysis for general FIVPs is represented in section 4. In Section 5, the use of the Bezier curve method is defuzzied in order to solve the proposed FIVPs. The Bezier curve method is implemented to solve numerical examples for different high order linear FIVPs is displayed in section 5. At last, in section 6, the summary of this work is presented.

# 2. Fuzzy Initial Value Problem (FIVP)

Here, the discussion related to FIVP is presented briefly. Note that, for the Basic concepts of fuzzy sets theory related to this study can be referred to:

First, the n-th order linear IVP is given by,

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t), t \in [t_0, T] \\ x(t_0) = k_1, x'(t_0) = k_2, \ldots, x^{(n-1)}(t_0) = k_n \end{cases} \tag{1}$$

as $f:[t_0,T] \text{ x } \Re \Rightarrow \Re$ is a continuous function definite on such as $T> 0$ and $k_1, k_2, \ldots, k_n \in \Re$. According to [16-18] if the initial condition in (1) is uncertain and can be demonstrated by a fuzzy number, then we have the following FIVP:

$$\begin{cases} \tilde{x}^{(n)}(t;r) = \tilde{f}(t, \tilde{x}(t;r), \ldots, \tilde{x}^{(n-1)}(t;r), t \in [t_0, T] \\ \tilde{x}(t_0) = \tilde{k}_1, \tilde{x}'(t_0;r) = \tilde{k}_2, \ldots, \tilde{x}^{(n-1)}(t_0;r) = \tilde{k}_n \end{cases} \tag{2}$$

where $\tilde{f}:[t_0,T] \text{ x } F(\Re) \to F(\Re)$ is a fuzzy valued function [19] on $[t_0, T]$ with $T>0$, $\tilde{k}_i = (\underline{k_i}(t_0), \overline{k}_i(t_0)) \in F(\Re), i = 1, \ldots, n.$ is a triangular shape fuzzy number and $r \in [0, 1]$ is a fuzzy level set [20,21,23].

# 3. Bezier Curves in Fuzzy Domain

The explanation of the Bezier curve polynomial of $m$ degree in the fuzzy domain (Farin, 1997) is given by the

$$\tilde{P}(t;r) = \sum_{j=0}^{m} \tilde{P}_j \tilde{B}_j^m \left(\frac{t-b_1}{b_2-b_1};r\right), t \in [b_1, b_2]. \tag{3}$$

$$\tilde{B}_j^m \left(\frac{t-b_1}{b_2-b_1};r\right) = \frac{m!}{j!\,(m-j)!}\left(\frac{t-b_1}{b_2-b_1};r\right)^j \left(\frac{b_2-t}{b_2-b_1};r\right)^{m-j}$$

$P_j$ is control points of Bezier coefficient and $\tilde{B}_j^m$ are the polynomial of Bernstein on interval [$a_1$, $a_2$] per every fuzzy level set $r \in [0,1]$. In correctly

$$\tilde{P}(t;r) = \sum_{j=0}^{m} P_j \tilde{B}_j^m(t;r), t \in [0,1]. \tag{4}$$

$$\tilde{B}_j^m(t;r) = \frac{m!}{j!\,(m-j)!}(t;r)^j (1-t;r)^{m-j}$$

where $\tilde{P}(t;r)$ is the vector polynomial that is valued, and the fuzzy parametric Bezier curve and $P_j$, j = $0, 1, \ldots,$ m are the Bezier control points. If $\tilde{P}(x;r)$ polynomial of a scalar valued, the purpose is to call $\tilde{y} = \tilde{P}(t;r)$ then an explicit Bezier curve is presented by $\left(t, \tilde{P}(t;r)\right)$.

# 4. Least Square Method for Solving FIVPs

Considering FIVP as in (2), our goal is to find a polynomial or piecewise polynomial function in fuzzy domain $\tilde{x}(t;r)$ such that $\tilde{x}(t;r)$ satisfies the initial condition and minimizes the residual function

$$\tilde{R}(t;r) = \tilde{x}^{(n)}(t;r) - \tilde{f}(t, \tilde{x}(t;r), \tilde{x}'(t;r) \ldots, \tilde{x}^{(n-1/}(t;r)) \tag{5}$$

where $t \in domain\ \Omega$.

