Mathematics and Statistics 9(2): 151-158, 2021 http://www.hrpub.org
DOI: 10.13189/ms.2021.090209

On Three-Dimensional Mixing Geometric Quadratic Stochastic Operators

Ftameh Khaled*, **Pah Chin Hee**

Department of Computational and Theoretical Sciences, International Islamic University Malaysia (IIUM), Kuantan Campus, Pahang-Indera Mahkota, Jalan Sultan Ahmad Shah 25200, Malaysia

Received December 19, 2020; Revised January 29, 2021; Accepted March 12, 2021

Cite This Paper in the following Citation Styles

(a): *[1] Ftameh Khaled, Pah Chin Hee, "On Three-Dimensional Mixing Geometric Quadratic Stochastic Operators," Mathematics and Statistics, Vol. 9, No. 2, pp. 151 - 158, 2021. DOI: 10.13189/ms.2021.090209.*

(b): *Ftameh Khaled, Pah Chin Hee (2021). On Three-Dimensional Mixing Geometric Quadratic Stochastic Operators. Mathematics and Statistics, 9(2), 151 - 158. DOI: 10.13189/ms.2021.090209.*

Abstract It is widely recognized that the theory of quadratic stochastic operator frequently arises due to its enormous contribution as a source of analysis for the investigation of dynamical properties and modeling in diverse domains. In this paper, we are motivated to construct a class of quadratic stochastic operators called mixing quadratic stochastic operators generated by geometric distribution on infinite state space X. We also study regularity of such operators by investigating of the limit behavior for each case of the parameter. Some of non-regular cases proved for a new definition of mixing operators by using the shifting definition, where the new parameters satisfy the shifted conditions. A mixing quadratic stochastic operator was established on 3-partitions of the state space X and considered for a special case of the parameter ε. We found that the mixing quadratic stochastic operator is a regular transformation for $\frac{1}{2}<\varepsilon<\frac{1}{4}$ and is a non-regular for $\varepsilon<\frac{1}{4}$. Also, the trajectories converge to one of the fixed points. Stability and instability of the fixed points were investigated by finding of the eigenvalues of Jacobian matrix at these fixed points. We approximate the parameter ε by the parameter r_6, where we established the regularity of the quadratic stochastic operators for some inequalities that satisfy r_6. We conclude this paper by comparing with previous studies where we found some of such quadratic stochastic operators will be non-regular.

Keywords Quadratic Stochastic Operator, Mixture Geometric Distribution, Regular Transformation, Jacobian Matrix

1. Introduction

The study of quadratic stochastic operator (QSO) is rooted in the work of Bernstein [1]. Also, the study triggered the idea of heredity in QSO where it is also known as "evolutionary operator". At present, QSO can be reinterpreted as an operator describing the dynamics of gene frequencies for a set of given laws of heredity in mathematical population genetics [2-4]. Therefore, it gives significant results to both biological and mathematical areas. These QSOs are defined on $m-1$ dimensional simplex.

Let $S^{m-1}=\{x=(x_1,x_2,...,x_m)\in R^m\ \}$ for any i and $\sum_{i=1}^{m}x_i=1$ be a set of all probability distributions on measurable space $(X,P(X))$, where $P(X)$ is a power set of X, that is the set of all subset of X.

For arbitrary $(i,j)\in X\times X$ we specify a discrete probability $P(i,j,k); k=1,2,...$, that is $P(i,j,k)\geq 0$ for any k and $\sum_{k=1}^{m}P(i,j,k)=1$. The scale of the biotypes is such that the biotypes of the parents i and j unambiguously determine the probability $P(i,j,k)$ for every species k for the first generation of direct descendants of the i and j. This probability is called the heredity coefficient and denoted as $P(i,j,k)=P_{ij,k}$.

