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Regression Analysis Following Levenberg - Marquardt Algorithm to Estimate Elastic Modulus of Sandy Clay Embankment

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Abstract Road design must consider the factors affecting the elastic modulus of the cohesive soil used for the roadbed. Accurate determination of the elastic modulus of the pavement will help calculate properly the deformation of the pavement and prevent the appearance of cracks in the pavement; especially when the roadbed is flooded. This study is based on the laboratory data, using Levenberg-Marquardt algorithm and building a program for regression analysis, proposing the coefficients to estimate the elastic modulus of the roadbed. From the research results obtained, the way to choose the adjustment coefficients has been improved by adding the coefficient μ, the Levenberg-Marquardt algorithm has solved the problem with only 1,351 iterations, proving the simplicity and efficiency to solve the problem of nonlinear minimum squares that sometimes cannot be solved by the Gauss-Newton method. This algorithm is applied to regression analysis of experimental results, adding the values of coefficients a_9 and a_{10} to propose the values of the remaining coefficients a_n and b_n of the formula Dong-Gyou Kim, MS, 2004 with correlation coefficient R^2 is 0.8929. As a result, the article proposes the appropriate value of the regression coefficients into the formula to estimate the elastic modulus of the sandy clay embankment according to the humidity and material characteristics of land, instead of using the Benkelman method which is time-consuming, expensive and difficult to implement.

Keywords Elastic Modulus, Regression Analysis, Levenberg - Marquardt Algorithm, Roadbed, Moisture

1. Introduction

Elastic Modulus (EMS) is determined on the basis of elastic deformation. For road works, EMS is used to calculate the settlement of the foundation and road surface. Due to the immediate nature of the work load, the loading and unloading time is very fast, repeated many times, after a number of times of applied load, the cumulative residual deformation decreases gradually and eliminates, the settlement of the road works depends mainly on the elastic deformation of the roadbed and road surface structure [7].

EMS is defined as the ratio between deviator stress and elastic strain, determined according to the AASHTO T294-94 standard guidelines [1,3,5]. Or EMS can be determined from the point of view of stress as stated by Hicks and Monismith (1971), Uzan (Universal) (1985), Johnson (1986), Rafael Pezo (1993), Louay (1999) [16,17,29].

Some other authors propose the formulas to determine the EMS of the foundation with a linear and nonlinear relationship with the deviator stress. Carmichael and Stuart (1986) proposed the USDA formula, in which soil type is the most important factor affecting EMS and the effect of pressure on EMS is also of interest [8]. Although the formula considers the effect of lateral pressure on EMS, the relationship between EMS and deviator stress is linear. In fact, EMS of cohesive soil tends to be nonlinear with deviator stress while keeping lateral pressure. EMS decreases as the plasticity index increases, in fact EMS of cohesive soils increases as the plasticity index increases. The EMS of high ductility cohesive soil (CH or MH) is determined to be greater for low plastic soils, which are not practical and therefore limited in application [11-14,33,34].

Drumm et al. (1990) proposed the Hyperbolic formula to determine the EMS of the roadbed, but the effect of lateral pressure on EMS was not considered in this equation [10]. EMS decreases as dry density increases, in fact EMS increases as the dry density of soil increases. The relationship between EMS and deviator stress is nonlinear, but this nonlinear tendency is significantly different from the test results of Kim.MS on soil samples ATH-50-222, 228, 413 type A-6. [21]

Santha (1994) proposed the GDOT formula. In this formula, the stress-strain relationship is nonlinear with many terms, the relationship between EMS and the soil physical characteristics is of interest. However, the formulation has many problems when evaluating the EMS of the roadbed. Only deviator stresses are considered. The effect of lateral pressure is neglected, in fact EMS increases with increased lateral pressure. In fact, EMS increases as the tightness and plasticity index increase, so the coefficients for tightness and plasticity index must be positive but in the above formula is negative [30].

Pezo and Hudson (1994) have proposed the formula TDOT, taking into account the effects of lateral pressure, deviator stress, moisture, density, and date of sample age on the EMS of the cohesive soils. In it, the coefficient that adjusts the EMS value according to the moisture decreases nonlinearly with the increase in humidity, but this formula has a linear relationship and is only applicable to the range of input parameters for this case study [29].

Lee et al, (1995) [22] proposed the formula UCS, which is the simplest formula, considering the influence of the lateral pressure and deviator stress on the EMS value through the coefficient determined by the experiment of triaxial compression test. However, the EMS determined from this formula is highly dependent on the axial stress causing 1% of the axial strain in the compression test, and in fact EMS is also related to many other basic soil features [22].

The Ohio Department of Transportation proposed an ODOT formula for determining the EMS of the roadbed. This formula does not consider the effects of side and deviator stress, so it is not possible to determine the exact EMS of cohesive soils for a series of soil stress states [28].

This article studies on regression analysis following Levenberg - Marquardt algorithm to estimate elastic modulus of sandy clay embankment.

2. Materials and Methods

2.1. Regression Equation Identification Basis

AASHTO T294-94 determines EMS for cohesive soil on the basis of simple linear regression following the formula [2]:

$$M_r = k_1(\sigma_d)^{k_2} \tag{1}$$

Equation (1) only considers the effect of deviator stress and does not consider lateral pressure. Dong-Gyou Kim. MS improves by considering the effect of lateral pressure on EMS. The EMS depends on the ratio of the octahedral stresses to octahedral shear stresses and the coefficients k_1, k_2. The EMS values for the cohesive soil use are estimated in equations (2) and (3).

$$\frac{M_r}{P_a} = k_1\left[\frac{\frac{\sigma_{oct}}{P_a}}{\left(\frac{\tau_{oct}}{P_a}\right)^2}\right]^{k2} \tag{2}$$

$$= k_1\left[\frac{P_a\sigma_{oct}}{\tau_{oct}^2}\right]^{k2} \tag{3}$$

$$= k_1\left[\frac{9P_a}{2}\left(\frac{1}{3\sigma_d}+\frac{\sigma_3}{\sigma_d^2}\right)\right]^{k2} \tag{4}$$

Assuming the two principal stresses σ_2 and σ_3 are equal under the conditions of axial symmetry, Equation (2) becomes Equation (4). Air pressure Pa (101 kPa) is applied so that the coefficients k_1 and k_2 are dimensionless.