Let the approximate solution of (2) using Finite Element Method is given by sum of weighted function, $\tilde{\varphi}_i(t;r), 1 \le i \le M$ is expressed as

$$\tilde{x}(t;r) = \sum_{1}^{M} w_i \tilde{\varphi}_i(t;r)$$

where $w_i's$ are coefficients (weights) need to be determined. The least square error of residual can be minimized by setting,

$$\widetilde{E} = \int_{\Omega} \widetilde{R}^2(t;r)\, dt \tag{6}$$

The minimum value of $\widetilde{E}$ can be determined by,

$$\frac{d\widetilde{E}}{dw_i} = 0 \ , \ i = 1, ...., M \tag{7}$$

From (6), (7) are presented as follows

$$\int_{\Omega} \widetilde{R}(t;r)\frac{d\widetilde{R}(t;r)}{dw_i}\, dt = 0\,, \tag{8}$$

or

$$\left(\widetilde{R}(t;r), \frac{d\widetilde{R}(t;r)}{dw_i}\right) = 0,\ i = 1,2,\ldots,M \tag{9}$$

lead to a linear system which can be solved for $w_i{}'s$.

### 4.1. The Method Using Bezier Curve

We choose polynomial solution of (2) in degree-m Bezier curve define on fuzzy domain given in form of (4) i.e $\widetilde{x}(t;r) = \widetilde{P}(t;r)$ where $\widetilde{x}(t;r) = (\underline{x}(t;r), \overline{x}(t,r))$. We have,

$$\begin{cases} \underline{x}(t;r) = \sum_{i=0}^{m} \underline{a}_i B_i^m(t) \\ \overline{x}(t;r) = \sum_{i=0}^{m} \overline{a}_i B_i^m(t), \quad 0 \le r, t \le 1. \end{cases} \tag{10}$$

where $\underline{a}_i$ and $\overline{a}_i$ are necessary to evaluate the Bezier control points. Substitute (10) into (5) and the residual purposes can be retrieved., i.e.

$$\begin{cases} \underline{R}(x;r) = \frac{d^n}{dt^n}(\sum_{i=0}^{m} \underline{a}_i B_i^m(x)) - \\ \quad - f(\sum_{i=0}^{m} \underline{a}_i B_i^m(x), \frac{d}{dt}(\sum_{i=0}^{m} \underline{a}_i B_i^m(x)), \ldots, \frac{d^{n-1}}{dt^{n-1}}(\sum_{i=0}^{m} \underline{a}_i B_i^m(x))) \\ \overline{R}(x;r) = \frac{d^n}{dt^n}(\sum_{i=0}^{m} \overline{a}_i B_i^m(x)) - \\ \quad - f(\sum_{i=0}^{m} \overline{a}_i B_i^m(x), \frac{d}{dt}(\sum_{i=0}^{m} \overline{a}_i B_i^m(x)), \ldots, \frac{d^{n-1}}{dt^{n-1}}(\sum_{i=0}^{m} \overline{a}_i B_i^m(x))) \end{cases} \tag{11}$$

From (8) & (9), the least square error of residual can be minimized by setting,

$$\begin{cases} \int_0^1 \underline{R}(t;r)\frac{d\underline{R}(t;r)}{d\underline{a}_1}\, dt = 0 \\ \int_0^1 \overline{R}(t;r)\frac{d\overline{R}(t;r)}{d\overline{a}_i}\, dt = 0 \end{cases} \tag{12}$$

or

$$\begin{cases} \left(\underline{R}(t;r), \frac{d\underline{R}(t;r)}{d\underline{a}_1}\right) = 0 \\ \left(\overline{R}(t;r), \frac{d\overline{R}(t;r)}{d\overline{a}_i}\right) = 0 \end{cases} \tag{13}$$

These will lead to a linear system which can be solved for $\underline{a}_i{}'s$ and $\overline{a}_i{}'s$.

Substitute the coefficients in (9), approximate solution of (2) satisfying triangular fuzzy number properties will be obtained.

## 5. Numerical Example

Three numerical examples are introduced to demonstrate the ability of Bezier curve representation and applied FIVPs.

