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Numerical Treatment for Solving Fuzzy Volterra Integral Equation by Sixth Order Runge-Kutta Method

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Abstract There has recently been considerable focus on finding reliable and more effective numerical methods for solving different mathematical problems with integral equations. The Runge–Kutta methods in numerical analysis are a family of iterative methods, both implicit and explicit, with different orders of accuracy, used in temporal and modification for the numerical solutions of integral equations. Fuzzy Integral equations (known as FIEs) make extensive use of many scientific analysis and engineering applications. They appear because of the incomplete information from their mathematical models and their parameters under fuzzy domain. In this paper, the sixth order Runge-Kutta is used to solve second-kind fuzzy Volterra integral equations numerically. The proposed method is reformulated and updated for solving fuzzy second-kind Volterra integral equations in general form by using properties and descriptions of fuzzy set theory. Furthermore a Volterra fuzzy integral equation, based on the parametric form of a fuzzy numbers, transforms into two integral equations of the second kind in the crisp case under fuzzy properties. We apply our modified method using the specific example with a linear fuzzy integral Volterra equation to illustrate the strengths and accurateness of this process. A comparison of evaluated numerical results with the exact solution for each fuzzy level set is displayed in the form of table and figures. Such results indicate that the proposed approach is remarkably feasible and easy to use.

Keywords Fuzzy Numbers, Fuzzy α-Level Sets, Fuzzy Integrals, Second-Class Fuzzy Volterra Integral Equations (FVIE2), Sixth Order Range -Kutta Method (RKM6)

1. Introduction

Integral equations discover specific pertinence in numerous logical and numerical mathematical models. In reality, it isn't continuously basic, but too outlandish to demonstrate the deficient physical models utilizing Integral equations to determine certain parameters in real life models so in that case fuzzy set theory be a powerful tool to treat these models mathematically. One of the significant components of a fuzzy analytical theory that plays a key role in numerical analysis is fuzzy integral equations. Zadeh introduced the concept of the fuzzy sets [1]. The theory has developed since, and is now a separate branch of applied mathematics and used to solve many mathematical problems in the filed on differential equations [2,3]. The elementary fuzzy calculation based on the theory of extension and integration was studied first by Dubois and Prade [4]. An integrated Rieman-type method was advocated by Goetschal and Voxman [5], and Kaleva [6] defined the integration of fuzzy functions using the Lebesgue-type integration theory. Fuzzy integral equations with arbitrary kernels were studied by Lakshmikantham and Mohapatra [7]. A number of researchers have been explored by the theoretical properties of fuzzy integral equations [7-13]. Friedman et.al investigated that core of numerical approaches to solving fuzzy integral equations with arbitrary kernels [14].

Although some numerical approaches in last decade were used to solve FVIE2 such as trapezoidal quadrature rule method [15] solved several types of FVIE2, in [16] a numerical method based on quadrature rules is suggested to solve nonlinear FVIE2, Iterative numerical method [17] is introduced to solve some examples involved nonlinear FVIE2 and so on. From the previous analysis, it would be important to analyze the RKM6 which was not used to solve FVIE2.

The general numerical solution of Runge-Kutta methods for solving crisp FVIE2 were discussed in [18, 19]. In this work, we present the use fuzzy set theory properties to formulate RKM6 for solving and analyze numerical solution FVIE2 involving linear test example in the form of table and figures.

The paper is structured accordingly: Some significant meanings and basic concepts to be used in this paper are given in Section 2. In section3, we introduce the fuzzy analysis of FVIE2. We are addressing RKM6 in section 4 to find a numerical solution for FVIE2In section 5, a numerical illustration illustrates the proposed procedure. Finally, in section 6, a short description of this study is provided.