Assume that the reproduction for considered population of biotypes $X=\{1,2,...,m\}$ is a sexual one, that is $P_{ij,k}=P_{ji,k}$ for any $i,j,k\in X$. Then an evolution operator $V:X\to X$ is defined as follows:

$$(V_x)_k=\sum_{i,j=1}^{m}P_{ij,k}x_ix_j \qquad (1)$$

Such operator (1) is called quadratic stochastic operator. Note that each element $x\in S^{m-1}$ is a probability distribution on X and the population evolves by starting from an arbitrary state $x\in S^{m-1}$ then passing to the state V^x in the next generation, then to the state $V(V(x))=V^2(x)$, and so on.

Moreover, it can be redefined on the set of all probability measures on a measurable state space. For example, a QSO generated by a family of pure geometric distributions was studied in [5,6], while QSO generated by pure Poisson distributions was studied in [7,8], and QSO generated by Gaussian distributions was studied in [9-11], and QSO generated by Lebesgue distributions was studied in [12,13] respectively.

Some of applications of such QSO were established to generate algebra named as Rock-Paper-Scissor algebra [14]. Transitions of blood groups on population based on QSO were investigated in [14]. There are many other applications of QSO, e.g., evolutionary games theory [15-17], gene conversion theory [3,4] and statistical physics [18,19].

One of the main problems in the theory of QSO is to study the asymptotic behaviour of the trajectories $V^{n+1}(\lambda)=V(V^n(\lambda))$ for all $n=0,1,...$ for arbitrary $xa??S^{m-1}$.

2. Three-Dimensional Mixing Geometric QSO

Let (X,F) be a measurable space and $S(X,F)$ be the set of all probability measures on (X,F), where X is a state space and F is $\sigma-$ algebra of subsets of X.

Let $\{P(x,y,A):\ x,y\in X,\ A\in F\}$ be a family of functions on $X\times X\times F$ that satisfy the following conditions:

$P(x,y,.)\in S(X,F)$ for any fixed $x,y\in X$, that is, $P(x,y,.):F\to[0,1]$ is the probability measure on F,

1. $P(x,y,A)$ regarded as a function of two variables x and y with fixed $A\in F$ is measurable function on $(X\times X,F\otimes F)$;
2. $P(x,y,A)=P(y,x,A)\ \forall x,y\in X, A\in F$.

We consider a nonlinear transformation called quadratic stochastic operator (QSO), $V:S(X,F)\to S(X,F)$ defined by

$$(V\lambda)(A)\iint P(x,y,A)d\lambda(x)d\lambda(y) \qquad (2)$$

Where $\lambda\in S(X,F)$ is an arbitrary initial probability measurable of m partitions of the set X and $\zeta=\{B_{ij}\ :\ i,j=1,2,...,m\}$ be a corresponding partition of the cartesian square of $X\times X$, where $B_{ij}=(A_i\times A_j)$ for $i=j$ and $(A_i\times A_j)\cup(A_j\times A_i)$ if $i\neq j$. It is evident that $B_{ij}=B_{ji}$.

For any measure $\mu\in S(X,F)$, let.

$$A_s(\mu)=\sum_{n\in N_s}\mu(n), \qquad (3)$$

where $\sum_{s=0}^{m-1}A_s(\mu)=1$ and N_s be a partition of the X with $N_s=\{n\in X:n=s(\text{mod } m)\}$ with $0\le s\le m-1$

Definition 1: A geometric distribution G_r with a real parameter r; $0\le r\le 1$, is defined on X by the equation.

$$G_r(k)=(1-r)r^k \qquad (4)$$

Where $k\in X$.

If we consider $A_s(\mu)$ on geometric distribution G_r, we have.

$$A_s(G_r)=(1-r)r^s\left[\frac{1}{1-r^m}\right] \qquad (5)$$

We select a family $\{\mu_{ij}:i,j=1,2,...,m\}$ of pure probability measures on (X,F) and define probability measure $P(x,y,A)$ as follows:

$$P(x,y,A)=\mu_{ij}(A)\ \ if\ \ (x,y)\in B_{ij}$$

for arbitrary $A\in F$.

