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No Finite Time Blowup for 3D Incompressible Navier Stokes Equations via Scaling Invariance

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Abstract The problem to The Clay Math Institute "Navier-Stokes, breakdown of smooth solutions here on an arbitrary cube subset of three dimensional space with periodic boundary conditions is examined. The incompressible Navier-Stokes Equations are presented in a new and conventionally different way here, by naturally reducing them to an operator form which is then further analyzed. It is shown that a reduction to a general 2D N-S system decoupled from a 1D non-linear partial differential equation is possible to obtain. This is executed using integration over n-dimensional compact intervals which allows decoupling. The operator form is considered in a physical geometric vorticity case, and a more general case. In the general case, the solution is revealed to have smooth solutions which exhibit finite-time blowup on a fine measure zero set and using the Prékopa-Leindler and Gagliardo-Nirenberg inequalities it is shown that for any non zero measure set in the form of cube subset of 3D there is no finite time blowup for the starred velocity for large dimension of cube and small δ. In particular vortices are shown to exist and it is shown that zero is in the attractor of the 3D Navier-Stokes equations.

Keywords Incompressible, Navier-Stokes, Blowup, Gagliardo-Nirenberg, Millenium Problem

1 Introduction

The question of whether the solutions to the 3-D- Incompressible N-S equations are globally regular or demonstrate finite time blowup has been a long going debate in mathematics and in general the scientific communities. The Millennium problem posed by the Clay Institute [1] is asking for a proof of one of the above conjectures. Seminal papers conducted by Jean Leray [2, 3, 4] proved that there exists a global (in time) weak solution and a local strong solution of the initial value problem when the domain is all of $\mathbb{R}^3$, that is solutions up to some finite T^* on an interval $[0,T^*]$. While specific cases have approached answers in unique cases, the question of whether there is a unique solution for all instants of time, (ie. a global unique solution) is presently open. It has been shown that there exists a unique global solution for the 2D plane-parallel N-S equations[5, 6]. While at first glance, the NSEs appear as a compact set of PDEs, the fascination with these Partial Differential Equations is only increased by the fact that the nonlinearity of the ensuing expanded equations, appear to be connected with notions of highly chaotic turbulence and vorticity[7, 8, 9]. Since the announcement of the Millennium Problem, several results have attempted to comment on the existence and uniqueness of the NSEs. One particularly recent result by Kyritsis noted that there existed indications towards establishing a regularity of solutions regarding the Euler Equations and NSEs more generally; this utilized the conservation of particles[10].In the present work, to the best of the authors' knowledge, the procedure revealed here has not been previously observed in the literature on the question of Incompressible N-S 3-D existence of unique global solutions, except for compressible flows in [11, 12, 13]. First, a cube in $\mathbb{R}^3$ with boundary conditions that generate a vortex is considered , and an attempt has been made to naturally reduce the 3-D incompressible NSEs to a one component decoupled velocity field solution under scale invariant transformations, with a separate 2-component velocity field solution. For the variable z-component, in particular, a form of solution is extracted in the analysis presented using the divergence form of Green's identity, Ostogradsky's theorem. The decomposition method using Geometric Algebra is used together with a main result that a bound for $\|u_z\|_\infty$ implies bounds for all derivatives. [14] [18]

2 Model

The 3D incompressible unsteady Navier-Stokes Equations (NSEs) in Cartesian coordinates may be listed below in compactified form for the velocity field $\mathbf{u}^* = u^{*i}\vec{\mathbf{e}}_i$, $u^{*i} = \{u_x^*, u_y^*, u_z^*\}$:

$$\rho\left(\frac{\partial}{\partial t^*} + u^{*j}\nabla_{*j}\right)u_i^* - \mu\nabla_*^2 u_i^* + \nabla_{*i}P^* = \rho F_i^* \tag{1}$$

where ρ is constant density, μ is dynamic viscosity , $\mathbf{F}^* = F^{*i}\vec{\mathbf{e}}_i$ are the body forces on the fluid. In some cases, it may be elected to reparametrize the components of the velocity vector, and pressure to $\mathbf{u} = (u)^i\vec{\mathbf{e}}_i$, $\mathbf{P} = (P)^i\vec{\mathbf{e}}_i$, coordinates $\mathbf{x}_i$ and time t according to the following form utilizing the non-dimensional quantity δ(assumed negative):

$$u_i^* = \frac{1}{\delta}u_i \ , \ P_i^* = \frac{1}{\delta^2}P_i \ , \ x_i^* = \delta x_i \ , \ t^* = \delta^2 t \tag{2}$$

The continuity equation in Cartesian co-ordinates, is

$$\nabla^i u_i = 0 \tag{3}$$

2.1 Data

Eq.(1), together with Eq.(3) and using the initial condition of $\vec{u}^*(\vec{x}^*, 0) = \vec{\xi}(\vec{x}^*)$ such that $\nabla \cdot \vec{\xi} = 0$ encompass the NSEs along with an incompressible initial condition. Ensuring periodic boundary conditions specified in [1] defined on a cube domain Ω in $\mathbb{R}^3$ is referred to as the periodic BVP for the NSEs in $\mathbb{R}^3$. See Fig. 1 below for lattice geometry of flow problem. I take the cube to be centred at the origin.

3 Application

Using Eq.2 above, multiplying the first two components of scale invariant Eq.1 by Cartesian unit vectors $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ respectively and adding modified equations within the set Eq.1 give the following equations, for the resulting composite vector $\vec{b} = \frac{1}{\delta}u_x\vec{i} + \frac{1}{\delta}u_y\vec{j}$,

$$\begin{aligned} &\tfrac{1}{\delta^2}\tfrac{\partial \vec{b}}{\partial t} + \tfrac{u_x}{\delta^2}\tfrac{\partial \vec{b}}{\partial x} + \tfrac{u_y}{\delta^2}\tfrac{\partial \vec{b}}{\partial y} - \tfrac{\mu}{\rho\delta^2}\nabla^2\vec{b} \\ &+\tfrac{1}{\delta^3\rho}\tfrac{\partial P}{\partial x}\vec{i} + \tfrac{1}{\delta^3\rho}\tfrac{\partial P}{\partial y}\vec{j} = \vec{F}_T \end{aligned} \tag{4}$$

where $\vec{F}_T = \frac{1}{\delta^2}$. Multiplying Eq. 4 by δ^3 and by u_z^* and, the z^* component of Eq.1 by δ^3 and by $\vec{b}$,(again using Eq.2), the addition of the resulting equations [11, 12, 13] recalling the product rule, produces a form as displayed below in Eq.5, where $\vec{a} = u_z\vec{b}$. The nonlinear inertial term when added to $\vec{b}\nabla u_z \cdot \vec{b}$ and factoring out $\vec{b}$ gives, $\vec{b} \cdot \nabla\vec{a}$. Here $\nabla u_z\vec{b}$ is a dyadic.