EMS is dependent on the stress state and basic physical characteristics of the soil. Thus, the coefficients k_1, k_2 must contain parameters representing the main stress state and physical characteristics of the soil; The regression coefficients a_n, b_n in equations (5), (6) show the influence of deviator stress, lateral pressure and major physical characteristics of the soil on EMS. Equation (18) considers the stress state condition of cohesive soil that can actually occur in the roadbed, considering the influence of side pressure and deviator stress.

k_1, k_2: is recognized as equation (5), (6), (7) and (8).

$$k_1 = a_1 a_3^{a_2} + a_3\left(\frac{S}{100}\right)^{a_4} + a_5 q_u + a_6 PI + a_7(LL - w) + a_8\left(w_{opt} - w\right) + a_9(P200 - a_{10}) \tag{5}$$

$$k_2 = b_1\sigma_3^{b_2} + b_3\left(\frac{S}{100}\right)^{b_4} + b_5 q_u^{b_6} + b_7 PI + b_8 LL \tag{6}$$

$$a_1 = a_{11} + a_{12}\left(\frac{w_{opt}-w}{w_{opt}}\right) \tag{7}$$

$$b_1 = b_{11} + b_{12}\left(w - w_{opt}\right) \tag{8}$$

The coefficients a_n and b_n in Equation (5), (6), (7) and (8) look up Table 1 and Table 2.

Table 1. Coefficients of a_n for cohesive soils according to the experiment of Kim. MS

k_1			
Cofficient	A-4	A-6	A-7-6
a_{11}	6.46	8.320	9.28
a_{12}	44.41	71.960	39.98
a_2	0.73	0.700	0.64
a_3	-120.40	-29.800	-193.39
a_4	19.24	6.500	2.02
a_5	0.11	0.886	0.73
a_6	28.60	5.300	2.57
a_7	0.00	4.800	10.43
a_8	57.27	30.070	23.28
a_9	2.66	0.000	0.00
a_{10}	54.27	0.000	0.00

Table 2. Coefficients of b_n for cohesive soils according to the experiment of Kim. MS

k_2			
Cofficient	A-4	A-6	A-7-6
b_{11}	0.00240	0.00753	0.01
b_{12}	0.00390	0.00270	0.00
b_2	0.35100	0.52300	0.46
b_3	0.04300	0.20500	0.08
b_4	24.00000	13.40000	15.30
b_5	3.17000	1.13000	2.58
b_6	-0.63800	-0.61200	-0.60
b_7	-0.00016	-0.00021	0.00
b_8	-0.00028	-0.00016	0.00

The regression coefficients a_1, a_2 in equations (5) and (7) show the effect of the change in humidity value compared with the optimum moisture value and pressure on the EMS, a_3 and a_4 values. The influence of saturation on the EMS value, a_5 the effect of the compressive strength on the EMS value, a6 effect of the plasticity index on the EMS value, a_7 the effect of the difference in moisture from the liquid limit on the EMS value, a_8 the effect of the change in moisture value around the optimum moisture value on the EMS, a_9 and a_{10} values affect the percentage of particles passing through sieve No.200 on the EMS value, through k_1. The regression coefficients b_1, b_2 in equations (6) and (8) show the effect of the change of w value around the optimum moisture value and pressure on the EMS value, b_3 and b_4 influence of saturation on the EMS value, b_5 and b_6 affect the influence of the compressive strength on the EMS value, b_7 the effect of the plasticity on the EMS value, b_8 the effect of the liquid limit on the EMS value, through k_2.

k_1 and k_2 are established on the basis of the relationship between the stress state, physical characteristics of the soil, and the EMS tested in the laboratory. The coefficients of an and bn in Equation (5), (6) are determined by regression analysis based on the least squares method. Specific values of the coefficients of a_n and b_n obtained from the experimental results are listed in Table 1 and Table 2.

The physical characteristics and classification of the soil samples studied by Kim. MS are presented in Table 3. The experimental sample is named by the research project according to the locality, route, and the collection process. [4]

Table 3. Physical characteristics and classification of Kim. MS soil samples

Soil type		Form name	Liquid limit (%)	Plasticity index (%)	Sand (%)	Dust (%)	Clay (%)
AASHTO	USCS						
A-4	SC-SM	MUSK-60-21	29	6	29	34	8
	CL	GREEN-35-21.13,320,400	24	8	22	51	13
	CL	WAS-7-Mari	29	10	3	46	17
	CL	SHE-SR47	26	9	17	66	14
A-6	CL	WAR-741-3	28	11	23	49	12
	CL	WAS-821-113, 13216	32	11	13	55	21
	CL	BEL-SR147, 265	35	11	4	72	19
	CL	ATH-50-Cool	33	13	8	45	39
	CL	ATH-50-222, 228,413	31	12	18	42	14
A-7-6	CH	ATH-SR7	59	32	0	66	34
	CH	FAI-170	55	36	7	53	39
	CL	CRAW-Beal	41	21	7	81	12
	CL	HEN-SR6, 24	41	20	5	55	40

The characteristics of Kim. MS's EMS estimation formula (2) are illustrated in the diagram in Fig. 1. EMS varies with five different stress classes of 14, 28, 41, 55, 69 kPa and three pressure classes. The lateral pressures are 0, 21, 41 kPa at the k_1 value is 1 and k_2 is 0.1. EMS increases as lateral pressure increases and deviator stress decreases. EMS calculated from Equation (4) decreases with a nonlinear trend as deviator stress increases; in accordance with the results of three-axis compression test.

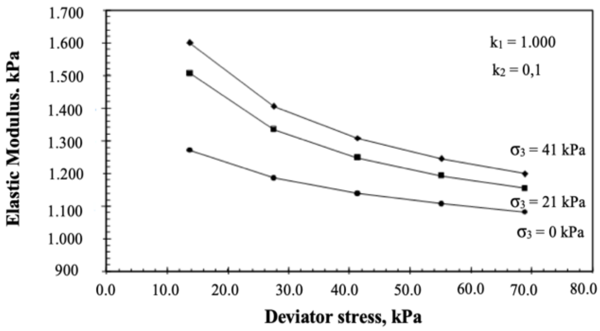

Figure 1. EMS changes with stress and lateral pressure [3]

Fig. 2 illustrates the graph of EMS calculated using Equation (4) at a pressure value of 21 kPa with a coefficient k_1 of 100 respectively; 500 and k_2 equals 0.1.