Then for arbitrary $\lambda\in S(X,F)$ we have that.

$$(V\lambda)(A) = \iint_X P(x,y,A)d\lambda(x)d\lambda(y)$$
$$= \sum_{i,j=1}^{m} \int_{A_i}\int_{A_j} \mu_{ij}(A)d\lambda(x)d\lambda(y) \quad (6)$$
$$= \sum_{i,j=1}^{m} \mu_{ij}(A)\lambda(A_i)\lambda(A_j)$$

where $A \in F$ is an arbitrary measurable set.

Let $\{V^n\lambda : n = 1,2,...\}$ be the trajectory of the initial point $\lambda \in S(X,F)$ where $V^{n+1}(\lambda) = V(V^n(\lambda))$ for all $n = 0,1,...$ with $V^0(\lambda) = 0$. Then

$$(V^{n+1}\lambda)(A) = \iint_X P(x,y,A)dV^n\lambda(x)dV^n\lambda(y)$$
$$= \sum_{i,j=1}^{m} \int_{A_i}\int_{A_j} \mu_{ij}(A)dV^n\lambda(x)dV^n\lambda(y) \quad (7)$$
$$= \sum_{i,j=1}^{m} \mu_{ij}(A)(V^n\lambda)(A_i)(V^n\lambda)(A_j)$$

with

$$(V^{n+1}\lambda)(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)(V^n\lambda)(A_i)(V^n\lambda)(A_j) \quad (8)$$

with $k = 1,2,...,m.$

Let $x_k^{(n)} = (V^n\lambda)(A_k)$ and $p_{ij,k} = \mu_{ij}(A_k).$ Then, $(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}) \in S^{m-1},$

where

$$S^{m-1} = \{x = (x_1, x_2, ..., x_m) \in R^m \text{ for any } i,j,\ x_i \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1\} \quad (9)$$

be the $m-1$-dimensional simplex, and one can rewrite the system of equations (3) as follows.

$$(Wx)_k = \sum_{i,j=1}^{m} P_{ij,k}x_ix_j$$

for all with

a) $P_{ij,k} \geq 0,$

b) $P_{ij,k} = P_{ji,k}$ for all $i,j,k,$

c) $\sum_{k=1}^{m} P_{ij,k} = 1.$

Thus, for fixed measurable $m-$partition $\xi = \{A_1, A_2, ..., A_m\}$ and selected probability measures $\{\mu_{ij} : i,j = 1,2,...,m\}$ one can approximate QSO V (1) by finite dimensional QSO W (5).

Assume $\{x^{(n)} \in S^{m-1} : n = 0,1,...\}$ is the trajectory of the initial point $x \in S^{m-1}$ where $x^{n+1} = W(x^{(n)})$ for all $n = 0,1,...$, with $x^{(0)} = x.$

Definition 2. A point $a \in S^{m-1}$ is called a fixed point of a QSO W if $W(a) = a.$ let $Fix(W)$ be a set of all fixed points of QSO $W.$

Definition 3 A QSO W is called regular if for any initial point $x \in S^{m-1}$ the limit.

$$\lim_{n\to\infty} W^{(n)}(x) \quad (10)$$

exists.

Note that the limit point is a fixed point of a QSO W. Thus, the fixed points of QSO describe limit or long-run behaviour the trajectories of any initial point. Limiting behaviour of trajectories and the fixed points of QSO play important role in many applied problems [8,9,13,14]. Recall ergodic hypothesis for quadratic stochastic operators [16].

Definition 4. A QSO W is said to be ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} V^k(x) \quad (11)$$

exists for any $x \in S^{m-1}$.

Note that a regular QSO W is ergodic, however in general, ergodicity does not imply regularity. For the counter examples on finite dimensional QSO, one can refer [1,8,9,10,13,14,15,17].