$$\begin{aligned} &\tfrac{\partial \vec{a}}{\partial t} + \vec{b}\cdot\nabla\vec{a} - \tfrac{\mu}{\rho}u_z\nabla^2\vec{b} + \tfrac{1}{\delta}\tfrac{1}{\rho}u_z\nabla_{xy}P \\ &+\vec{b}\tfrac{1}{\rho}\tfrac{\partial P}{\partial z} + \delta^2 u_z\vec{F}_T + \delta^3 F_z\vec{b} - \tfrac{\mu}{\rho}\vec{b}\nabla^2 u_z = 0 \end{aligned} \tag{5}$$

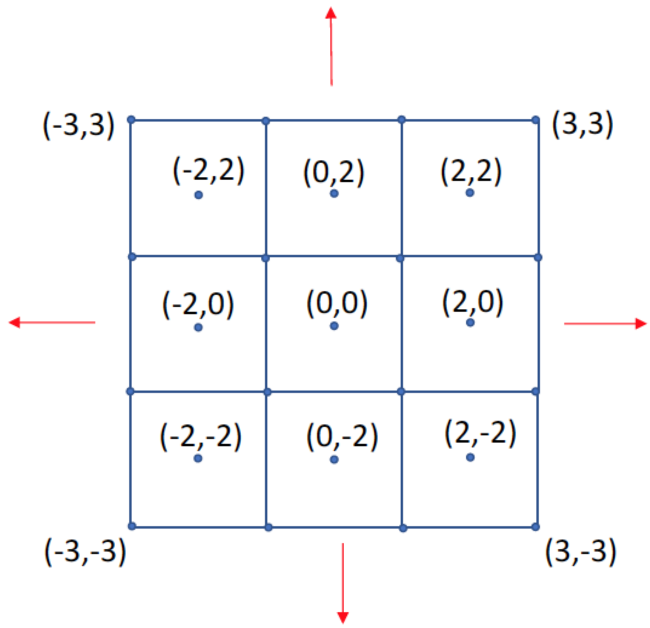


Figure 1. Vortex generation in a 2-D Lattice.

Multiplying Eq.5 by u_z and adding to it $u_z\vec{b}\nabla\cdot\vec{a}$ gives:

$$\begin{aligned} &u_z\tfrac{\partial \vec{a}}{\partial t} + \vec{a}\cdot\nabla\vec{a} - \tfrac{\mu}{\rho}u_z^2\nabla^2\vec{b} + \tfrac{1}{\delta\rho}u_z^2\nabla_{xy}P + \\ &\vec{b}\tfrac{1}{\rho}u_z\tfrac{\partial P}{\partial z} + \delta^2 u_z^2\vec{F}_T + \delta^3 u_z F_z\vec{b} + \\ &u_z\vec{b}\nabla\cdot\vec{a} - \tfrac{\mu}{\rho}u_z\vec{b}\nabla^2 u_z = 0 \end{aligned} \tag{6}$$

At this point the z component of the external force, F_z, is assumed to be offset exactly by F_{T_1}, F_{T_2}. In this paper Eq.6 is solved, instead of Eq.5. For the $\varepsilon-$periodic solution, it is proposed that integration of divergence or curl of Eq.6 over an arbitrary small volume is equivalent to integration of divergence or curl of Eq.5 for the same volume. That is the extra term's divergence or curl in Eq.6 when integrated is negligible on set of measure zero. A necessary condition that the form of Eq. 5, call it $L_1 = 0$, is that both the divergence and curl of L_1 be zero and upon integrating over a volume U, we have the function of t which we call $C_s(t)$ which consists of the force term F_T. The same is true for the non $\varepsilon-$ periodic case where the justification of using Eq 6 instead of 5 will be based on the periodicity of the flow on an interval in $\mathbb{R}^3$. It can be seen that for the general non zero measure set, upon separately taking the divergence and curl of Eq. 6 and integrating over the volume U and using first Ostogradsky's theorem and the fact that the velocities are periodic on the interval in $\mathbb{R}^3$, we have that the extra term $u_z\vec{b}\nabla\cdot\vec{a} = \vec{a}\nabla\cdot\vec{a}$ has an integral of it's divergence equal to zero. Proof is straightforward upon taking the divergence and integrating. Next for the curl of the same term and integrating over the same volume we use the fact that $\int_U \nabla\times\vec{F}\ dV = \iint_{\partial U}\vec{n}\times\vec{F}\ dS$. This contribution is also zero due to periodicity of velocities on interval.

Theorem 1. *Reduced structure form of 3-D Navier-Stokes Equations*

The 3-D Incompressible Navier Stokes equations can be reduced to a simple form as,

$$\mathscr{G} = \frac{\vec{f}}{\vec{f}\cdot\vec{b}}\cdot\boldsymbol{Q}$$

where $\mathscr{G}$ is the nonlinear partial differential operator given by Eq.12, $\boldsymbol{Q}$ is the two dimensional Navier-Stokes operator acting on the vector $\vec{b}$ defined in the brackets of Eq.17 and $\vec{f} = \vec{a}\cdot\nabla\vec{a}$. The terms Ω_i contained in $\mathscr{G}$ are defined after Eq.10 and are part of the proof of present Theorem 1.