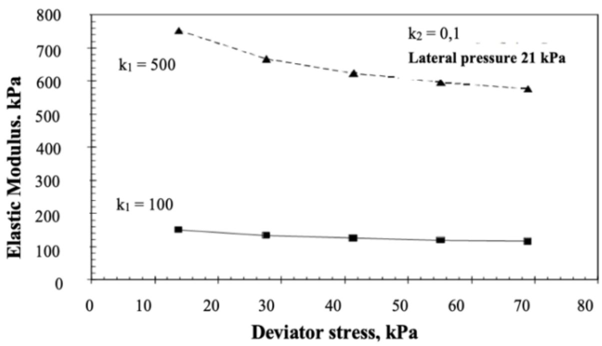

Figure 2. EMS according to deviator stress as the coefficient k1 increases

Keep the same value of lateral pressure and coefficient k_2, with five grades of deviator stress, as coefficient k_1 increases, EMS increases. Coefficients k_1 increase as lateral pressure, compressive strength, liquid limit, plasticity index, optimum humidity, percentage of soil passing through sieve No.200 increase and saturation and humidity decrease. EMS depends mainly on the coefficient k_1.

In Fig. 3, keeping the value of the lateral pressure and the coefficient k_1 with five levels of deviator stress, when the coefficient k_2 increases, EMS increases. Coefficients k_2 increase as lateral pressure, flexural strength, loose limit, plasticity index increase and sample saturation decreases. When the coefficient k_2 decreases, the EMS decreases linearly with the deviator stress. The EMS value does not depend much on the coefficient k_2.

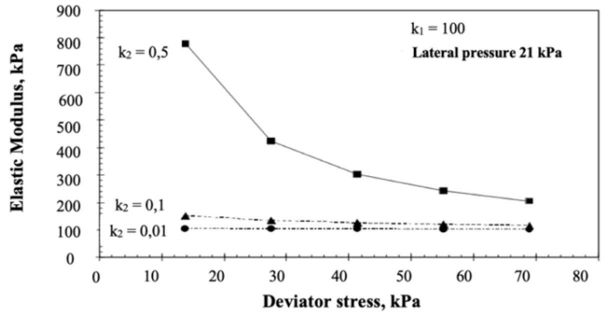

Figure 3. EMS according to deviator stress as the coefficient k2 increases

To evaluate the reliability of formula (4), Kim, MS compared the EMS value estimated by formula (4) with EMS value according to the results of triaxial compression test. The different humidity (in the dry, wet, optimal and saturated branch) of the 13 samples listed in Table 3. Fig. 4 shows the comparison results between EMS according to the results of triaxial compression test in room and estimated EMS calculated using formula (4). Correlation coefficient $R^2 = 0.996$. Accuracy is very reliable.

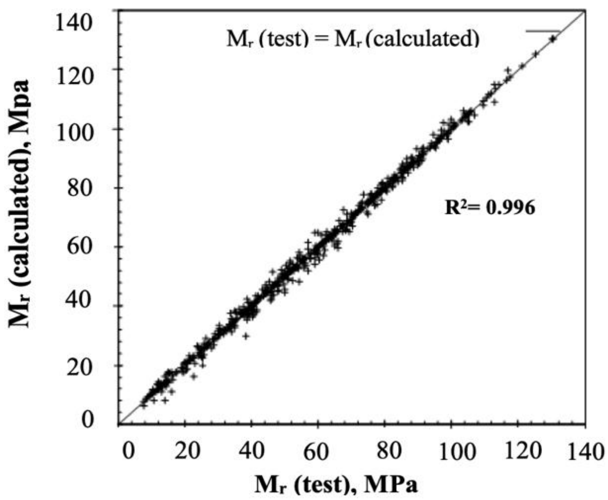

Figure 4. Comparison of M_r value according to results of tri-axial compression test of 13 samples with formula (4) for all humidity [9].

2.2. Identify Equation of Regression Analysis

Soil samples studied by Kim.MS have quite similar mechanical properties with sandy clay samples collected in the annual flood routes in the Mekong Delta. Mechanical and physical characteristics of soil samples type A-6 studied by Kim. MS: liquid limit from 28% to 35%, plasticity index from 11% to 13%, sand particle content from 4% to 23%, dust from 42% to 72%, clay from 12% to 39%, optimum humidity from 13% to 16%, as Table 3. Physical and mechanical characteristics of sandy clay samples collected on annual flood routes in the Mekong Delta: liquid limit from 29% to 37.6%, plasticity index from 11.3% to 16.4%, sand particle composition from 5.8 % to 36.8%, dust from 22.9% to 53.8%, clay from 31.2% to 41., 4%, optimum humidity from 14.8% to 21.2%. Compare the physical properties of the soil samples as shown in Table 4.

Table 4. Comparison of Atterberg limit and grain composition of soil samples

Parameter	Kim. MS	This research
Liquid limit (%)	28 ÷35	29 ÷38
Plasticity index (%)	11 ÷13	11 ÷16
Sand (%)	4 ÷23	6 ÷37
Dust (%)	42 ÷72	23 ÷54
Clay (%)	12 ÷39	31 ÷41

Soil samples were tested by Kim.MS with humidity from 10.3% to 27.2%, the range of changes is relatively wide, suitable for the study of changes in humidity of the flooded roadbed in the Mekong Delta. Formula by Kim. MS gives more accurate results than 6 formulas popular in America. Therefore, the regression analysis to determine the EMS of the sandy clay embankment in the Mekong Delta can choose the formula of Kim. MS (4); with k_1 is still calculated according to formula (5), regression coefficients a_1, a_2, consider the influence of lateral pressure; a_3, a_4, consider the influence of saturation; a_5, considering the impact of compressive strength in the chamber; a_6, considering the influence of the plasticity index; a_7, consider the effect of the difference of moisture on the liquid limit; a_8, a_{11}, a_{12}, considering the effect of the difference in humidity on the optimum humidity; a_9, a_{10}, taking into account the effect of the No.200 pass-through content on the EMS value; k_2 calculated according to the formula (6) regression coefficients b_1, b_2, considering the influence of lateral pressure; b_3, b_4, consider the influence of saturation; b_5, b_6, consider the impact of compressive strength in chamber; b_7, considering the influence of the plasticity index; b_8, considering the influence of liquid limit; b_{11}, b_{12}, consider the effect of the difference in moisture with the optimum humidity on the EMS value.

In regression analysis, use the equation (4), (5), (6), (7), including 11 regression coefficients: a_{11}, a_{12}, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} and 9 regression coefficients: b_{11}, b_{12}, b_2, b_3, b_4, b_5, b_6, b_7, b_8.

2.3. Regression Analysis

2.3.1. Data gathering

Collect the test results to determine the EMS value according to humidity corresponding to three levels of lateral pressure, five levels of deviator stress, particle content of sieve No.200, liquid limit, plasticity index, moisture content, the advantages, saturation and compressive strength of the soil samples.

2.3.2. Use regression analysis algorithm

This section presents the algorithm for determining the regression coefficients of the nonlinear formula by the method of least squares [15,18-20,24,25,31].