2. Preliminaries

In this section, some basic concepts for fuzzy calculus are presented as follows:

Definition 2.1 [20]. "Fuzzy numbers constitute a subset of the real numbers, reflecting unknown values. Fuzzy numbers are correlated with membership degrees that state how valid it is to tell if anything belongs to a given set or not. If the fuzzy number is described by three numbers $\alpha < \beta < \gamma$ then μ is called a triangular fuzzy number as in Figure 1, where $\mu(x)$ graph is a triangle with an interval base and vertex $[\alpha, \beta]$ of $x = \beta$ with membership function as follows:

$$\mu(x;\alpha,\beta,\gamma) = \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \le x \le \beta \\ \dfrac{\gamma-x}{\gamma-\beta}, & \text{if } \beta \le x \le \gamma \\ 0, & \text{if } x > \gamma \end{cases}$$

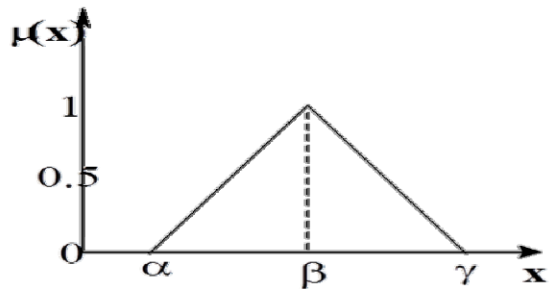

Figure 1. Triangular Fuzzy Number

Definition 2.2 [20]. "A trapezoidal fuzzy number is defined by four real numbers $\alpha < \beta < \gamma < \delta$ .The base of the trapezoid is the interval $[\alpha, \delta]$ with vertices at $x = \beta$ and $x = \gamma$. A trapezoidal fuzzy number as showed in Figure 2 will be denoted by:

$\mu = (\alpha, \beta, \gamma, \delta)$, the membership function is defined as the follows:

$$\mu(x;\alpha,\beta,\gamma,\delta) = \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \le x \le \beta \\ 1, & \text{if } \beta \le x \le \gamma \\ \dfrac{\delta-x}{\delta-\gamma}, & \text{if } \gamma \le x \le \delta \\ 0, & \text{if } x > \delta \end{cases}$$

Note that:

(1) $\mu > 0 \; if \; \alpha > 0$;
(2) $\mu > 0 \; if \; \beta > 0$;
(3) $\mu > 0 \; if \; \gamma > 0$; and
(4) $\mu > 0 \; if \; \delta > 0$.

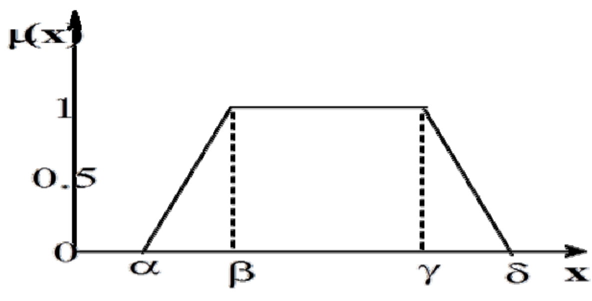

Figure 2. Trapezoidal Fuzzy Number

Definition 2.3 [20]. "A fuzzy number $\tilde{\boldsymbol{y}}: \boldsymbol{R} \to [\boldsymbol{0}, \boldsymbol{1}]$ which satisfies the following requirements:

(i) $\tilde{y}$ is half-continuous upper function,
(ii) $\tilde{y}(x) = 0$ outside some interval [c, d],
(iii) There are real numbers a, b such that $c \le a \le b \le d$ for which

(a) $\underline{y}(x)$ is increased monotonously at $[c, a]$,
(b) $\overline{\overline{y}}(x)$ is decreased monotonously at $[b, d]$,
(c) $\tilde{y}(x) = 1$ on $[a, b]$.

We will let R_F denote the set of fuzzy numbers on R. Obviously $R \subset R_F$, where $R \cong R\chi = \{\chi_{\{x\}} : x \in R\} \subset R_F$. The α -level represent of a fuzzy number$\tilde{y}$, denoted by $[\tilde{y}]^\alpha$, is defined as $\{\tilde{y}(x;\alpha)\}$ It is clear that the α -Level representation of a fuzzy number $\tilde{y}$ is a compact convex subset of R. Thus, if $\tilde{y}$ is a fuzzy number, then $[\tilde{y}]^\alpha = [\underline{y}(\alpha), \overline{y}(\alpha)]$, where $\underline{y}(\alpha)$=min$\{s: s \in [y]^\alpha\}$ and $\overline{y}(\alpha)$=max$\{s: s \in [y]^\alpha\} \forall \alpha \in [0, 1]$. Sometimes, we will write $\underline{y}_\alpha$ and $\overline{y}_\alpha$ replace of $\underline{y}(\alpha)$ and $\underline{y}(\alpha)$, respectively, $\forall \alpha \in [0, 1]$.