Proof. Taking the geometric product with the inertial vector term in the previous equation Eq.6 given by $\vec{f} = \vec{a}\cdot\nabla\vec{a}$, it can be shown that in the context of Geometric Algebra [11, 13], the following scalar and vector grade equations arise:

SCALAR

$$\begin{gathered}\vec{f}\cdot\left(u_z^2\frac{\partial\vec{b}}{\partial t}+u_z\vec{b}\frac{\partial u_z}{\partial t}\right)+\left\|\vec{f}\right\|^2+u_z(\vec{f}\cdot\vec{b})\nabla\cdot\vec{a}=\\ \frac{\mu}{\rho}u_z^2\vec{f}\cdot\nabla^2\vec{b}-\frac{1}{\delta\rho}u_z^2\vec{f}\cdot\nabla_{xy}P-\\ \vec{f}\cdot\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z}+\frac{\mu}{\rho}u_z\nabla^2u_z(\vec{f}\cdot\vec{b})+\vec{f}\cdot(\text{Force Terms})\end{gathered} \tag{7}$$

VECTOR

$$\begin{gathered}u_z^2\frac{\partial\vec{b}}{\partial t}+\vec{a}\frac{\partial u_z}{\partial t}+u_z\vec{b}\nabla\cdot\vec{a}=\frac{\mu}{\rho}u_z^2\nabla^2\vec{b}-\\ \frac{1}{\delta\rho}u_z^2\nabla_{xy}P-\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z}+\\ \frac{\mu}{\rho}u_z\vec{b}\nabla^2u_z+\text{Force Terms}\end{gathered} \tag{8}$$

Taking the divergence of the vector equation Eq.8, recalling the product rule, and defining the new term $H = (u_z\vec{b}\cdot\vec{f})/(\partial_t u_z)$(After taking divergence multiply new equation by H), results in an expression which may be combined with the usage of the scalar equation Eq.7 to produce:

$$\begin{gathered}H\nabla u_z^2\cdot\frac{\partial\vec{b}}{\partial t}+u_z^2H\frac{\partial}{\partial t}(\nabla\cdot\vec{b})+\\ H\vec{a}\cdot\nabla(\frac{\partial u_z}{\partial t})+H\nabla\cdot(\vec{b}u_z\nabla\cdot\vec{a})-\frac{\mu}{\rho}H\nabla\cdot(u_z^2\nabla^2\vec{b})+\\ H\nabla\cdot(\frac{1}{\delta\rho}u_z^2\nabla_{xy}P+\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z})-\\ H\frac{\mu}{\rho}u_z\nabla^2u_z\nabla\cdot\vec{b}-H\frac{\mu}{\rho}\nabla(u_z\nabla^2u_z)\cdot\vec{b}+H\text{ Div of Force Terms}=\\ \vec{f}\cdot\left(u_z^2\frac{\partial\vec{b}}{\partial t}+u_z\vec{b}\frac{\partial u_z}{\partial t}-\frac{\mu}{\rho}u_z\vec{b}\nabla^2u_z\right)+\\ \left\|\vec{f}\right\|^2-\frac{\mu}{\rho}u_z^2\vec{f}\cdot\nabla^2\vec{b}+\frac{1}{\delta\rho}u_z^2\vec{f}\cdot\nabla_{xy}P+\frac{1}{\rho}\vec{f}\cdot\vec{b}u_z\frac{\partial P}{\partial z}+\vec{f}\cdot\text{Force Terms}\end{gathered} \tag{9}$$

Continuing with the previous paragraph we use the common term $u_z\left(\vec{f}\cdot\vec{b}\right)\nabla\cdot\vec{a}$ appearing in Eq.7,and in the new equation where we took divergence of Eq.8 and multiplied by H. This term in Eq.8 is $\vec{a}\frac{\partial u_z}{\partial t}$. Upon a division of the preceding equation 9 by u_z^2H, it can be seen to result in the general form:

$$\begin{gathered}\Omega_5+\frac{\partial Y}{\partial t}-G+\Omega_1-\\ \frac{\vec{f}}{\vec{b}\cdot\vec{f}}\cdot\left(F\vec{f}+\frac{1}{\delta\rho}\frac{\partial u_z}{\partial t}u_z^{-1}\nabla_{xy}P+\frac{\partial u_z}{\partial t}u_z^{-1}\frac{\partial\vec{b}}{\partial t}-\right.\\ \left.\frac{\mu}{\rho}\frac{\partial u_z}{\partial t}u_z^{-1}\nabla^2\vec{b}\right)-\Omega_3+\Omega_4+\\ \Omega_6+\Omega_7-\Omega_8+u_z^{-2}\nabla\cdot(\frac{1}{\delta\rho}u_z^2\nabla_{xy}P+\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z})-\\ u_z^{-2}\frac{\mu}{\rho}\nabla\cdot\left(u_z\vec{b}\nabla^2u_z\right)+\text{Force terms}=0\end{gathered} \tag{10}$$

where for brevity, the following symbols have been defined:

$$\begin{gathered}\Omega_1=u_z^{-2}\vec{a}\cdot\nabla\left(\frac{\partial u_z}{\partial t}\right),\Omega_3=u_z^{-2}\frac{\mu}{\rho}\nabla\cdot\left(u_z^2\nabla^2\vec{b}\right),\\ \Omega_4=u_z^{-2}\nabla\cdot\left(\vec{b}u_z\nabla\cdot\vec{a}\right),\Omega_5=u_z^{-2}\nabla u_z^2\cdot\frac{\partial\vec{b}}{\partial t},Y=\nabla\cdot\vec{b}\\ F=u_z^{-3}\frac{\partial u_z}{\partial t}\ ,\ G=\frac{1}{u_z^2}\left(\frac{\partial u_z}{\partial t}\right)^2,\\ \Omega_6=\frac{\mu}{\rho}\frac{\partial u_z}{\partial t}u_z^{-2}\nabla^2u_z,\Omega_7=u_z^{-2}\nabla\cdot\left(\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z}\right),\\ \Omega_8=\frac{1}{\rho}u_z^{-2}\frac{\partial u_z}{\partial t}\frac{\partial P}{\partial z}\end{gathered}$$