The least squares problem is defined as follows:

Find x* that the following F(x) function reaches the minimum value:

$$F(x) = \frac{1}{2}\sum (f_i(x))^2 \qquad (9)$$

In which f_i: $R^n \rightarrow R$, i=1,…,m

An important application of the least squares problem is to find formulas based on a given set of data. Example of finding the formula for a set of m points (t1, y1), (t2, y2), ..., (t_m, y_m) in Fig. 5. The set of points is approximately equal to the formula:

$$M(x,t) = x_3 e^{x_1 t} + x_4 e^{x_2 t} \qquad (10)$$

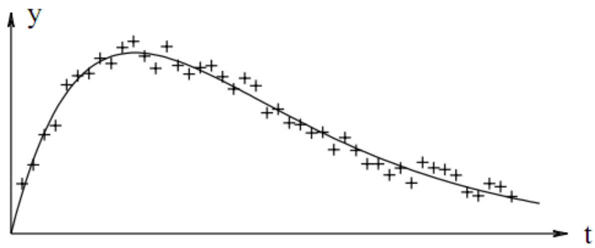

Figure 5. Graph of M (x, t) is a continuous line (The set of points (t_m, y_m) is marked as +)

The formula M(x, t) depends on the parameters x = [x_1, x_2, x_3, x_4] T. Whatever value of x we can calculate the remainder:

$$f_i(x) = y_i - M(x, t_{i)} \qquad (11)$$

$$= y_i - x_3 e^{x_1 t} + x_4 e^{x_2 t} \text{ , i = 1,…,m} \qquad (12)$$

F(x) is the sum of the sum of the squares of these remainders. The problem is to find x* such that F(x) is the smallest, x* is the regression coefficients of the formula M(x, t). In some cases, F(x) is also known as the objective function or the cost function.

2.3.3. Levenberg-Marquardt algorithm

The least squares problem can be solved by many different methods. This section describes the method for calculating the coefficients x*

Formula (9) can be rewritten as follows:

$$F(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}(f_i(\mathbf{x}))^2 = \frac{1}{2}\|\mathbf{f}(\mathbf{x})\|^2 = \frac{1}{2}\mathbf{f}(\mathbf{x})^\top \mathbf{f}(\mathbf{x}) \qquad (13)$$

Assuming f is a continuous function, f can expand the Taylor series as follows:

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\mathbf{h} + O(\|\mathbf{h}\|^2) \qquad (14)$$

In which, J is the Jacobi matrix calculated as follows:

$$(\mathbf{J}(\mathbf{x}))_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \qquad (15)$$

From (15), the first-order partial derivative of F would be:

$$\frac{\partial F}{\partial x_j}(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x}) \frac{\partial f_i}{\partial x_j}(\mathbf{x})\ . \qquad (16)$$

Thus, the gradient of F is:

$$\mathbf{F}'(\mathbf{x}) = \mathbf{J}(\mathbf{x})^{\top}\mathbf{f}(\mathbf{x}) \qquad (17)$$

We can calculate Hesse matrix of F. From equation (13), the element in position (j,k) in Hesse matrix would be:

$$\frac{\partial^2 F}{\partial x_j \partial x_k}(\mathbf{x}) = \sum_{i=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{x})\frac{\partial f_i}{\partial x_k}(\mathbf{x}) + f_i(\mathbf{x})\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\mathbf{x})\right) \qquad (18)$$

Thus:

$$\mathbf{F}''(\mathbf{x}) = \mathbf{J}(\mathbf{x})^{\top}\mathbf{J}(\mathbf{x}) + \sum_{i=1}^{m} f_i(x)\mathbf{f}_i''(\mathbf{x}) \qquad (19)$$

The Gauss-Newton method is based on the linear approximation of the f function from Taylor expansion [6,23]:

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) \simeq \boldsymbol{\ell}(\mathbf{h}) \equiv \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\mathbf{h} \qquad (20)$$

From definition of (9), F (x) is rewritten as follows:

$$\begin{aligned} F(\mathbf{x}+\mathbf{h}) \simeq L(\mathbf{h}) &\equiv \tfrac{1}{2}\boldsymbol{\ell}(\mathbf{h})^{\top}\boldsymbol{\ell}(\mathbf{h}) \\ &= \tfrac{1}{2}\mathbf{f}^{\top}\mathbf{f} + \mathbf{h}^{\top}\mathbf{J}^{\top}\mathbf{f} + \tfrac{1}{2}\mathbf{h}^{\top}\mathbf{J}^{\top}\mathbf{J}\mathbf{h} \\ &= F(\mathbf{x}) + \mathbf{h}^{\top}\mathbf{J}^{\top}\mathbf{f} + \tfrac{1}{2}\mathbf{h}^{\top}\mathbf{J}^{\top}\mathbf{J}\mathbf{h} \end{aligned} \qquad (21)$$

A^T is the transposition of matrix A, has the following 2 properties:

$$(A + B)^T = A^T + B^T \text{ and } (cA)^T = c(A^T) \qquad (22)$$

$$(AB)^T = (B^T)(A^T) \qquad (23)$$

From formula (13)

$$F(x) = ½\ f(x)^T f(x) = ½\ f^T f \qquad (24)$$

According to Equation (20) then

$$f(x+h) = f(x)+J(x)h = f + Jh \qquad (25)$$

From these 2 expressions calculate F(x + h) by replacing (20) in (13).

(The operations are matrix addition, multiplication, and transposition)

$$F(x+h) = ½\ f(x+h)^T f(x+h)$$

$$= ½\ (f+Jh)^T(f+Jh)$$

$$= ½\ (f^T+(Jh)^T)(f+(Jh))$$

$$=½\ (f^Tf + f^T(Jh)+ (Jh)^Tf + (Jh)^T(Jh))$$

$$=½\ (f^Tf + f^T(Jh)+ h^TJ^Tf + h^TJ^TJh)$$

$$= ½\ (f^Tf + f^T(Jh)+ h^TJ^Tf + h^TJ^TJh)$$

$$\text{But: } f^T(Jh) = f^T(Jh)^T)^T = (Jh)^Tf = h^TJ^Tf$$

So:

$$F(x+h) = ½\ (f^Tf + h^TJ^Tf + h^TJ^Tf + h^TJ^TJh)$$

$$F(x+h) = ½\ (f^Tf + 2h^TJ^Tf + h^TJ\ Jh)$$

$$F(x+h) = ½\ f^Tf + h^TJ^Tf + ½\ h^TJ^TJh$$

This is the formula (21).

Where f = f (x) and J = J (x).