Theorem 2.1 [4]. "Suppose that $\underline{\mathbf{y}}$: [0, 1] → R and $\overline{\mathbf{y}}$: [0, 1] → R satisfies the following conditions:

(i) $\underline{y}(\alpha)$ is a bounded increasing left continuous function on (0, 1],
(ii) $\overline{y}(\alpha)$ is a bounded decreasing left continuous function on (0, 1],
(iii) $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ are right continuous functions at $\alpha = 0$,
(iv) $\underline{y}(\alpha) \le \overline{y}(\alpha)$ on [0, 1], Then $\tilde{y}$: R → [0, 1] defined by:"

$$\tilde{y}(s) = \sup\{\alpha: \underline{y}(\alpha) \le s \le \overline{y}(\alpha)\} \qquad (2.1)$$

Moreover, if $\tilde{y}$: R → [0, 1] is a fuzzy number given by $[\underline{y}(\alpha), \overline{y}(\alpha)]$, then the functions $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ satisfy conditions (i-iv).

Definition 2.4 [4,14]. "(Integral of fuzzy function) Let $\boldsymbol{F}: [\boldsymbol{a}, \boldsymbol{b}] \to \boldsymbol{E^1}$ then for each partition $\mathbf{P} = \{\boldsymbol{x_0}, \boldsymbol{x_1}, \dots . \boldsymbol{x_n}\}$ of $[\mathbf{a}, \mathbf{b}]$ and for arbitrary $\xi_{\mathbf{i}} \in [\mathbf{x_{i-1}}, \mathbf{x_i}]$, $\mathbf{1 \le i \le n}$ we assume

$$R_P = \sum_{i=1}^{n} F(\xi_i)(x_i - x_{i-1}), \ \Delta := \{|x_i - x_{i-1}|, \ i = 1, \dots \dots, n\}$$

The definite integral of $F(x)$ over [a, b] is $\int_a^b F(x)dx = \lim_{\Delta \to 0} R_P$The limit is in metric D given if the fuzzy function $F(x)$ in metric D is continuous [3], then the following integral exists:

$$\overline{\left(\int_a^b F(x;\alpha)dx\right)} = \int_a^b \overline{F}(x;\alpha)dx, \quad \underline{\left(\int_a^b F(x;\alpha)dx\right)} = \int_a^b \underline{F}(x;\alpha)dx$$

3. Analysis of FVIE2

The general form of FVIE2 is defined in the following form

$$y(x;\alpha) = f(x;\alpha) + \lambda \int_a^x K(x, t, y(t;\alpha))dt, \qquad (3.1)$$

$$t \in I = [a, x], a \le x \le b,$$

where $\lambda > 0, K(x, t)$ is an arbitrary kernel function over the interval $S = \{(x, t): a \le t \le x \le b\}$ and the function $f(x;\alpha)$ is a continuous fuzzy function on the interval $[a, b]$.

To build a numerical method for Eq. (3.1) then the given fizzy integral equations parametric shape as follows:

$$\begin{cases} \underline{y}(x;\alpha) = \underline{f}(x;\alpha) + \lambda \int_a^x K_1(x, t, \underline{y}(t;\alpha), \overline{y}(t;\alpha))\, dt \\ \overline{y}(x;\alpha) = \overline{f}(x;\alpha) + \lambda \int_a^x K_2(x, t, \underline{y}(t;\alpha), \overline{y}(t;\alpha))\, dt \end{cases} \qquad (3.2)$$

where

$$K_1(x, t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)) = \begin{cases} K(x, t, \underline{y}(x;\alpha)), & K(x,t) \ge 0 \\ K(x, t, \overline{y}(x;\alpha)), & K(x,t) < 0 \end{cases},$$

and

$$K_2(x, t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)) = \begin{cases} K(x, t, \overline{y}(x;\alpha)), & K(x,t) \ge 0 \\ K(x, t, \underline{y}(x;\alpha)), & K(x,t) < 0 \end{cases}.$$

Clearly, Eq. (3.2) is a system of fuzzy Volterra Integral Equations for each $0 \le \alpha \le 1$ and $a \le x \le b$.