The divergence of $\vec{F_T}$ is assumed to be zero. It can be seen that the expression beginning with parentheses may be abbreviated into a nonlinear vector operator **Q** and so Eq.10 can be written compactly as:

$$\mathscr{G}(u_z,\vec{b})=\frac{\vec{f}\cdot\mathbf{Q}(u_z,\frac{\partial u_z}{\partial t},\vec{b},\vec{f},\frac{\partial\vec{b}}{\partial t},\nabla_{xy}P,\vec{F_T})}{\vec{f}\cdot\vec{b}} \tag{11}$$

where $\mathscr{G}$ is the non-linear operator associated with remaining part of Eq.10. Utilizing the continuity Eq.3, it can be seen that the operator $\mathscr{G}$ is given by the following expression:

$$\begin{gathered}\mathscr{G}(u_z,\vec{b})=\frac{\partial}{\partial t}\left(-\frac{\partial u_z}{\partial z}\right)+\Omega_1-\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7-\\ \Omega_8-G+\frac{1}{u_z^2}\nabla\cdot(\frac{1}{\rho}u_z^2\nabla_{xy}P+\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z})=\frac{\|\mathbf{Q}\|}{\left\|\vec{b}\right\|}\end{gathered} \tag{12}$$

The nonlinear operator form of the NSEs presented is:

$$\mathscr{G}=\frac{\vec{f}}{\vec{f}\cdot\vec{b}}\cdot\mathbf{Q} \tag{13}$$

This completes the proof. □

It is observed that Eq.(13) can be expressed as,

$$\mathscr{G}=\frac{\vec{f}}{\vec{f}\cdot\vec{b}}\cdot\mathbf{Q}=\frac{\|\mathbf{Q}\|}{\left\|\vec{b}\right\|}$$

An important observation is that in Eq.12,

$$\begin{gathered}\Omega_1+\Omega_6-\Omega_8-G=(\frac{1}{\delta}-1)\frac{1}{u_z^2}\left(\frac{\partial u_z}{\partial t}\right)^2+\\ \frac{\mu}{\rho}\frac{1}{u_z^2}(1-\frac{1}{\delta})\frac{\partial u_z}{\partial t}\nabla^2u_z+\frac{1}{\rho u_z^2}(\frac{1}{\delta}-1)\frac{\partial u_z}{\partial t}\frac{\partial P}{\partial z}\end{gathered} \tag{14}$$

Equation 13 displays a general form which may be expanded and analyzed by allowing a geometric assumption to be undergone, or the general case may also be considered.

4 Two Cases

4.1 The Geometric Case

As a special case, one may consider the case where:

$$\vec{f}\cdot\mathbf{Q}=0 \tag{15}$$

This condition means that the Lie Product of the velocity inertia is entirely perpendicular to the Force terms, and thus refers to a vortex fluid scenario. This condition automatically implies $\mathscr{G}=0$ and so,

$$\mathscr{G}(u_z,\vec{b})=\frac{\partial}{\partial t}\left(\frac{\partial u_z}{\partial z}\right)+\Omega_1-\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7-\Omega_8- G+\frac{1}{u_z^2}\nabla\cdot\left(\frac{1}{\delta\rho}u_z^2\nabla_{xy}P+\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z}\right)=0 \tag{16}$$

4.2 The General Case

For $\mathbf{Q}$, the expressions with u_z and it's derivative with respect to t factor out with the exception of tensor product of velocity inertial term in $\vec{b}$ and ∇u_z. Thus $\vec{Q}$ is:

$$\frac{u_z\frac{\partial u_z}{\partial t}}{u_z^2}\times\left[\frac{\partial\vec{b}}{\partial t}+\vec{b}\cdot\nabla\vec{b}-\frac{\mu}{\rho}\nabla^2\vec{b}+\frac{1}{\delta\rho}\nabla_{xy}P-\delta^2\vec{F_T}+\frac{1}{u_z}\vec{b}\cdot\left(\vec{b}\otimes\nabla u_z\right)\right]=\frac{1}{u_z^2}\frac{\partial u_z}{\partial t}\vec{b}\cdot\left(\vec{b}\otimes\nabla u_z\right) \tag{17}$$

The expression in brackets in Eq.17,(for $z^*\in\mathbb{R}$, consists of the 2-D "plane-parallel" Navier-Stokes Equations and it is well known that if all data of problem are independent of one of x,y,z,then the BVP in Eqs.1, Eqs.3, has a unique solution for all instants of time with no restrictions on smallness of $\vec{F_T}$, $\vec{\xi}$ or the domain Ω.[6]. As a result $\mathbf{Q}=\frac{1}{u_z^2}\frac{\partial u_z}{\partial t}\vec{b}\cdot\left(\vec{b}\otimes\nabla u_z\right)$.

Using Green's identity, for divergence, also known as Ostogradsky's Theorem, it is known that for an arbitrary vector field $\vec{\Gamma}$ and scalar field ψ, the following identity holds:

$$\int_U\left(\psi\nabla\cdot\vec{\Gamma}+\vec{\Gamma}\cdot\nabla\psi\right)dV=\oint_{\partial U}\psi(\vec{\Gamma}\cdot\vec{n})dS \tag{18}$$

Eq.16 can be expressed due to integration on a general rectangular volume U as:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial u_z}{\partial z}\right)+\Omega_5+\left(\frac{1}{\delta}-1\right)\frac{1}{u_z^2}\left(\frac{\partial u_z}{\partial t}\right)^2+\frac{1}{u_z^2}\left(u_z\frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial t}\right)+\\&\frac{\mu}{\rho}\frac{1}{u_z^2}\left(1-\frac{1}{\delta}\right)\frac{\partial u_z}{\partial t}\nabla^2u_z+\left(\frac{1}{\delta}-1\right)\frac{1}{\rho}\frac{1}{u_z^2}\frac{\partial u_z}{\partial t}\frac{\partial P}{\partial z}\\&+\frac{1}{u_z^2}\iint_S\left(\frac{1}{\delta\rho}u_z^2\nabla_{xy}P+\vec{b}\frac{1}{\rho}u_z\frac{\partial P}{\partial z}\right)\cdot\vec{n}\,dS+\\&\delta^2\frac{1}{u_z^2}\vec{F_T}\cdot\nabla u_z^2-\frac{1}{u_z^2}\delta^3u_z\frac{\partial u_z}{\partial t}\frac{\partial u_z}{\partial z}F_z+\\&\delta^3\vec{b}\cdot\nabla(u_zF_z)=\frac{1}{\left\|\vec{b}\right\|u_z^2}\int_U\left\|\frac{\partial u_z}{\partial t}\vec{b}\cdot\left(\vec{b}\otimes\nabla u_z\right)\right\|dV\end{aligned} \tag{19}$$