The Gauss-Newton repetition step determines h_{gn} such that L(h) reaches minimum.

It is easy to see that the gradient and Hesse matrix of L are as follows:

$$L'(h) = J^Tf + J^TJh \qquad (26)$$

$$L''(h) = J^TJ \qquad (27)$$

From (17) and (26), deduce L'(0) = F' (x). Moreover, from (19), L"(h) is independent from h, symmetrical and always positive. This leads to L(h) reaching minimum when L'(h) = 0. The correction coefficients h_{gn} are determined by solving the following system of equations:

$$(J^TJ)h_{gn} = -J^Tf \qquad (28)$$

Equations of (28) can be solved by the Cholesky method [32].

Gauss-Newton algorithm for x* is presented as follows:

k = 0; // number of repetitions .
x = x_0 // initial value for x*
found = **false**;
while (**not** found) **and** (k < kmax)
begin
Solve $(J^TJ)\ h_{gn} = -\ J^Tf$. // Solve equations 28
if (not exist h_{gn})
found = **true**;
else
x= x + h_{gn};
k= k+1;
end.

In some cases, the Gauss-Newton Method will fail because h_{gn} cannot be found due to rank (J(x)) < m.

The Levenberg-Marquardt method is based on the Gauss-Newton method, improving the selection of the h_{lm} correction coefficients.

Levenberg-Marquardt added the formula (28) coefficient μ as follows:

$$(J^TJ + \mu I)h_{lm} = -g \qquad (29)$$

Where, J = J(x), f = f(x), $g = -J^Tf$, $\mu > 0$. I is the unit matrix.

For μ to be small, hlm is chosen equal to h_{gn}, in contrast with is large, h_{lm} is chosen by the formula:

$$h_{lm} \cong -\frac{1}{\mu}g = -\frac{1}{\mu}F'(x) \qquad (30)$$

The initial value μ_0 is chosen as follows:

$$\mu_0 = \tau.\,max_i\left(a_{ii}^{(0)}\right) \qquad (31)$$

For a_{ij} belonging to the matrix A = J (x) T J (x) and τ chosen by the user, normally $\tau = 10^{-6}$.

During the iteration, the factor μ can be updated by the ratio:

$$\varrho = \frac{F(x)-F(x+h_{lm})}{L(0)-L(h_{lm})} \quad (32)$$

The denominator of this ratio is calculated using the following formula:

$$\begin{aligned} L(\mathbf{0}) - L(\mathbf{h}_{\text{lm}}) &= -\mathbf{h}_{\text{lm}}^{\top}\mathbf{J}^{\top}\mathbf{f} - \tfrac{1}{2}\mathbf{h}_{\text{lm}}^{\top}\mathbf{J}^{\top}\mathbf{J}\mathbf{h}_{\text{lm}} \\ &= -\tfrac{1}{2}\mathbf{h}_{\text{lm}}^{\top}\left(2\mathbf{g} + (\mathbf{J}^{\top}\mathbf{J} + \mu\mathbf{I} - \mu\mathbf{I})\mathbf{h}_{\text{lm}}\right) \\ &= \tfrac{1}{2}\mathbf{h}_{\text{lm}}^{\top}(\mu\mathbf{h}_{\text{lm}} - \mathbf{g}) \;. \end{aligned} \quad (33)$$

A larger value ϱ means that $L(h_{lm})$ is closer to $F(x + h_{lm})$, so μ can be reduced, otherwise, it is small and possibly negative, so μ must be increased.

The iterations of the Levenberg-Marquardt method will stop when:

+ Reach the minimum global value: F'(x*) = g(x*) = 0, we use the condition:

$$\|\mathbf{g}\|_{\infty} \leq \varepsilon_1 \quad (34)$$

Where ε_1 is a positive, very small number chosen by the user.

+ The changeover x is very small; the following condition is used:

$$\|\mathbf{x}_{\text{new}} - \mathbf{x}\| \leq \varepsilon_2(\|\mathbf{x}\| + \varepsilon_2) \quad (35)$$

ε_2 is also a positive number and is chosen by the user.

+ Number of iterations reaches limit value k_{max} to limit infinite loop.

$$k \geq k_{max} \quad (36)$$

k_{max} is selected by the user.

The Levenberg-Marquardt algorithm is presented as follows:

```
k := 0;   ν := 2;   x := x_0
A := J(x)^T J(x);   g := J(x)^T f(x)
found := (||g||_∞ ≤ ε_1);   μ := τ * max{a_ii}
while (not found) and (k < k_max)
begin
  k := k+1;   Solve (A + μI)h_lm = −g
  if ||h_lm|| ≤ ε_2(||x|| + ε_2)
    found := true
  else
    x_new := x + h_lm
    ϱ := (F(x) − F(x_new))/(L(0) − L(h_lm))
    if ϱ > 0
      x := x_new
      A := J(x)^T J(x);   g := J(x)^T f(x)
      found := (||g||_∞ ≤ ε_1)
      μ := μ * max{1/3, 1 − (2ϱ − 1)^3}; ν := 2
    else
      μ := μ * ν;   ν := 2 * ν
end
```

This section presented the Levenberg-Marquardt algorithm to determine the regression coefficients x* for a formula. The execution time of the algorithm depends on the selection of initial values x0 and input parameters τ, ε_1, ε_2, k_{max}. In some cases, the algorithm ends with the condition (36), when k is greater than k_{max}, it is necessary to review the regression results because then F(x) has not reached the minimum at x_k. However, the Levenberg-Marquard algorithm is still a good one to solve the nonlinear minima-squared problem.

The Levenberg-Marquard algorithm has improved efficiency to solve non-linear least squares problems. This algorithm can be widely applied in many technical fields needing regression analysis to determine the coefficients of predetermined formulas.

3. Results and Discussion

To determine the regression coefficients, use the Levenberg - Marquardt algorithm in the LAPACK library (Linner Algebra Package) and write a program to analyze the regression coefficients using Visual C++ to analyze the regression coefficients for the fomula (4).

Regression coefficient analysis program is performed on the basis of experimental results to determine the values of parameters of 30 soil samples as input data including: moisture content, grain content passing sieve No.200, storm harmony, optimum humidity, plasticity index, liquid limit, flexural strength, deviator stress, lateral pressure and EMS value. Regression coefficient analysis program includes the following programs:

C.1 - Lemachieuall program performs calculation including the following steps: from the input data, calculation and determination of k_1 by formula (5), (7); calculate the exponent base by formula (4); calculate and determine k_2 by formula (6), (8); calculate M_r/P_a value by formula (26); Output the values of the regression coefficients a_{11}, a_{12}, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}; b_{11}, b_{12}, b_2, b_3, b_4, b_5, b_6, b_7, b_8 [32].