In 1-dimensional case, i.e., $m = 2$ the behaviour of the iterations was found to be rather simple in [14,15].

For 2-dimensional case, a quadratic stochastic operator W on S^1 has the following form

$$(Wx)_1 = ax_1^2 + 2bx_1x_2 + cx_2^2$$
$$(Wx)_1 = (1-a)x_1^2 + 2(1-b)x_1x_2 + (1-c)x_2^2 \quad (12)$$

where $a = p_{11,1},\ b = p_{12,1} = p_{21,1},\ c = p_{22,1}$ are arbitrary coefficients with $0 < a,b,c < 1$.

Eliminating the second coordinate by the identity $x_2 = 1 - x_1$ we have

$$y = (a - 2b + c)x^2 + 2(b - c)x + c \quad (13)$$

It is evident that a function in term of x in (8) maps the segment $[0,1]$ (1-dimensional simplex) into itself with

$$y|_{x=0} = c,\ \ y|_{x=1} = a. \quad (14)$$

In order to avoid the analysis of particularities, by considering that $a < 1$ and $c > 0.$ Then, the following statements are valid.

Theorem 1 [20]. A fixed point of the transformation (8) is unique and belongs to the open interval $(0,1)$.
We consider the discriminant of the quadratic equation.

$$(a-2b+c)x^2+2(b-c)x+c=0 \quad (15)$$

to investigate the local characteristic of the fixed point:

$$\Delta = 4(1-a)c+(1-2b)^2. \quad (16)$$

Theorem 2 [20]: A fixed point of the transformation (8) is attractive when $0<\Delta<4$, and if $4<\Delta<5$. then it is repelling.

Corollary: [20]: All trajectories converge to a fixed point when $0<\Delta<4$.

Theorem 3 [20]: If $4<\Delta<5$, then there exists a cycle of second order and all trajectories tend to this cycle except the stationary starting with fixed point.

Let us consider in detail when $0<\Delta<4$ and when $4<\Delta<5$.

Theorem 4 [2]: one has that $4<\Delta<5$ if and only if $a\in\left[0,\frac{1}{4}\right], c>\frac{3}{4(1-a)}$ and

$$b<\frac{1}{2}-\sqrt{1-(1-a)c} \text{ or } b>\frac{1}{2}+\sqrt{1-(1-a)c} \quad (17)$$

We construct the family of mixture geometric nonlinear transformations defined on the countable sample space of nonnegative integers generated by 3-partitions.

Let

$A_1=\{3k \text{ for all } k\in N\}, A_2=\{3k+1 \text{ for all } k\in N\}$, $A_3=\{3k+2 \text{ for all } k\in N\}$.

We consider a mixture distribution as follows:

$$\begin{aligned}
P_{ij,k}&=(1-r_1)r_1^k : i,j\in A_1\times A_1,\\
P_{ij,k}&=(1-r_2)r_2^k : i,j\in A_2\times A_2,\\
P_{ij,k}&=(1-r_3)r_3^k : i,j\in A_3\times A_3,\\
P_{ij,k}&=(1-r_4)r_4^k : i,j\in A_1\times A_2\cup A_2\times A_1,\\
P_{ij,k}&=(1-r_5)r_5^k : i,j\in A_1\times A_3\cup A_3\times A_1,\\
P_{ij,k}&=(1-r_6)r_6^k : i,j\in A_2\times A_3\cup A_3\times A_2.
\end{aligned} \quad (18)$$

with,

$$A_1(r)=\frac{1}{r^2+r+1}, A_1(r)=\frac{r}{r^2+r+1}, A_1(r)=\frac{r^2}{r^2+r+1}.$$