Note that the surface integral in Eq.18 is zero since it is taken over six faces of a general rectangular volume, and $\vec{b}$ is a periodic vector field with normals pointing in opposite directions on opposite sides. Here $\psi=u_z$ whose surface integral is assumed to be equal on the four vertical faces of cube and not equal between the top and bottom face. Also Ω_1 is zero using Eq.18 and continuity Eq.3, Ω_3 vanishes using Eq.18 as well due to periodic boundary conditions on the cube's surface. The pressure is expressed using the divergence theorem as a surface integral over the surface. In addition Ω_4 and Ω_7 vanish using the divergence theorem and the fact that $\vec{b}$ is periodic on the surface of U. The velocities u_x and u_y satisfy the 2-D Navier Stokes equations and are given in [20] as,

$$u_x=\sin(4t)x+\big(\cos(4t)+2\big)y$$

$$u_y=\big(\cos(4t)-2\big)x-\sin(4t)y$$

The velocity is a kinematic benchmark example for testing vortex criteria. Solving for u_z in Eq 19 (Maple) gives a class of solutions$\{F_6\}_i$.(See results in Fig 2) Here the defining equations are in terms of the vorticity components in 3D. Since u_x and u_y are z-independent, the vorticity component equations for a constant vorticity are, $\frac{\partial u_z}{\partial x}=-C$, $\frac{\partial u_z}{\partial y}=C$ and $\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}=C$. Setting this equality, $\frac{\partial u_z}{\partial y}=\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}$ gives a differential equation in terms of F_6 the unknown form of solution.

Lemma 1 (Maximum Principle For Eq.19)**.** *In this lemma we prove a maximum principle for Eq.19 in the general case where there are no restrictions on Φ and the most general Gagliardo [15]-Nirenberg [16] inequality (see also [17]) is used in this lemma and can be stated as follows, Let $1\le q\le\infty$ and $j,k\in\mathbb{N}$, $j<k$, and either*

$$\begin{cases}r=1\\ \frac{j}{k}\le\theta\le1\end{cases}$$

or

$$\begin{cases}1<r<\infty\\ k-j-\frac{n}{r}=0,1,2,...\\ \frac{j}{k}\le\theta<1\end{cases}$$

If we set

$$\frac{1}{p}=\frac{j}{n}+\theta\left(\frac{1}{r}-\frac{k}{n}\right)+\frac{1-\theta}{q},$$

then there exists a constant C independent of u such that ,

$$\left\|\nabla^ju\right\|_p\le C\left\|\nabla^ku\right\|_r^\theta\|u\|_q^{1-\theta} \tag{20}$$

Suppose that $\frac{\partial P}{\partial z}=-\mid\Phi\mid$, for a general function Φ, $\mid\Omega\mid$ is volume of arbitrarily small positive measure of the cube interval, then u_z has no blowup in Ω, and in general for any side length of cube there is no blowup, We choose δ small with the x's,y's and z's large so that the starred spatial variables in Eq(2) are finite and increasing.

Proof. Using the z momentum equation again and multiplying by $C\|u_z\|_q^{1-\theta}$, for the case,

$$\begin{cases} r=1 \\ \frac{j}{k} \le \theta \le 1 \end{cases}$$

we obtain for $j=1$, $k=2$, with $q=\infty$ and $\delta \to 0$,

$$\begin{aligned}
&C \mid \Omega \mid \|u_z\|_q^{1-\theta} \sup_{x\in\Omega} \mid \tfrac{\partial u_z}{\partial t} \mid \\
&\ge C \int_\Omega \left(\|u_z\|_q^{1-\theta} \mid \tfrac{\partial u_z}{\partial t} \mid \right) dv \\
&\ge C \int_\Omega \left(\|u_z\|_q^{1-\theta} \tfrac{\partial u_z}{\partial t} \right) dv \\
&= C \int_\Omega \|u_z\|_q^{1-\theta} \left(\nabla^2 u_z + \mid \Phi \mid - u_z \tfrac{\partial u_z}{\partial z} \right) dv \\
&= C \int_\Omega \|u_z\|_q^{1-\theta} \left(\nabla^2 u_z + \mid \Phi \mid - \vec{u} \cdot \nabla u_z \right) dv \\
&> C\|u_z\|_q^{1-\theta} \int_\Omega \nabla^2 u_z dv \\
&\ge C\|u_z\|_q^{1-\theta} \left[\mid \Omega \mid^{\frac{1}{2}} \left\| \nabla^2 u_z \right\|_r^{\theta} \right] \\
&\ge \mid \Omega \mid^{\frac{1}{2}} \|\nabla u_z\|_2
\end{aligned} \tag{21}$$

where in the last two lines of inequality 21 the Gagliardo-Nirenberg [15],[16] inequality, has been used. To go from line 5 to line 6 the term $\vec{b}\cdot\nabla u_z = 0$ on the surface in Fig. 2. Also as $q\to\infty$, $p\to 2$. To go from line 6 to 7 of inequality 21 the Prékopa-Leindler inequality has been used. See theorem 7.1, page 367 of [21] for the special case of $\lambda = \frac{1}{2}$. In inequality (21), $\frac{\partial u_z}{\partial z} = 0$, using the 2-D formulas for u_x and u_y where I write the partial of u_z wrt z in terms of the negative of the divergence of $\vec{b}$. Next $\vec{b}\cdot\nabla u_z = C(u_y - u_x)$ which is in general not equal to zero and is bounded on the finite cube set Ω. Inequalities follow by choosing the sup of $|\Phi|$ on Ω larger than this term. I abbreviate to the following expression involving the pressure and velocity terms,

$$\Psi_1 = \iint_S \left(\frac{1}{\delta\rho} u_z^2 \nabla_{xy} P + \vec{b}\frac{1}{\rho} u_z \frac{\partial P}{\partial z}\right)\cdot n dS$$