The Lemachieuall program includes subroutines:

C.2 - Lemakk is used to determine k_1 and k_2, including the steps: calculating and determining k_1 by formula (5), (7) in two cases where the lateral pressure is greater than 0 or the pressure not equal to 0; calculate the exponent base using Equation (26); k_2 is calculated and determined by the formula (6), (8) in two cases where the lateral pressure is greater than 0 or the lateral pressure is 0; calculate M_r/P_a value by formula (4); calculate the partial derivative k_1; calculate the partial derivative k_2; Output the results k_1 and k_2 values [32].

C.3 - Lemak1Ai is used to determine the coefficients of an, including the following steps: calculating and determining M_r/P_a value by formula (26); calculate the partial derivative for the coefficients a_{11}, a_{12}, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}; Output the results of the values of the coefficients a_{11}, a_{12}, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} [32].

C.4 - Lemak2Bi is used to determine the b_n coefficients including the following steps: calculating M_r/P_a value according to formula (26); calculating the partial derivative for the coefficients b_{11}, b_{12}, b_2, b_3, b_4, b_5, b_6, b_7, b_8; Output the values of the coefficients b_{11}, b_{12}, b_2, b_3, b_4, b_5, b_6, b_7, b_8. (The results of coefficients a_n and b_n are shown in Table 5) [32].

Run the Lemakk program, select the values k_1 and k_2 with correlation coefficient R^2 greater than or equal to 0.80 and the sum of squares SS less than or equal to 1 with lateral pressure greater than 0, obtain Get the EMS values [32].

Using the Lemachieuall program with 1,351 iterative steps, for convergence results, the results obtained the values of the coefficients of soil a_n and b_n with correlation coefficient R^2 equal to 0.8929 as shown in Table 4 [32].

Compare the proposed research coefficients with Kim. MS coefficients for soil type A-6 is shown in Table 6.

Coefficients a_9 equal to - 0.072 (negative number) and a_{10} equal to 3,650 (positive number) are additionally determined to take into account the adverse effect of finer particles than 0.075mm on the value of EMS, when humidity increases. The greater the fine particle percentage, the higher the percentage change of EMS value.

Table 5. a_n and b_n coefficients for sandy clay in the Mekong Delta

Coefficient	k_1	Coefficient	k_2
a_{11}	2.007	b_{11}	0.000007
a_{12}	13.612	b_{12}	0.000002
a_2	0.945	b_2	2.364
a_3	-101.047	b_3	0.360
a_4	19.444	b_4	17.900
a_5	1.044	b_5	4.371
a_6	6.951	b_6	-0.712
a_7	3.996	b_7	-0.004
a_8	17.776	b_8	-0.001
a_9	-0.072		
a_{10}	3.650		

Coefficients a_{12} equal to 13,612 and a_8 equal to 17,776 are determined to be positive to consider the adverse effect when the humidity exceeds the optimum humidity k_1 decreases, EMS decreases.

Coefficients a_3 equal to -101.047 (negative numbers) and a_4 equal to 19.444 (positive numbers) are relatively large in the correlation between the coefficients, considering the effect of saturation, saturation increases, EMS decreases.

Coefficient a_5 equal to 1.044 is defined as a positive correlation, considering the effect of the compressive strength on the value of EMS, the compressive strength decreases, the EMS decreases.

Coefficient a_6 equal to 6.951 is defined as a positive number with appropriate correlation, considering the effect of the plasticity index on the value of EMS, the plastic index increases, the EMS increases.

The coefficient a_7 is 3.996 is determined to be a positively correlated positive number, considering the effect of the difference between the liquid limit and the moisture content on the value of EMS, the closer the humidity increases to the liquid limit, EMS more and less.

The coefficients a_{11} equal to 2007 and a_2 equal to 0.945 are positive numbers, which are well correlated considering the beneficial effect of lateral pressure on the value of EMS, increased lateral pressure, increased EMS.

Similarly, a factor b_{12} of 0.000002 is defined as a positive number to consider the effect of the difference in humidity relative to the optimum humidity. The b_{12} value is very small, has almost no effect on EMS.

The coefficients b_3 equal to 0.360 and b_4 equal to 17,900 are relatively large positive numbers in the correlation between the bn coefficients, considering the significant effect of saturation on EMS, increased saturation, and decreased EMS.

The coefficients b_5 equal to 4,371 (positive number) and b6 equal to -0,712 (negative number) are correlated accordingly, considering the effect of the compressive strength on the value of EMS, the decrease in compressive strength, EMS reduction.

The coefficient b_7 equal to -0.004 is defined as a negative number and has a suitable correlation considering the effect of the plasticity index on the value of EMS, the plasticity index increases, the EMS increases.

Coefficient b_8 equal to -0.001 is defined as a negative number and has a suitable correlation considering the influence of liquid limit on EMS.

The coefficients b_{11} equal to 0.000007 and b_2 equal to 2.364 are positive numbers, which are well correlated considering the beneficial effect of lateral pressure on the value of EMS, increased lateral pressure, increased EMS. Comparing with the coefficients of Kim. MS, a_9 coefficient equal to - 0.072 (negative number) and a_{10} equal to 3,650 (positive number) is added, demonstrating the adverse effect of the fine grain content smaller than 0.075 mm to the value of EMS, when humidity increases.

The coefficients a_{11} and a_{12} are found to be decreased, while a_2 is found to be increased, and the effect of lateral pressure on EMS is different; the coefficients a_3 and a_4 found to be increased, the effect of saturation on the EMS decreased; coefficient a_5 is found to increase, and the effect of compressive strength on EMS increases; the coefficient a_6 found increases, the effect of plasticity index on EMS increases; the coefficient a_7 found is reduced, the effect of the difference between the liquid limit and the moisture content on the EMS decreases; a_8 coefficient found to be reduced, the effect when the humidity exceeds the optimum humidity on the EMS decrease;

Table 6. Comparison of proposed coefficients with Kim.MS coefficients

Coefficient	Propose	Kim.MS	Increase (%)	Reduce (%)	Note
a11	2,007	8,320	75,9		
a12	13,612	71,960	81,1		
a2	0,945	0,700		35,0	
a3	-101,047	-29,800		239,0	
a4	19,444	6,500		199,1	
a5	1,044	0,886		17,8	
a6	6,951	5,300		31,2	
a7	3,996	4,800	16,7		
a8	17,776	30,070	40,9		
a9	-0,072	0,000			Additional
a10	3,650	0,000			Additional
b11	0,000007	0,007530	99,9		
b12	0,000002	0,002700	99,9		
b2	2,364	0,523		352,0	
b3	0,360	0,205		75,7	
b4	17,900	13,400		33,6	
b5	4,371	1,130		286,8	
b6	-0,712	-0,612		16,3	
b7	-0,004	-0,0002		2.011,3	
b8	-0,001	-0,0002		211,0	

The coefficients b_{11} and b_{12} are found to be reduced, b_2 found to be increased, the effect of lateral pressure on EMS is different; the coefficients b_3 and b_4 found to be increased, the effect of saturation on the EMS decreased; coefficients b_5 and b_6 are found to be increased, the effect of compressive strength on EMS increases; the coefficient b_7 is found to increase, the effect of compressive strength on EMS increases; the coefficient b8 found increases, the effect of the liquid limit on the EMS increases.