**Example 5.1 Assume the following 2nd – order linear FIVP [22]:**

$$\begin{cases} \tilde{y}'' - 4\tilde{y}' + 4\tilde{y} = 4t - 4, t \geq 0. \\ \tilde{y}(0) = (2 + r, 4 - r), \tilde{y}'(0) = (3 + 2r, 9 - 2r) \end{cases} \tag{14}$$

The accurate solution of equationis

$$\begin{cases} \underline{Y}(t,r) = (2 + r)e^{2t} + (-1 + r)te^{2t} + t. \\ \overline{Y}(t,r) = (4 - r)e^{2t} + (1 - r)te^{2t} + t, \end{cases} \tag{15}$$

We use third-degree Bezier curves and four control points to consider the approximate solution to numerical execution.

We found the lower control points

$\underline{a1} = 1.0r + 2.0,\quad \underline{a2} = 1.666666667r + 3.0,$

$\underline{a3} = 2.637284701r + 5.03343465,$

$\underline{a4} = 5.358662614r + 7.419452888.$

the upper control points

$\overline{a1} = 4.0 - 1.0r, \overline{a2} = 7.0 - 2.333333333r,$

$\overline{a3} = 11.21580547 - 3.54508612r,$

$\overline{a4} = 22.43465046 - 9.656534954r.$

From the above control points the approximate solution function of BCM is obtained via Matlab 14a for all $r \in [0.1]$ is given below:

$$\underline{y}(t,r) = (1.4468085106382978723404255319149r - 0.68085106382978723404255319148936)t^3 + (0.91185410334346504559270516717325r + 3.1003039513677811550151975683891)t^2 + (2.0r + 3.0)t + 1.0r + 2.0.$$

$$\overline{y}(t,r) = (5.7872340425531914893617021276596 - 5.0212765957446808510638297872 34r)t^3 + (0.36474164133738601823708206686 93r + 3.6474164133738601823708206686 93)t^2 + (9.0 - 4.0r)t - 1.0r + 4.0.$$

In Tables 1-2 the third degree approximate solution by BCM is displayed and compared with the exact solution and third degree undetermined fuzzy coefficients method [22] for some $r \in [0.1]$ when at $t = 0.001$ as follows (table 1-2).

From Table 1-2, the third degree BCM approximate solution and is more accurate than the third degree, which undetermined fuzzy coefficients method [22] for some $r \in [0.1]$, when $t = 0.001$ and presented in triangular fuzzy number form as bellow in Figure 1:

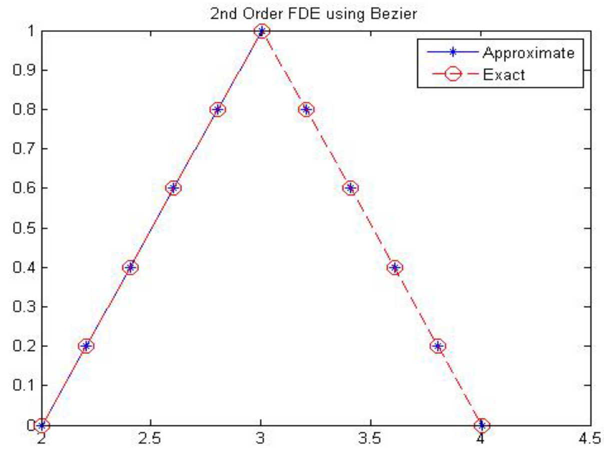


**Figure 1.** The accurate and the approximate solution of Eq. (14)at t=0.001

**Table 1.** Comparisons between the lower solution for the accurate and the approximate solution of Eq. (14) at $t = 0.001$.

| r | Approximate solution | Accurate solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | 2.0030030996231 | 2.00299999866533 | 3.10095776745456e-06 | 0.00099871222831 |
| 0.2 | 2.20340328228328 | 2.20340039893213 | 2.88335114939642e-06 | 0.00119975860278 |
| 0.4 | 2.40380346494347 | 2.40380079919893 | 2.66574453222646e-06 | 0.00140080497724 |
| 0.6 | 2.60420364760365 | 2.60420119946573 | 2.44813791416831e-06 | 0.00160185135171 |
| 0.8 | 2.80460383026383 | 2.80460159973253 | 2.23053129699835e-06 | 0.00180289772617 |
| 1 | 3.00500401292401 | 3.00500199999933 | 2.01292467938430e-06 | 0.00200394410063 |