4. Analysis of RKM6 in Fuzzy Domain

The exact and approximate solution at x_n , $1 \le n \le N$ are denoted by $[F(x)]^\alpha = [\underline{F}(x;\alpha), \overline{F}(x;\alpha)]$ and $[y(x)]^\alpha = [\underline{y}(x;\alpha), \overline{y}(x;\alpha)]$ respectively. We replace the interval [a, b]. by a set of discrete equally spaced grid point a=$x_0 \le x_1 \dots \dots \le x_N$=b at which the exact solution $[F(x)]^\alpha = [\underline{F}(x;\alpha), \overline{F}(x;\alpha)]$ is approximated by some $[y(x)]^\alpha = [\underline{y}(x;\alpha), \overline{y}(x;\alpha)]$ respectively. Runge-kutta method for the solution of Eq. (3.2) computes a numerical solution taking time steps of size $h = \Delta x$ with the relation $x_n = x_0 + nh$ at the point $x_n = a + nh, 1 \le n \le N$.

By generating approximations at some intermediate points in $[x_n, x_{n+1}]$, $1 \le n \le N$

$$x_n + \theta_r h \ , \quad n = 1,2, \dots . N-1 \quad , r = 1,2, \dots . P-1$$

$$\text{Where } 0 = \theta_0 \le \theta_1 \le \dots \le \theta_{p-1} \le 1 \qquad (4.1)$$

and the value h is called a step size (mish point) $h = \frac{b-a}{N}$ .The explicit Runge-kutta (RK) methods for FVIE, given by the formula

$$\begin{cases} \underline{y}(x_{n+1};\alpha) = \underline{y}(x_n;\alpha) + h \sum_{i=0}^{p-1} A_{pi} \underline{k}_i \\ \overline{y}(x_{n+1};\alpha) = \overline{y}(x_n;\alpha) + h \sum_{i=0}^{p-1} A_{pi} \overline{k}_i \end{cases} \qquad (4.2)$$

where

$$\underline{k}_0 = \underline{f}\left(a+nh\,,\underline{y}\,(x_n;\alpha)\right),$$

$$\overline{k}_0 = \overline{f}(a+nh\,,\overline{y}\,(\,x_n;\alpha)),$$

$$\underline{k}_r = \underline{f}\;\left(a+(n+\theta_r)h\,,\;\underline{y}\,(x_n;\alpha)+h\sum_{i=0}^{r-1}A_{ri}\,\underline{k}_i\right)\;\;,r=1,2,\dots,p-1,$$

$$\overline{k}_r = \overline{f}\left(a+(n+\theta_r)h\;,\overline{y}\,(x_n;\alpha)+h\sum_{i=0}^{r-1}A_{ri}\,\overline{k}_i\;\right)\;\;,r=1,2,\dots,p-1,$$

$$\text{and}\;\;\sum_{i=0}^{r-1}A_{ri}=\begin{cases}\theta_r\;,\;r=1,2,\dots,p-1\\ 1\,, \qquad r=p\end{cases} \tag{4.3}$$

With $\underline{y}_r\,,\overline{y}_r$ is an approximation to the solution at $x=x_r=a+rh$ .The second argument of $\underline{k}_r$ and $\overline{k}_r$ may be regarded as an approximation to $\underline{y}(a+(n+\theta_r)h)$ and $\overline{y}(a+(n+\theta_r)h)$ respectively we rewrite Eq. (4.2) as

$$\left\{\begin{array}{c}\underline{y}\,(x_{n+1};\alpha)=\underline{y}(\,x_n;\alpha)+\\ h\,\sum_{i=0}^{p-1}A_{pi}\,\underline{f}(\,x_n+\theta_i h\;\;,\underline{y}\,(x_{n+\theta_i};\alpha\,))\\ \overline{y}\,(x_{n+1};\alpha)=\overline{y}(\,x_n;\alpha)+\\ h\,\sum_{i=0}^{p-1}A_{pi}\,\overline{f}(\,x_n+\theta_i h\;\,,\overline{y}(x_{n+\theta_i};\alpha\,))\end{array}\right\} \tag{4.4}$$

To specify a particular method, we need to provide the integer p (the number of stages), and the coefficients θ_i (for i =1, 2,.., p-1), A_{pi} (for $1\le i\le p$).these data are usually arranged in a co-called Butcher tableau [18].