Let,

$$\begin{aligned} L_4 = \tfrac{\mu}{\rho}(1-\tfrac{1}{\delta})\tfrac{\partial u_z}{\partial t}\nabla^2 u_z + (\tfrac{1}{\delta}-1)\tfrac{1}{\rho}\tfrac{\partial u_z}{\partial t}\tfrac{\partial P}{\partial z} + \\ \delta^2 \vec{F}_T\cdot\nabla u_z^2 - \delta^3 u_z \tfrac{\partial u_z}{\partial t}\tfrac{\partial u_z}{\partial z} F_z + \delta^3 \vec{b}\cdot\nabla(u_z F_z) \end{aligned} \tag{22}$$

Next it follows using 14, 19 and 21 where in 21 the start at the third line of the inequality is considered and gives ,

$$\begin{aligned}
&|\Omega| \|\nabla u_z\|_2^2 \le \\
&C^2 \left(\int_\Omega \|u_z\|_\infty^{\frac{1}{2}} \left(L_1 + L_2 - L_3 + L_4 \right.\right. \\
&\left.\left. + \tfrac{1}{\delta}\left(\tfrac{\partial u_z}{\partial t}\right)^2 + \Psi_1 + C_s(t) \right)^{1/2} dv \right)^2 \\
&\le C^2 \|u_z\|_\infty \times \\
&\int_\Omega \left(L_1 + L_2 - L_3 + L_4 + \tfrac{1}{\delta}\left(\tfrac{\partial u_z}{\partial t}\right)^2 + \Psi_1 + C_s(t) \right) dv
\end{aligned} \tag{23}$$

Multiplying by negative one gives for constants $N \ge 0$, $M \ge 0$,

$$\begin{aligned}
&M \ge N - \|\nabla u_z\|_2^2 \ge -\|\nabla u_z\|_2^2 \ge C^2 |\Omega|^{-1} \times \\
&\|u_z\|_\infty \int_\Omega \left(-L_1 - L_2 + L_3 - L_4 - \tfrac{1}{\delta}\left(\tfrac{\partial u_z}{\partial t}\right)^2 - C_s(t) \right) dv - \\
&\int_\Omega \Psi_1 dv = \|u_z\|_\infty \times \\
&\left[C^2 |\Omega|^{-1} \int_\Omega \left(-L_1 - L_2 + L_3 - L_4 - \tfrac{1}{\delta}\left(\tfrac{\partial u_z}{\partial t}\right)^2 - C_s(t) \right) dv - \right. \\
&\left. C^2 |\Omega|^{-1} \Psi_1(t) \right]
\end{aligned} \tag{24}$$

where $C_s(t) = \int_\Omega \delta^2 \nabla u_z^2 \cdot \vec{F} dV$, $L_1 = u_z^2 \frac{\partial^2 u_z}{\partial t \partial z}$, $L_2 = u_z \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial t}$ and $L_3 = \frac{\partial u_z}{\partial t}\vec{b}\cdot\left(\vec{b}\otimes\nabla u_z\right)$ and $-\Psi_1(t) \ge 0$

$$\begin{aligned}
&C^2 |\Omega|^{-1} \|u_z\|_\infty \times \\
&\int_\Omega \left(-L_1 + L_2 + L_3 + L_4 - \tfrac{1}{\delta}\left(\tfrac{\partial u_z}{\partial t}\right)^2 - C_s(t) \right) dv - \\
&C^2 |\Omega|^{-1} \|u_z\|_\infty \Psi_1(t) \le M
\end{aligned} \tag{25}$$

Here ∇u_z is bounded if $\|u_z\|_\infty$ is bounded [14],[18],
The gradient of pressure is negative in order to drive the flow, so that $-\Psi_1 > 0$. In the expression for Ψ_1 there is the vector $\vec{b} = (\frac{u_x \vec{i}}{\delta} + \frac{u_y \vec{j}}{\delta})$, so when it is multiplied by $-\delta$, u_x and u_y remain. Here in Ψ_1, $\frac{\partial P}{\partial z}$ is negative. There is a positive term on left side of inequality 25 since each term in square brackets is greater than or equal to $M=0$ which is true for δ small enough. In this case it becomes an equality and expression for Ψ_1 is possible to obtain. The remaining steps are to use the condition which replaces the pressure terms by u_z and derivative terms in the equation deduced by Inequality 25,

Lemma 2. *Considering starred variables for t, spatial variables and* ∇ *if,*

$$K = \frac{\partial u_z}{\partial t^*}(u_x, u_y)\cdot\left((u_x, u_y)\otimes\nabla^* u_z\right) \tag{26}$$

is Lebesgue integrable on Ω *and* Ψ_1^* *is the transformed surface integral in* Ψ_1 *(by Eq 2), then it follows that pressure terms can be replaced by transformed* L_3*, that is,*

$$\Psi_1^* = \left[\int_\Omega \frac{\partial u_z}{\partial t^*}(u_x, u_y)\cdot\left((u_x, u_y)\otimes\nabla^* u_z\right) dv\right]^* \tag{27}$$

Proof. Transforming t, spatial variables and ∇ to star variables by Eq.(2) it follows that using Eq.(25) for the L_3 term, that is the term

$$C^2 |\Omega|^{-1} \|u_z\|_\infty \int_\Omega L_3 dv = \mathscr{O}(\delta^6) \times \text{Star Variables} \tag{28}$$

Then transforming to star variables for Ψ_1 which contains the pressure term P it follows that,

$$C^2 |\Omega|^{-1} \|u_z\|_\infty \Psi_1(t) = \mathscr{O}(\delta^6) \times \text{Star Variables} \tag{29}$$

Transforming L_4 and the rest of the terms in equation derived from inequality 25 it can be shown these are $\mathscr{O}(\delta^5)$ (L_1 and L_2

cancel each other by integration by parts) Dividing by δ^5 and letting $\delta \to 0$ it follows that two terms, that is L_3^* and Ψ_1^* go to zero through δ. It now follows that, the conclusion of Lemma 2 follows that is

$$\Psi_1^* = [\int_\Omega \frac{\partial u_z}{\partial t^*}(u_x, u_y) \cdot \left((u_x, u_y) \otimes \nabla^* u_z\right) dv]^* \tag{30}$$