Then for any initial measure $\lambda\in S(X,F)$ we have that,

$$\begin{aligned}
V^{n+1}\mu(k)&=(1-r_1)r_1^k\left[A_1^2\left(V^n\mu\right)\right]+(1-r_2)r_2^k\left[A_2^2\left(V^n\mu\right)\right]\\
&+(1-r_3)r_3^k\left[A_3^2\left(V^n\mu\right)\right]\\
&+(1-r_4)r_4^k\left[2A_1\left(V^n\mu\right)A_2\left(V^n\mu\right)\right]\\
&+(1-r_5)r_5^k\left[2A_1\left(V^n\mu\right)A_3\left(V^n\mu\right)\right]\\
&+(1-r_6)r_6^k\left[2A_2\left(V^n\mu\right)A_3\left(V^n\mu\right)\right]
\end{aligned} \quad (19)$$

with,

$$\begin{aligned}
A_l(V^{n+1}\mu)&=A_l(r_1)\left[A_1^2\left(V^n\mu\right)\right]+A_l(r_2)\left[A_2^2\left(V^n\mu\right)\right]\\
&+A_l(r_3)\left[A_3^2\left(V^n\mu\right)\right]\\
&+A_l(r_4)\left[2A_1\left(V^n\mu\right)A_2\left(V^n\mu\right)\right]\\
&+A_l(r_5)\left[2A_1\left(V^n\mu\right)A_3\left(V^n\mu\right)\right]\\
&+A_l(r_6)\left[2A_2\left(V^n\mu\right)A_3\left(V^n\mu\right)\right]
\end{aligned} \quad (20)$$

where $l=1,2,3$.

It is true that the limiting behaviour of the recurrence equation in (19) is fully determined by limiting behaviour of recurrence equations in (20).

Since $\left[A_1\left(V^n\mu\right)\right]+\left[A_2\left(V^n\mu\right)\right]+\left[A_3\left(V^n\mu\right)\right]=1$ where $\left[A_1\left(V^n\mu\right)\right],\left[A_2\left(V^n\mu\right)\right],\left[A_3\left(V^n\mu\right)\right]>0.$

The recurrence equations in (13) can be rewrite as

$$\begin{aligned}
x_1^{'}&=A_1(r_1)x_1^2+A_1(r_2)x_2^2+A_1(r_3)x_3^2\\
&\quad+2A_1(r_4)x_1x_2+2A_1(r_5)x_1x_3+2A_1(r_6)x_2x_3,\\
x_2^{'}&=A_2(r_1)x_1^2+A_2(r_2)x_2^2+A_2(r_3)x_3^2\\
&\quad+2A_2(r_4)x_1x_2+2A_2(r_5)x_1x_3+2A_2(r_6)x_2x_3,\\
x_3^{'}&=A_3(r_1)x_1^2+A_3(r_2)x_2^2+A_3(r_3)x_3^2\\
&\quad+2A_3(r_4)x_1x_2+2A_3(r_5)x_1x_3+2A_3(r_6)x_2x_3,
\end{aligned} \quad (21)$$

with

$x_1=\left[A_1\left(V^n\mu\right)\right], x_2=\left[A_2\left(V^n\mu\right)\right], x_3=\left[A_3\left(V^n\mu\right)\right]>0.$,

and $x_1+x_2+x_3=1$.

According to the previous representation, a 3-dimension simplex S^2 (case $m=3$) for an arbitrary QSO has the following form:

$$\begin{aligned}
x_1^{'}&=a_1x_1^2+a_2x_2^2+a_3x_3^2+2a_4x_1x_2+2a_5x_1x_3+2a_6x_2x_3,\\
x_2^{'}&=b_1x_1^2+b_2x_2^2+b_3x_3^2+2b_4x_1x_2+2b_5x_1x_3+2b_6x_2x_3,\\
x_3^{'}&=c_1x_1^2+c_2x_2^2+c_3x_3^2+2c_4x_1x_2+2c_5x_1x_3+2c_6x_2x_3,
\end{aligned} \quad (22)$$

where a_i, b_i, c_i are arbitrary parameters with

$$\sum_{i=1}^{6} a_i + b_i + c_i = 1. \quad (23)$$

Remark: if $r_1 = r_4; r_2 = r_5; r_3 = r_6$, then the QSO W was investigated in [15], where the authors proved the regularity of such operator.