Now, the method defined in Eq. (4.4) can be extended to give a class of Runge-Kutta method for the solution of Eq. (3.2). Suppose that the kernel in Eq. (3.2). Can rewrite as

$$\begin{cases}K(x,t,\underline{y}(t;\alpha))=\sum_r u_r(x)\;\;v_r(t\;\;,\underline{y}(t;\alpha)),\\ K(x,t,\overline{y}(t;\alpha))=\sum_r u_r(x)\;\;v_r(t\;\;,\overline{y}(t;\alpha)).\end{cases} \tag{4.5}$$

Then we have

$$\begin{cases}\underline{y}(x;\alpha)=\underline{f}(x;\alpha)+\sum_r u_r(x)\underline{Z}_r(x;\alpha)\;,\;\;x>0\\ \overline{y}(x;\alpha)=\overline{f}(x;\alpha)+\sum_r u_r(x)\overline{z}_r(x;\alpha)\;,\;\;x>0\end{cases} \tag{4.6}$$

Let $\underline{Z}_r(x;\alpha)=v_r\left(t,\underline{y}(t;\alpha)\right)$, $\underline{Z}_r(0;\alpha)=0$,

$$\overline{Z}_r(x;\alpha)=v_r\big(t,\overline{y}(t;\alpha)\big)\,,\;\overline{Z}_r(0;\alpha)=0.$$

Now we recall Runge –Kuta formulas as in Eq. (4.4) we get

$$\left\{\begin{array}{ll}\underline{Z}_r(\theta_P h;\alpha)\;=h\,\sum_{i=0}^{p-1}A_{pi}\,v_r\left(\theta_i h,\underline{y}(\theta_i h;\alpha)\right) & ,p=1,2,3,\dots,m\\ \overline{Z}_r(\theta_P h;\alpha)\;=h\,\sum_{i=0}^{p-1}A_{pi}\,v_r(\theta_i h,\overline{y}(\theta_i h;\alpha)) & ,p=1,2,3,\dots,m\end{array}\right\} \tag{4.7}$$

where θ_p and A_{pi} satisfy Eq. (4.1) and Eq. (4.3) respectively.

The fuzzy number $\underline{Z}_r(h)$ and $\overline{Z}_r(h)$ is the required for error order $O(h^{m+1})$ that approximation to $\underline{Z}_r(h)$ and $\overline{Z}_r(h)$ respectively for the values $m\le 6$. By Substituting (4.7) in Eq. (4.6), yields

$$\underline{y}(\theta_P h;\alpha)=\underline{f}(\theta_P h;\alpha)+\sum_r u_r(\theta_P h)\;\underline{Z}_r(\theta_P h;\alpha)$$

From

$$\underline{y}(\theta_P h;\alpha)=\underline{f}(\theta_P h;\alpha)+h\sum_r u_r(\theta_P h)\;\sum_{i=0}^{p-1}A_{pi}\,v_r\left(\theta_i h,\underline{y}(\theta_i h;\alpha)\right),$$

We have

$$\underline{y}(\theta_P h;\alpha)=\underline{f}(\theta_P h;\alpha)+h\sum_{i=0}^{p-1}A_{pi}\,K\left(\theta_p h\,,\theta_i h\;\;,\underline{y}(\theta_i h;\alpha)\right)\;\;,$$

Now rewrite Eq. (3.2) as:

$$\underline{y}(\theta_P h;\alpha)=\underline{f}(\theta_P h;\alpha)+h\sum_{j=0}^{p-2}\int_{\theta_{j=0}h}^{\theta_{p-1}h}K\left(\theta_p h\,,t\;\;,\underline{y}(t;\alpha)\right)dt+\int_{\theta_{p-1}h}^{\theta_p h}K\left(\theta_p h\,,t\;\;,\underline{y}(t;\alpha)\right)\,dt\,,$$

$$\theta_j h\;\le t\le\theta_{j-1}$$

Therefore,

$$\underline{y}(n,p)=\underline{Y_n}(\theta_P h;\alpha)+h\sum_{i=0}^{p-1}A_{pi}\,K\left(\theta_p h\,,\theta_i h\;\;,\underline{y}(\theta_i h;\alpha)\right) \tag{4.8}$$

where

$$\underline{Y_n}(\theta_P h;\alpha)=\underline{f}(\theta_P h;\alpha)+\sum_{j=0}^{n-1}h\left(\sum_{i=0}^{m}A_{pi}\,K\left(\theta_p h\,,\theta_i h\;\;,\underline{y}(\theta_i h;\alpha)\right)\right)\,,p=1,2,\dots,m \tag{4.9}$$