□

Next It is possible to obtain Poisson's equation, that is,

$$\frac{1}{\rho} \triangle \left(P + \frac{1}{2}\rho \|\vec{u}\|^2 \right) = \left\|\vec{\omega}\right\|^2 - \vec{u} \cdot \left(\nabla \times \vec{\omega} \right) \tag{31}$$

where the Vorticity and Enstrophy are $\vec{\omega}$ and $\nabla \times \vec{\omega}$ respectively. The result follows by taking the divergence of the full Navier Stokes equations and using the continuity equation repeatedly. Also the vector calculus identity, $(\vec{u} \cdot \nabla)\vec{u} = \frac{1}{2}\nabla\|\vec{u}\|^2 - \vec{u} \times \vec{\omega}$, is used together with the identity, $\nabla \cdot (\vec{u} \times \vec{\omega}) = \vec{\omega} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{\omega}) = \left\|\vec{\omega}\right\|^2 - \vec{u} \cdot (\nabla \times \vec{\omega})$, where $\vec{\omega} = \nabla \times \vec{u}$ has been used repeatedly to obtain,

$$\nabla \cdot ((\vec{u} \cdot \nabla)\vec{u}) = \frac{1}{2} \triangle \|\vec{u}\|^2 - \left\|\vec{\omega}\right\|^2 + \vec{u} \cdot (\nabla \times \vec{\omega})$$

Next the velocity inside the boundary layers in the cube (region named Ω_2) is taken to be $\vec{u} = (u_x(x,y,z,t), u_y(x,y,z,t), u_z(x,y,z,t))$. It can be shown that the Enstrophy $\vec{E}$ is equal to,

$$\vec{E} = (\frac{\partial^2 u_z}{\partial z \partial x}, \frac{\partial^2 u_z}{\partial z \partial y}, -\nabla_{xy}^2 u_z)$$

Substituting into Poisson's equation, gives,

$$\frac{1}{\rho} \triangle P = -\triangle(\frac{1}{2}u_z^2) + \left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2 - u_z \nabla_{xy}^2 u_z$$

which reduces to,

$$\frac{1}{\rho} P_{zz} = -\triangle(\frac{1}{2}u_z^2) + \left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2 - u_z \nabla_{xy}^2 u_z - \left(\frac{\partial^2}{\partial z \partial t} uz + \frac{\partial^3}{\partial z^3} uz\right)$$

in Ω_2. Using Eq.30 to eliminate pressure(δ arbitrarily small) and substituting into Eq.19 it is noted that P_z still occurs in the equation. Solving for it algebraically and then differentiating the resulting equation wrt z gives P_{zz}. First $P_{xx} + P_{yy} = \frac{\partial^2}{\partial z \partial t} uz(x,y,z,t) + \frac{\partial^3}{\partial z^3} uz(x,y,z,t)$, where this has been derived by using the continuity Eq.3 and the Prandtl Equation, Eq.(1.1) in [19] where u_x and u_y are of order δ in the boundary layer. (Recall $b_1 = \frac{u_x}{\delta}$ and $b_2 = \frac{u_y}{\delta}$ in current paper) Furthermore the result from Poisson's equation in terms of P_{zz} can be substituted to have a complete equation in u_z alone.

5 Results and Discussion

It can be shown that u_z is,

$$u_z = F_3 \times F_6\left(y, 8\int c_3 y \left(\sin(4t) - \cos(4t)\right) e^{-2c_3\left(\sin(4t)-\cos(4t)\right)} dt + x e^{2c_3\left(\cos(4t)-\sin(4t)\right)}\right) \tag{32}$$

where,

$$F_3 = C_1 + C_2 e^{\frac{z}{\sqrt{c_3}}} + C_3 e^{-\frac{z}{\sqrt{c_3}}} \tag{33}$$

where C_1, C_2, C_3 are arbitrary constants and F_6 belongs to a general family of functions. Recall δ is negative by assumption. The family of solutions F_6 is assumed to have a second partial derivative set to zero and since there is a separation of variables we look for either inflexion points or an undefined condition for u_z. The full expression for F_6 is shown here to be exactly in the form,

$$F_6 = \int \frac{\left(\frac{1}{64} - \frac{i}{64}\right) e^{\frac{1}{C3\left(xe^{8it} + iye^{8it} + y + ix\right)} \sqrt{\frac{-ix + 2C3\left(xe^{8it} + iye^{8it} + y + ix\right) - y}{x + iy}}}}{y + ix - C3\left(xe^{8it} + iye^{8it} + y + ix\right)} dt \tag{34}$$

It can be observed that since F_6 is a general form of solution that $\exp(iF_6)$ will also be a solution and can be made periodic on the lattice of Fig 1. Differentiating twice wrt t and since exponential is always positive this implies that the second derivative of F_6 is zero. That the solution $\exp(iF_6)$ is periodic in spatial variables on the lattice can be seen in Fig 2 below for the case where $t = 0.5, C3 = 300$ and $x = 2\pi$. It is a straightforward exercise to show that as $C3$ approaches infinity, $\exp(iF_6)$ approaches unity.

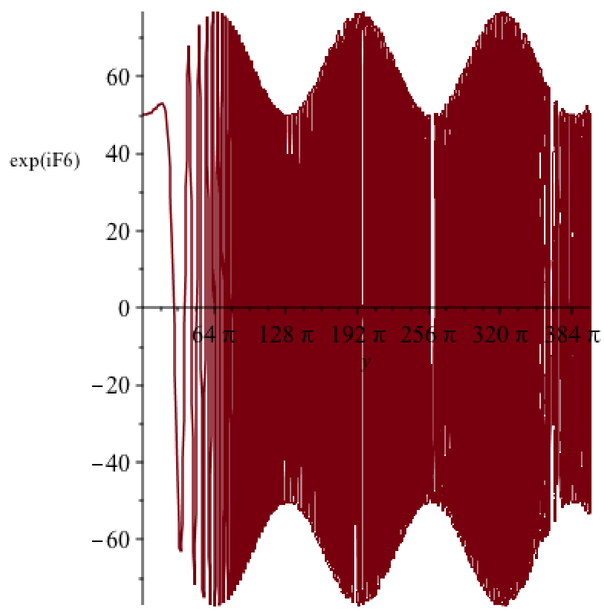
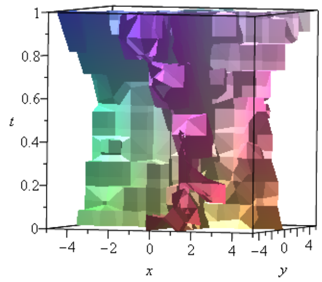
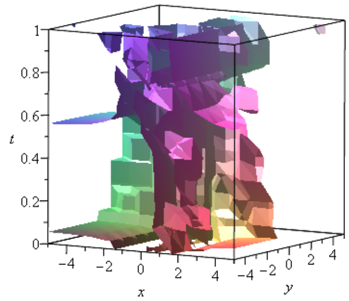