To avoid the complexity of study of the regularity of W we consider the QSO W with

$a_1 = b_2 = c_3 = a_4 = c_5 = 1;$

$a_2 = a_3 = a_5 = b_1 = b_3 = b_4 = b_5 = c_1 = c_2 = c_4 = c_6 = 0;$

$a_6 = b_6 = \varepsilon.$

Then, the QSO W can be rewrite as

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2\varepsilon x_2x_3, \\ x_2' &= x_2^2 + 2(1-2\varepsilon)x_2x_3, \\ x_3' &= x_3^2 + 2x_1x_3 + 2\varepsilon x_2x_3, \end{aligned} \quad (24)$$

Remark: for $\varepsilon = 0$, the QSO W has the following form:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2, \\ x_2' &= x_2^2 + 2x_2x_3, \\ x_3' &= x_3^2 + 2x_1x_3. \end{aligned} \quad (25)$$

It was studied by Zakharevich [21], where the author investigated ergodicity and regularity of such operators.

Next, we study the regularity of QSO W in terms of ε.

3. Regularity

In this section, the general form of the QSO W has the form:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2A_1(r_6)x_2x_3, \\ x_2' &= x_2^2 + 2(1-2A_1(r_6))x_2x_3, \\ x_3' &= x_3^2 + 2x_1x_3 + 2A_1(r_6)x_2x_3, \end{aligned} \quad (26)$$

We consider the measure.

$A_1(r_6) = P(x,y,3k) = \sum_{k=1}^{\infty}(1-r_6)r_6^{3k} = \frac{1}{r_6^2 + r_6 + 1}$ is a shifted geometric distribution defined as follows:

$$A_1(r_6) = P(x,y,3k) = \sum_{k=1}^{\infty}(1-r_6)r_6^{3k} = 0 \text{ for } k = 0,1,$$

and

$$A_1(r_6) = P(x,y,3k) = \sum_{k=1}^{\infty}(1-r_6)r_6^{3k} = \frac{r_6^2}{r_6^2 + r_6 + 1} \text{ for}$$

To investigate the regularity of the QSO, we consider the following cases:

3.1. Case One: For $\varepsilon = \frac{1}{4}$

In this part, we consider the QSO W for $\varepsilon = \frac{1}{4}$ so that $\frac{r_6^2}{r_6^2 + r_6 + 1}$ then $r_6 = \frac{1 \pm \sqrt{13}}{6}$.

We can rewrite the QSO W in the following form:

For $\varepsilon = \frac{1}{4}$, one can rewrite the QSO W as the following form:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + \frac{1}{2}x_2x_3, \\ x_2' &= x_2^2 + x_2x_3, \\ x_3' &= x_3^2 + 2x_1x_3 + \frac{1}{2}x_2x_3, \end{aligned} \quad (27)$$

The QSO W has four fixed points namely, $a_1 = (1,0,0), a_2 = (0,1,0), a_3 = (0,0,1), a_4 = (0,0,0),$

For $0 \le x_i \le 1$ and $\sum_{i=1}^{3} x_i = 1,$ the function $f(x_2) = x_2(1-x_1)$ is a decreasing function and $x_2 = 0$ is a fixed point of $f(x_2)$ thus, $\lim_{n\to\infty} f^{(n)}(x_2) = 0,$ where $f^{(n)}(x_2) = f\left(f^{(n)}(x_2)\right).$

It is easy to see for $f(x_1) = x_1(1 + x_2 - x_3) + \frac{1}{2}x_2x_3,$ we have that.

$$\lim_{n\to\infty} f^{(n)}(x_1) = 0. \quad (28)$$

Similarly,

$$\lim_{n\to\infty} f^{(n)}(x_3) = 1 \quad (29)$$

According to above calculations, we have proved the following theorem.