**Table 2.** Comparisons between the upper solution for the accurate and the approximate solution of Eq. (14) at $t = 0.001$.

| r | Approximate solution | Accurate solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | 4.00900365320365 | 4.009008005336 | 4.35213235494558e-06 | 0.00101009737569 |
| 0.2 | 3.80820372514772 | 3.80820680426867 | 3.07912094799079e-06 | 0.00080905100122 |
| 0.4 | 3.60740379709179 | 3.60740560320133 | 1.80610954103599e-06 | 0.00060800462676 |
| 0.6 | 3.40660386903587 | 3.40660440213400 | 5.33098134525289e-07 | 0.00040695825230 |
| 0.8 | 3.20580394097994 | 3.20580320106667 | 7.39913272429504e-07 | 0.00020591187783 |
| 1 | 3.00500401292401 | 3.00500199999933 | 2.01292467894021e-06 | 0.00000486550337 |

**Example 5.2 Assume the following third – order linear FDE [22]:**

$$\begin{cases} t^3\tilde{y}''' - 3t^2\tilde{y}'' + 6t\tilde{y}' - 6y = 0, t \geq 1. \\ \tilde{y}(1) = (r, 2-r), \\ \tilde{y}'(1) = (-1+r, 1-r), \\ \tilde{y}''(1) = (2+2r, 6-2r), \end{cases} \quad (16)$$

in [22] the accurate solution of equation is given by

$$\begin{cases} \underline{Y}(t,r) = (3+2r)t + (-5-2r)t^2 + (2+r)t^3, \\ \overline{Y}(t,r) = (7-2r)t + (-9+2r)t^2 + (4-r)t^3. \end{cases} \quad (17)$$

We use third-degree Bezier curves and four control points to consider the approximate solution for numerical execution.

We found the lower control points

$$\underline{a1} = 1.0r, \underline{a2} = 1.333333333r - 0.3333333333,$$

$$\underline{a3} = 2.0r - 0.3333333333,$$

$$\underline{a4} = 2.902777778r - 0.3888888889.$$

and the upper control points

$$\overline{a1} = 2.0 - 1.0r, \overline{a2} = 0.3333333333 - 1.333333333r,$$

$$\overline{a3} = 3.666666667 - 2.0r$$

$$\overline{a4} = 5.416666667 - 2.902777778r.$$

From the above control points the approximate solution function of BCM is obtained via Matlab 14a for all $r \in [0.1]$ is given bellow:

$$\underline{y}(t,r) = (-0.0972222222222222222222222222222222r - 0.388888888888888888888888888888889)t^3 + (1.0r+1.0)t^2 + (1.0r-1.0)t + 1.0r.$$

$$\overline{y}(t,r) = (0.0972222222222222222222222222222228r - 0.583333333333333333333333333333332)t^3 + (3.0 - 1.0r)t^2 + (1.0 - 1.0r)t - 1.0r + 2.0.$$

In Tables 3-4, the third-degree approximate solution by BCM is displayed and compared with the exact solution and third degree undetermined fuzzy coefficients method [22] for some $r \in [0.1]$ when at $t = 1.01$ as follows (table 3-4).

From Table 1-2, the third degree BCM approximate solution is more accurate than third degree, which undetermined fuzzy coefficients method [22] for some $r \in [0.1]$ when at $t = 1.01$ and presented as triangular fuzzy number as below in Figure 2.

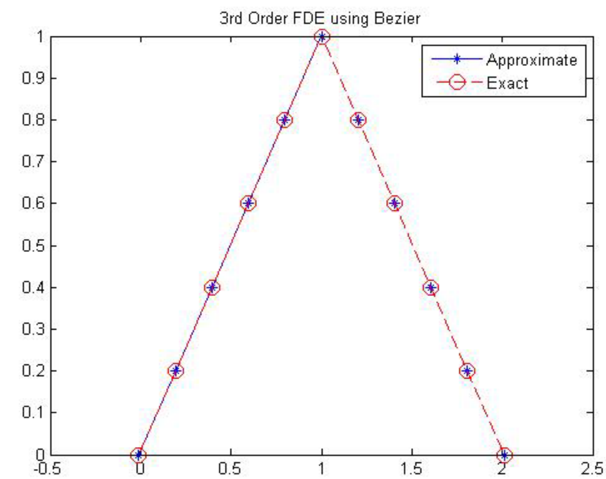


**Figure 2.** The accurate and the approximate solution of Eq. (16) at $t = 1.01$

**Table 3.** Comparisons between the lower solution for the accurate and the approximate solution of Eq. (16) at $t = 1.01$