Similarly, for the upper solution $\overline{Y_n}$, consider the problem:

$$\overline{Y_n}(\theta_P h;\alpha)=\overline{f}(\theta_P h;\alpha)+\sum_{j=0}^{n-1}h\left(\sum_{i=0}^{m}A_{pi}\,K\left(\theta_p h\,,\theta_i h\;\;,\overline{y}(\theta_i h;\alpha)\right)\right)\,,p=1,2,\dots,m \tag{4.10}$$

For more details (see [21,22]). A correct weight preference A_{mi} with the parameter θ_m of equations Eq. (4.9) and Eq. (4.10), the following classification is given. Now, the common sixth order Runge –Kutta formula with $m=7$ is given by see [23,24]:-

$$\theta_0=0,\theta_1=\theta_3=\frac{1}{3},\theta_2=\frac{2}{3}\;,\theta_4=\frac{5}{6},\theta_5=\frac{1}{6},$$

$$\theta_1=\theta_3=1,A_{10}=\frac{1}{3},A_{20}=0,A_{21}=\frac{2}{3},A_{30}=\frac{1}{12},$$

$$A_{31}=\frac{1}{3},A_{32}=-\frac{1}{12},A_{40}=\frac{25}{48},A_{41}=-\frac{55}{24},$$

$$A_{42}=\frac{35}{48},A_{43}=\frac{15}{8},A_{50}=\frac{3}{20},A_{51}=-\frac{11}{20},$$

$$A_{52}=-\frac{1}{8},A_{53}=\frac{1}{2},A_{54}=\frac{1}{10},A_{60}=-\frac{26}{1260},$$

$$A_{61}=\frac{33}{13},A_{62}=\frac{43}{156},A_{63}=-\frac{11}{839},A_{64}=\frac{32}{195},A_{65}=\frac{80}{39}.$$

Substitute these values in to Eq. (4.8), and get

$$\underline{y}(n,0) = \underline{y}(n-1,6),$$

$$\underline{y}(n,1) = \underline{Y_n}(x_{n+1/3};\alpha) + \frac{h}{3}\, K[x_{n+1/3}, x_n, \underline{y}(n,0)],$$

$$\underline{y}(n,2) = \underline{Y_n}(x_{n+2/3};\alpha) + \frac{2h}{3}\, K[x_{n+2/3}, x_{n+1}, \underline{y}(n,1)],$$

$$\underline{y}(n,3) = \underline{Y_n}(x_{n+1/3};\alpha) + \frac{h}{12}\big(K[x_{n+\frac{1}{3}}, x_n, \underline{y}(n,0)] + 4K[x_{n+1/3}, x_{n+1/3}, \underline{y}(n,1)] - K[x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, \underline{y}(n,2)]\big),$$

$$\underline{y}(n,4) = \underline{Y_n}(x_{n+5/6};\alpha) + \frac{h}{48}\big(25K[x_{n+5/6}, x_n, \underline{y}(n,0)] - 110K[x_{n+\frac{5}{6}}, x_{n+\frac{1}{3}}, \underline{y}(n,1)] + 35K[x_{n+\frac{5}{6}}, x_{n+\frac{2}{3}}, \underline{y}(n,2)] + 90K[x_{n+5/6}, x_{n+1/3}, \underline{y}(n,3)]\big),$$

$$\underline{y}(n,5) = \underline{Y_n}(x_{n+1/6};\alpha) + \frac{h}{40}\big(6K[x_{n+1/6}, x_n, \underline{y}(n,0)] - 22K[x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, \underline{y}(n,1)] - 5K[x_{n+\frac{1}{6}}, x_{n+\frac{2}{3}}, \underline{y}(n,2)] + 20K[x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, \underline{y}(n,3)] + 4K[x_{n+1/6}, x_{n+5/6}, \underline{y}(n,4)]\big],$$