The regression coefficients an affect the EMS value a lot, for soil type A-6 (sandy clay), Kim. MS is considered to be finer particles than 0.075mm, does not affect EMS, the coefficient a_9 is equal to 0 and the coefficient a_{10} is 0. Therefore, if considering the effect of finer particles than 0.075mm on the EMS value, the coefficient a_9 is -0.072 and the coefficient a_{10} is 3,650. There must be a corresponding increase or decrease change.

The negative regression coefficients have an increase or decrease but have little effect on EMS value.

Values of coefficients k_1 and k_2 according to Lemachieuall program are compared with k_1 and k_2 according to Kim. MS for 63 cases. Correlation coefficient for k_1 is 0.9850; for k_2 it is 0.9836.

The results of this study are consistent with the conclusions of previous studies, although the same type of sandy clay, for the Mekong Delta, EMS has a much smaller value than EMS of sandy clay in Ohio, USA [28].

Evaluate the reliability of the coefficients:

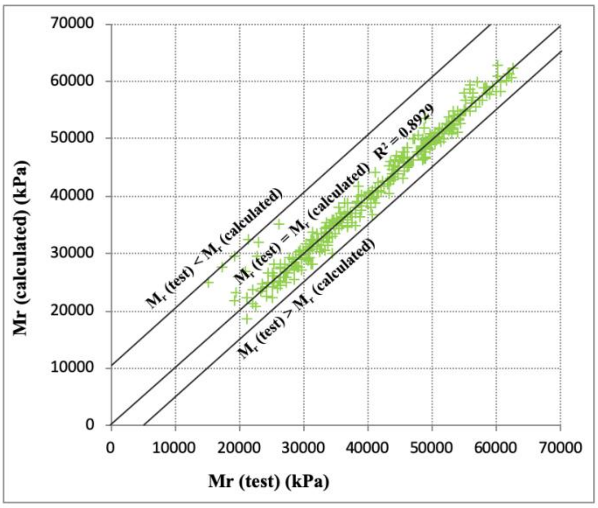

Figure 6. Correlation of the estimated M_r value by coefficient a_n and b_n, the results found with the results of the three-axis rapid compression test

To assess the reliability of the coefficients of a_n and b_n after conducting a regression analysis, it is necessary to consider the correlation coefficient between the EMS values of the experimental soil samples estimated by the formula (4), using the coefficients of a_n and b_n found by regression analysis by Levenberg - Marquardt algorithm (Table 5) with the results of rapid compression test using three-axis compression chamber on soil samples [26,27]. For each humidity, lateral pressure, deviator stress, the sample is estimated as M_r value and compared with the results of rapid compression test of that sample. Comparisons are made with the test moisture values. The results of assessing the EMS correlation between estimating EMS value according to the proposed regression coefficients compared with the results of rapid compression test using three-axis compression chamber are quite suitable, with correlation coefficient R^2 of 0.8929 acceptable as shown in Fig. 6.

4. Conclusions

1) Due to improving the selection of adjustment coefficients by adding the coefficient μ, the Levenberg-Marquardt algorithm has solved the problem with only 1,351 iterations, proving its simplicity and efficiency to solve the square problem. nonlinear minima that is sometimes not possible by the Gauss-Newton method. This algorithm is applied to regression analysis of experimental results, adding values of coefficients a_9 and a_{10}, proposing the values of the coefficients and the rest of the formula (26), with correlation coefficient R^2 equal to 0.8929, it is acceptable to apply the EMS value calculation of the sandy clay embankment in the Mekong Delta.
2) The results of regression analysis show that coefficients a_9 are and a_{10} have been added to consider the adverse effects of the percentage of grain sizes smaller than 0.075mm to the value of EMS. This result is consistent with the physical properties of sandy clay and overcome the limitation of formula (26). Comparing the differences of the research results with that of Kim. MS, the coefficients a_3, a_4, a_5, a_6, b_3, b_4, b_5, b_6, b_7 increase, indicate the degree of influence of the saturation, compressive strength and plasticity index of the soil to EMS more. And the coefficients a_{11}, a_{12}, b_{11}, b_{12} decreased, indicating the degree of influence of lateral pressure on EMS is different. The coefficients a_7, a_8 decrease, b_8 increase, determine the effect of the difference between limit of liquid and moisture, between optimum humidity and humidity on EMS decrease.
3) The regression coefficients found are only suitable for estimating EMS value for soil type A-6 (sandy clay) used as the roadbed of the Mekong Delta. However, the application of the regression coefficients found to estimate the EMS value is very simple. In this case, we just need to know the value of the input parameters including moisture content, particle content of more than 0.075mm, saturation., optimum humidity, plasticity index, liquid limit, flexural strength, lateral pressure, deviator stress. These parameters can be obtained through four laboratory experiments: standard compaction, Atterberg limit, particle composition analysis and compression. The results of estimating the EMS value by this method will save considerable time and cost compared to the field test using the Benkelman ring gauge, especially for long-distance roads.

5. Recommendation

1) The study results can be referenced to calculate deformation limitation and prevent landslide in the Mekong Delta during flooding. Designing the roadbed of the Mekong Delta region within slow-speed traffic vehicles, intersections, parking lots, etc.
2) It is necessary to continue to study and establish the formula for estimating the EMS value of the roadbeds of flooded Mekong Delta areas within the range of fast traffic vehicles, highways, national highways, etc. View Considering the effect of pH as well as the salt content on the deformation of the roadbed.

REFERENCES

[1] AASHTO T294-94, "Standard Method of Test for Resilent Modulus of Subgrade Soils and Untreated Base/Subbase Materials - SHRP Protocol P46," American Association of State Highway and Transportation Officials, Washington, D.C., 1995.