| r | Approximate solution | exact solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | -0.00990038888888889 | -0.00989799999999974 | 2.38888888914879e-06 | 0.21760000006310$e-5$ |
| 0.2 | 0.192119591666667 | 0.1921222 | 2.60833333332045e-06 | 0.17408000001495$e-5$ |
| 0.4 | 0.394139572222222 | 0.3941424 | 2.82777777793619e-06 | 0.13055999996681$e-5$ |
| 0.6 | 0.596159552777778 | 0.5961626 | 3.04722222210785e-06 | 0.08704000005189$e-5$ |
| 0.8 | 0.798179533333333 | 0.7981828 | 3.26666666672359e-06 | 0.04352000000374$e-5$ |
| 1 | 1.00019951388889 | 1.000203 | 3.48611111133934e-06 | 0.02176000011289$e-5$ |

**Table 4.** Comparisons between the upper solution for the accurate and the approximate solution of Eq. (16) at $t = 1.01$

| r | Approximate solution | exact solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | 2.01029941666667 | 2.010304 | 4.58333333464012e-06 | 0.2176000001519200$e-5$ |
| 0.2 | 1.80827943611111 | 1.8082838 | 4.36388888913619e-06 | 0.174079999970540$e-5$ |
| 0.4 | 1.60625945555556 | 1.6062636 | 4.14444444540862e-06 | 0.13056000001122$e-5$ |
| 0.6 | 1.404239475 | 1.4042434 | 3.9249999999047e-06 | 0.08704000005189$e-5$ |
| 0.8 | 1.20221949444444 | 1.2022232 | 3.70555555617713e-06 | 0.04352000004815$e-5$ |
| 1 | 1.00019951388889 | 1.000203 | 3.48611111156139e-06 | 0.02176000002407$e-5$ |

**Example 5.3 Assuming the second– order linear FDE [22] as follow:**

$$\begin{cases} \tilde{y}'' + \tilde{y}' = -t, \ t \geq 1. \\ \tilde{y}(0) = (-0.1 + 0.1r, 0.1 - 0.1r) \\ \tilde{y}'(1) = (0.088 + 0.1r, 0.288 - 0.1r) \end{cases} \quad (18)$$

The exact solution of equation is

$$\begin{cases} \underline{Y}(t,r) = (-0.1 + 0.1r)cost + (1.088 + 0.1r)sint - t. \\ \overline{Y}(t,r) = (0.1 - 0.1r)cost + (1.288 - 0.1r)sint + t. \end{cases} \quad (19)$$

We use third-degree Bezier curves and four control points to consider the approximate solution for numerical execution.

We found the lower control points

$$\underline{a1} = 0.1r - 0.1, \underline{a2} = 0.133333333r - 0.0706666667,$$

$$\underline{a3} = 0.1482685978r - 0.02902658939,$$

$$\underline{a4} = 0.1383251334r - 0.1380542926.$$

and the upper control points

$$\overline{a1} = 0.1 - 0.1r, \overline{a2} = 0.196 - 0.1333333333r,$$

$$\overline{a3} = 0.2675106062 - 0.1482685978r,$$

$$\overline{a4}\ 0.1385959741 - 0.1383251334r.$$

From the above control points the approximate solution function of BCM is obtained via Matlab 14a for all $r \in [0.1]$ is given bellow:

$$\underline{y}(t,r) = (-0.0064806599914735738400027653274053 * r - 0.16297452442129762988397147103896)t^3 + (0.036920231826613971816704881955083 - 0.055194206639090207284333268040876r)t^2 + (0.1r + 0.088)t + 0.1r - 0.1.$$

$$\overline{y}(t,r) = (0.0064806599914735738400027653274032r - 0.17593584440424477756397700169376)t^3 + (0.055194206639090207284333268040881r - 0.073468181451566442751961654126675)t^2 + (0.288 - 0.1r)t - 0.1r + 0.1.$$

In Tables 5-6 the third-degree approximate solution by BCM is displayed and compared with the exact solution and the third degree undetermined fuzzy coefficients method [22] for some $r \in [0.1]$ when at $t = 0.01$ as follows (table 5-6).