$$\underline{y}(n,6) = \underline{Y_n}(x_{n+1};\alpha) - \frac{261h}{260}K[x_{n+1}, x_n, \underline{y}(n,0)] + \frac{33h}{13}K[x_{n+1}, x_{n+1/3}, \underline{y}(n,1)] + \frac{43h}{156}K[x_{n+1}, x_{n+2/3}, \underline{y}(n,2)] - \frac{118h}{39}K[x_{n+1}, x_{n+1/3}, \underline{y}(n,3)] + \frac{32h}{195}K[x_{n+1}, x_{n+5/6}, \underline{y}(n,4)] + \frac{80h}{39}K[x_{n+1}, x_{n+1/6}, \underline{y}(n,5)],$$

$$\underline{y}(n,7) = \underline{Y_n}(x_{n+1};\alpha) + \frac{h}{200}\big(13K[x_{n+1}, x_n, \underline{y}(n,0)] + 55K[x_{n+1}, x_{n+\frac{2}{3}}, \underline{y}(n,2)] + 55K[x_{n+1}, x_{n+\frac{1}{3}}, \underline{y}(n,3)] + 32K[x_{n+1}, x_{n+\frac{5}{6}}, \underline{y}(n,4)] + 32K[x_{n+1}, x_{n+\frac{1}{6}}, \underline{y}(n,5)] + 13K[x_{n+1}, x_{n+1}, \underline{y}(n,6)]\big],$$

where

$$\underline{Y}(x_n;\alpha) = \underline{f}(x_n;\alpha) + \frac{h}{200}\sum_{j=0}^{n-1}\Big(13K\big[x_{j+1}, x_j, \underline{y}(j,0)\big] + 55K\big[x_{j+1}, x_{j+\frac{2}{3}}, \underline{y}(j,2)\big] + 55K\big[x_{j+1}, x_{j+\frac{1}{3}}, \underline{y}(j,3)\big] + 32K\big[x_{j+1}, x_{j+\frac{5}{6}}, \underline{y}(j,4)\big] + 32K\big[x_{j+1}, x_{j+\frac{1}{6}}, \underline{y}(j,5)\big] + 13K[x_{j+1}, x_{j+1}, \underline{y}(j,6)]\Big).$$

Similarly, for the upper solution $\overline{Y_n}$, consider the problem:

$$\overline{Y}(x_n;\alpha) = \overline{f}(x_n;\alpha) + \frac{h}{200}\sum_{j=0}^{n-1}\Big(13K\big[x_{j+1}, x_j, \overline{y}(j,0)\big] + 55K\big[x_{j+1}, x_{j+\frac{2}{3}}, \overline{y}(j,2)\big] + 55K\big[x_{j+1}, x_{j+\frac{1}{3}}, \overline{y}(j,3)\big] + 32K\big[x_{j+1}, x_{j+\frac{5}{6}}, \overline{y}(j,4)\big] + 32K\big[x_{j+1}, x_{j+\frac{1}{6}}, \overline{y}(j,5)\big] + 13K[x_{j+1}, x_{j+1}, \overline{y}(j,6)]\Big).$$

5. Numerical Example

Consider the fuzzy Volterra integral equation [26]

$$\tilde{y}(x) = \tilde{f}(x) + \int_0^x (x-t)\tilde{y}(t)dt, \quad x \in [0,1] \qquad (5.1)$$

where $\tilde{f}(x) = [3+\alpha, 8-2\alpha]\cot(x)$ for all fuzzy level sets $\alpha \in [0,1]$. From [26], the exact solution of Eq. (5.1) is given as:

$$\tilde{Y}(x;\alpha) = [3+\alpha, 8-2\alpha]Cosh(x). \qquad (5.2)$$

According to Section 2 and fuzzy set theory Eq. (5.1) can be defuzzified in to the followings:

$$\begin{cases} \underline{y}(x;\alpha) = (3+\alpha) + \int_0^x (x-t)\underline{y}(t;\alpha)dt \\ \overline{y}(x;\alpha) = (8-2\alpha) + \int_0^x (x-t)\underline{y}(t;\alpha)dt \end{cases} \qquad (5.3)$$

Define the absolute error $\tilde{E}(x;\alpha) = \left|\tilde{Y}(x;\alpha) - \tilde{y}(x;\alpha)\right|$, then by formulating the Runge –Kutta method in Section 4 to Eqs. (5.3) we obtain the following results displayed in Table 1, Figure 3 and Figure 4with step size $h = 10$:

Table 1. Sixth order Runge –Kutta method lower and upper solution and accuracy of Eq. (5.1) at $x = 1$ for all $\alpha \in [0,1]$

α	$\underline{y}(1;\alpha)$	$\underline{E}(1;\alpha)$	$\overline{y}(1;\alpha)$	$\overline{E}(1;\alpha)$
0	4.629241877699991	$2.674573984506878 \times 10^{-9}$	12.344645007199974	$7.132197588077816 \times 10^{-8}$
0.2	4.937858002879990	$2.852878999703989 \times 10^{-9}$	11.727412756839975	$6.775587735319277 \times 10^{-8}$
0.4	5.246474128059989	$3.031183926083258 \times 10^{-9}$	11.110180506479976	$6.418977882560739 \times 10^{-8}$
0.6	5.555090253239988	$3.209488941280369 \times 10^{-9}$	10.492948256119979	$6.062367852166517 \times 10^{-8}$
0.8	5.863706378419987	$3.387793867659638 \times 10^{-9}$	9.8757160057599800	$5.705757999407979 \times 10^{-8}$
1	6.172322503599987	$3.566098794038908 \times 10^{-9}$	9.2584837553999830	$5.349147969013757 \times 10^{-8}$

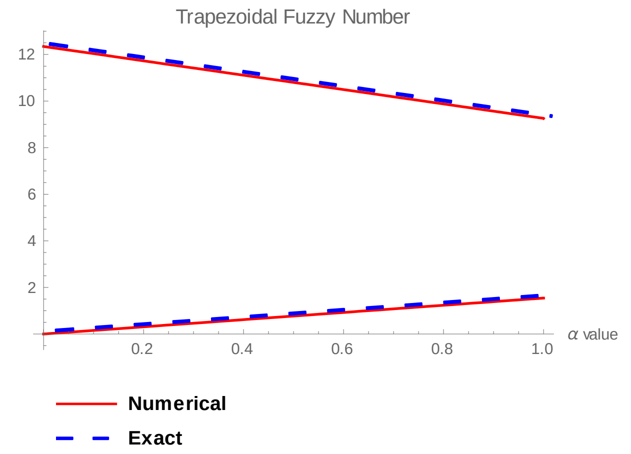

Figure 3. Sixth order Runge –Kutta method lower and upper solution and exact of Eq. (5.1) at $x = 1$ for all $\alpha \in [0,1]$

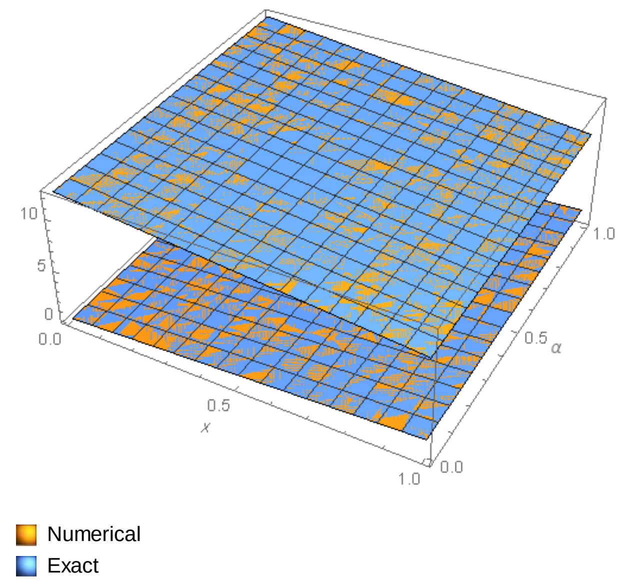

Figure 4. Sixth order Runge –Kutta method lower and upper solution and exact of Eq. (5.1) for all $\alpha \in [0,1]$ and $x \in [0,1]$

6. Conclusions

The RKM6 has been formulated into fuzzy domain and analyzed in order to solve fuzzy Volterra equation integral equation numerically. Comparison of the conclusion obtained from this method with the exact solution of a given test problem involving a linear FVIE2 displayed in and figures. The numerical results of the proposed Runge – Kutta method for FVIE2 indicate that this method is appropriate one for solving such problem. Furthermore, the obtain result was accurate, even with small step size and satisfy the fuzzy number properties.

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