[2] AASHTO M145- 91, "The Classification of Soil-Aggregate Mixtures for Highway Construction Puroses," American Association of State Highway and Transportation Officials, Washington, D.C., 1998.

[3] AASHTO T294-03, "Standard Method of Test for Resilient Modulus of Subgrade Soils and Untreated Base/Subbase Materials – SHRP Protocol P46," American Association of State Highway and Transportation Officials, Washington, D.C., 2003.

[4] AASHTO T88-04, "Particle Size Analysis of Soils," American Association of State Highway and Transportation Officials, Washington, D.C., 2004.

[5] AASHTO T274-03, "Standard Method of Test for Resilient Modulus of Subgrade Soils," American Association of State Highway and Transportation Officials, Washington, D.C., 2003.

[6] Björck, A. "Numerical methods for least squares problems" SIAM, Philadelphia. ISBN 0-89871-360-9, 1996.

[7] Burczyk, J. M. et al, "Factors Influencing Determination of a Subgrade Resilient Modulus Value," Transportation Research Record No 1462, Transportation Research Board, National Research Council, pp. 72-78, 1994.

[8] Carmichael, R. F. III and Stuart, E., "Predicting Resilient Modulus: A Study to Determine the Mechanical Properties of Subgrade Soils," Transportation Research Record No 1043,

Transportation Research Board, National Research Council, pp.145-148, 1986.

[9] Dong-Gyou Kim, M. S. “Development of a constitutive model for resilent modulus of cohesive soils”, The Ohio State University, 2004.

[10] Drumm, E. C. et al, “Estimation of Subgrade Resilient Modulus from Standard Tests,” Journal of Geotechnical Engineering, ASCE, Vol. 116, No. 5, May, pp. 774-789, 1990.

[11] Elliot, R. P., and Thornton, S. I. ‘‘Simplification of subgrade resilient modulus testing,’’ Transportation Research Record 1192, Transportation Research Board, National Research Council, Washington, DC, pp. 1–7, 1988.

[12] Erdem Çöleri, “Relationship between resilent modulus and soil index properties of unbound materials”, Thesis, 2007.

[13] Fredlund, D. G. et al, “Relation between Resilient Modulus and Stress Research Conditions for Cohesive Subgrade Soils,” Transportation Record No 642, Transportation Research Board, National Research Council, pp. 73-81, 1977.

[14] George, K., P. “Prediction of resilient modulus from soil properties index.” University Mississippy, Nov, 2004.

[15] Henri P. Gavin (2019), The Levenberg-Marquardt algorithm for nonlinear least squares curve-fitting problems, Computer Science, IEEE Journal of Solid-State Circuits, 2019.

[16] Hicks, R. and Monismith C.L., “Factors influencing the Resilient Response of Granular Materials”, Highway Research Record 345, Highway Research Record Board, Washington, D.C., 1971.

[17] Johnson, T., Berg R., and DiMillio A., “Frost Action Predictive Techniques: An Overview of Research Results”, TRR 1089, TRB, Washington, D.C., 1986.

[18] Jose Pujol (2007). "The solution of nonlinear inverse problems and the Levenberg-Marquardt method". Geophysics (SEG) 72 (4): W1–W16. doi:10.1190/1.2732552.

[19] K. Madsen, H.B. Neilsen, O. Tingleff, “Methods for Non-linear least squares problems” 2nd edition, April 2014.

[20] Keyvan Amini,Faramarz Rostami &Giuseppe Caristi, An efficient Levenberg–Marquardt method with a new LM parameter for systems of nonlinear equations, Optimization-A Journal of Mathematical Programming and Operations Research, 2018.

[21] Kim, D. S., Kweon, G. C., and Rhee, S, “Alternative method of determining resilient modulus of subbase soils using a static triaxial test,” Canadian Geotechnical Journal, Vol.38, No. 1, pp. 117-124, 2001.

[22] Lee, W. J. et al, “Resilient Modulus of Cohesive Soils and the Effect of Freeze-Thaw,” Canadian Geotechnical Journal, Vol. 32, pp. 559-568, 1995.

[23] Letcher, Roger. Practical methods of optimization (2nd ed.). New York: John Wiley & Sons. ISBN 978-0-471-91547-8, 1987.

[24] Mark K. Transtrum and James P. Sethna “Improvements to the Levenberg-Marquardt algorithm for nonlinear least-squares minimization,” Preprint submitted to Journal of Computational Physics, January 30, 2012.

[25] Masoud Ahookhosh Francisco J. Arag ́on Artacho, Ronan M.T. Fleming Phan T. Vuong, Local convergence of the Levenberg–Marquardt method under H ̈older metric subregularity, Advances in Computational Mathematics, 2019.

[26] Ministry of Transport, Standard 22TCN - 211- 06 “Soft dressing-Requirements and Design Instructions”, Ha Noi, Viet Nam, 2006.

[27] Ministry of Science and Technology, Standard TCVN 4200-2012 “Soils - Laboratory methods for determination of compressibility”, Ha Noi, Viet Nam, 2012.

[28] Ohio Department of Transportation, Pavement Design Concepts, 1999.

[29] Pezo, R., “A General method of Reporting Resilient Modulus Tests of Soils, A Pavement Engineer’s Point of View”, 72nd Annual meeting of Transportation Research Board, Jan. 12-14, Washington, D.C., 1993.

[30] Santha, B.L., “Resilient Modulus of Subgrade Soils: Comparison of Two Constitutive Equations,” Transportation Research Record No 1462, Transportation Research Board, National Research Council, pp. 79-90, 1994.

[31] Zhu, X., Lin, G.H.: Improved convergence results for a modified Levenberg–Marquardt method for nonlinear equations and applications in MPCC. Optim. Methods Softw. 31(4), 791–804 (2016).

[32] Chieu Quang Phan, Study on the effects of moisture and fine grain content on the elastic modulus of sandy clay embankment in the Mekong Delta, PhD. Thesis, Ho Chi Minh City University of Technology, 2018.

[33] Armand Augustin Fondjo , Elizabeth Theron , "Assessment of the Mineral Composition of Heaving Soils Using Geotechnical Properties," Civil Engineering and Architecture, Vol. 8, No. 4, pp. 619 - 631, 2020. DOI: 10.13189/cea.2020.080425.

[34] Tuan Anh Nguyen , Dat Thanh Nguyen , Tung Thanh Pham , Linh Truong Chau , "Study on Using Fly Ash for Fly Ash - Soil Piles in Reinforcing Soft Ground," Civil Engineering and Architecture, Vol. 8, No. 5, pp. 1074 - 1085, 2020. DOI: 10.13189/cea.2020.080534.