From Table 5-6 the third degree BCM approximate solution is more accurate than the third degree, which undetermined fuzzy coefficients method [22] for some $r \in [0.1]$ when at $t = 1.01$ and presented as triangular fuzzy number as below in Figure 3:

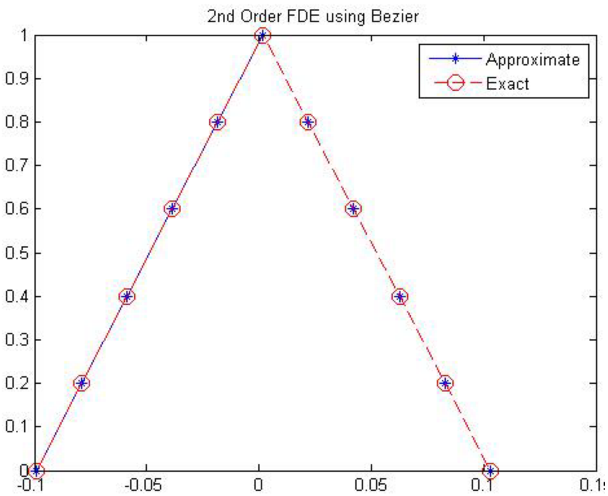


**Figure 3.** The accurate and the approximate solution of Eq. (18) at $t = 0.01$.

**Table 5.** Comparisons between the lower solution for the accurate and the approximate solution of Eq. (18) at $t = 0.01$

| r | Approximate solution | exact solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | -0.0991164709513418 | -0.0991151813740932 | 1.28957724855605e-06 | $0.11649297906799e-4$ |
| 0.2 | -0.0789175761316065 | -0.0789161846990765 | 1.39143252998741e-06 | $0.11107494923460e-4$ |
| 0.4 | -0.0587186813118713 | -0.0587171880240599 | 1.49328781139796e-06 | $0.10565691940079e-4$ |
| 0.6 | -0.0385197864921361 | -0.0385181913490433 | 1.5951430928085e-06 | $0.10023888956719e-4$ |
| 0.8 | -0.0183208916724009 | -0.0183191946740266 | 1.69699837424681e-06 | $0.09482085973359e-4$ |
| 1 | 0.00187800314733434 | 0.00187980200099 | 1.79885365565735e-06 | $0.08940282990000e-4$ |

**Table 6.** Comparisons between the upper solution for the accurate and the approximate solution of Eq. (18) at $t = 0.01$

| r | Approximate solution | exact solution | abs. error | abs. error [22] |
|---|---|---|---|---|
| 0 | 0.10287247724601 | 0.102874785376073 | 2.30813006280028e-06 | $0.62311660731923e-5$ |
| 0.2 | 0.0826735824262752 | 0.0826757887010566 | 2.2062747813828e-06 | $0.62311660731923e-5$ |
| 0.4 | 0.06247468760654 | 0.0624767920260399 | 2.1044194999445e-06 | $0.73147712399066e-5$ |
| 0.6 | 0.0422757927868047 | 0.0422777953510233 | 2.00256421854089e-06 | $0.78565738232741e-5$ |
| 0.8 | 0.0220768979670695 | 0.0220787986760066 | 1.90070893710259e-06 | $0.83983764066348e-5$ |
| 1 | 0.0018780031473343 | 0.00187980200099 | 1.79885365569551e-06 | $0.89401789899965e-5$ |

# 6. Conclusions

This paper demonstrates that the Bezier curve is a capable and effective methodology for dealing with high order fuzzy initial value problems including ordinary differential equations. Fuzzy sets properties of a general method structure were successfully introduced and evaluated in order to achieve an estimated solution to fuzzy differential equations involving high order FIVPs. The hypothesis of fuzzy linear differential equations using the Bezier curve representation has been strengthened by this research. The obtained results for the considered problems using the BCM method with an undetermined fuzzy coefficients method are presented in the form of tables and figures. The triangular fuzzy number properties have also been confirmed by the illustration of approximate solution. Overall, this paper demonstrates that the BMC can be used to solve nonlinear FIVPs and other types of fuzzy differential equations.

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