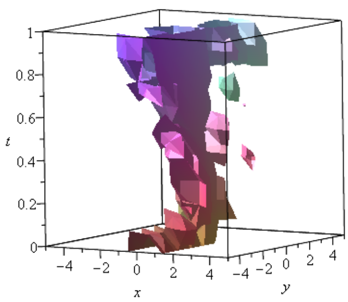
Figure 2. Spatially Periodic Solution of NS equations



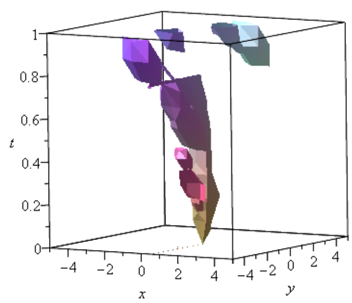
(a) $c_2 = 0.95$



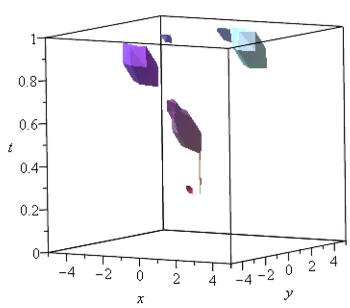
(b) $c_2 = 1.5$



(c) $c_2 = 2.0$



(d) $c_2 = 3.5$



(e) $c_2 = 6.0$

The basic energy integral is obtained on multiplying the original Navier Stokes equations (1) by $u_i^*(x,t)$ and integrating over Ω, that is in the form,

$$\frac{d}{dt}\|u^*\|_2^2 = -2\nu\|\nabla u^*\|_2^2 \tag{35}$$

where ν is viscosity term. Integrating wrt t is possible with Maple 18 software. This nontrivial task shows upon substitution of F_6 form in u_z and integrating the expression involved in F_6 that it can be integrated twice, one integral there and the other in the energy integral to obtain $\|u_z\|_2$. Results prove definitively that $\|u_z\|_2 = 0$ on the lattice in FIgure 1.

In derivation of the solution of F_6 the following facts were used. Taking the derivative of the L_2 norm $\|u\|_2^2$ the following is obtained,

$$\frac{d}{dt}\|u\|_2^2 = 2\|u\|_2\frac{d\|u\|_2}{dt}$$

$$= 2\|u\|_2\frac{1}{\|u\|_2}u^T\frac{du}{dt}$$

$$= 2u^T\frac{du}{dt}$$

where T is the transpose, it follows that,

$$\left(\frac{d}{dt}\|u\|_2^2\right)^2 = 4u_z^2\left(\frac{du_z}{dt}\right)^2$$

Dividing by a factor 2 and observing that the factor $2(\frac{du_z}{dt})^2$ appears in the expression leading to the solution F_6(used it to obtain F_6 with Maple), and the right side of the Energy equation is used in (terms of the norm of the gradient of the solution), we have,

$$\left(\frac{du_z}{dt}\right)^2 = \nu^2\left[\frac{\int_\Omega\left[\left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right]dx}{u_z}\right]^2$$

It is seen that the term is negligible wrt to term $u_z\frac{u_z}{\partial t}$. By controlling this growth term in terms of the viscosity it is possible to conclude that the norm of u_z is zero on the periodic lattice and there is no blowup for the periodic Navier-Stokes equations. In Fig. 3a an implicit plot in 3d is shown for the surface generated by F_6. The u_z solution is obtained by multiplying F_6 by z-solution F_3. F_3 grows without bound as z increases to large values. For small values there is finite time blowup, however as $z^* \to \infty$,(due to arbitrarily small parameter δ) u_z has no finite time blowup as can seen in Figures a-e of Figure 3. It is interesting that for low z values there is a coherent symmetry surface of blowup and as z increases the symmetry is broken and the blowup is randomly distributed and eventually these areas disappear completely as δ approaches zero. The form of the solution discussed in section 4.2 for equal vorticity components and related to Fig.3 is,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}F_1(x) = c_1F_1(x)^2, \frac{\mathrm{d}}{\mathrm{d}y}F_2(y) = c_2,\ \frac{\mathrm{d}}{\mathrm{d}t}F_3(t) = -2\,\frac{\mathrm{e}^{-2c_3(-\cos(4t)+\sin(4t))}F_3(t)^2c_3\left(-\cos(4t)+\sin(4t)\right)c_2}{c_1}\right]$$

where the F_6 form is the product of the above functions.

□

6 Conclusion

Here there has been a natural reduction of the 3-D incompressible NSEs to a one component decoupled velocity field solution under scale invariant transformations, with a separate 2-component velocity field solution which is the solution of the 2-D Incompressible Navier-Stokes equations with data specified in section 2.1. For the variable z- component, in particular, a form of solution is extracted in the analysis presented using the divergence form of Green's identity, Ostogradsky's theorem. In the general case,the solution is revealed to have smooth solutions which exhibit finite-time blowup on a fine measure zero set and using the Prékopa-Leindler and Gagliardo-Nirenberg inequalities it is shown that for any non zero measure set in the form of cube subset of 3D there is no finite time blowup for the starred velocity for any dimension of cube and small δ. In particular vortices are shown to exist and it is shown that zero is in the attractor of 3D Navier Stokes equations.

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