Theorem 5: A quadratic stochastic operator W (26), is a regular transformation for $r_6 = \frac{1 \pm \sqrt{13}}{6}$. and the iterations converge to the fixed point (0,0,1), i.e.,

$$\lim_{n\to\infty} W^{(n)} = (0,0,1).$$

3.2. Case Two: For $\varepsilon = \frac{1}{2}$

For $\varepsilon = \frac{1}{4}$, we can rewrite the QSO W as follows:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + x_2x_3, \\ x_2' &= x_2^2, \\ x_3' &= x_3^2 + 2x_1x_3 + x_2x_3, \end{aligned} \tag{30}$$

The QSO W has four fixed points namely, $a_1 = (1,0,0), a_2 = (0,1,0), a_3 = (0,0,1), a_4 = (0,0,0),$.

Also, Jacobian matrix of the QSO W has the form $\begin{pmatrix} 2x_1+2x_2 & 2x_1+x_3 & x_2 \\ 0 & 2x_2 & 0 \\ 2x_3 & x_3 & 2x_3+2x_1+x_2 \end{pmatrix}$ and the eigenvalues at the point $(0,0,1)$ is $\lambda = 0$ then the point a_3 is a stable point. (it is easy check for instability of a_1, a_2). On the other hand, for a function $f(x) = x^2$ we have $\lim_{n\to\infty} f^{(n)}(x) = 0.$ where $x \in (0,1)$. Therefore, $\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} x_2^{(n)} = 0.$. Thus,

$$\lim_{n\to\infty} W^{(n)} = (0,0,1).$$

According to previous results the following theorem has been proved.

Theorem 6: The quadratic stochastic operator W in (26) is a regular transformation and its trajectory converges to the fixed point $(0,0,1)$ when the parameter $\varepsilon = \frac{1}{2}$.

Remark: For the parameter $\frac{r_6^2}{r_6^2 + r_6 + 1} = \frac{1}{2}$, we have, then the QSO W is a regular transformation for $r_6 = \frac{1 \pm \sqrt{5}}{2}$.

3.3. Case Three: For $\frac{1}{4} < \varepsilon < \frac{1}{2}$.

According to the first and second cases we conclude that for $\frac{1}{4} < \varepsilon < \frac{1}{2}$ leads to $\frac{1}{4} < \frac{r_6^2}{r_6^2 + r_6 + 1} < \frac{1}{2}$, therefore $\frac{1-\sqrt{5}}{2} < r_6 < \frac{1-\sqrt{13}}{6}$ or $\frac{1+\sqrt{13}}{6} < r_6 < \frac{1-\sqrt{5}}{2}$. The QSO W (26) has five fixed points. Namely,

$$a_1 = (1,0,0), a_2 = (0,1,0), a_3 = (0,0,1), a_4 = (0,0,0),$$

$$a_5 = \left(\frac{1-6\varepsilon+8\varepsilon^2}{3-12\varepsilon+8\varepsilon^2}, \frac{1-4\varepsilon}{3-12\varepsilon+8\varepsilon^2}, \frac{1-2\varepsilon}{3-12\varepsilon+8\varepsilon^2}\right).$$

A Jacobian matrix has the form:

$$\begin{pmatrix} 1+x_2-x_3 & x_1+2\varepsilon x_3 & -x_1+2\varepsilon x_2 \\ -x_2 & 1-x_1+(1-4\varepsilon)x_3 & (1-4\varepsilon)x_2 \\ x_3 & (2\varepsilon-1)x_3 & 1+x_1+(2\varepsilon-1)x_2 \end{pmatrix}$$

and eigenvalue of the Jacobian matrix at the point $(1,0,0)$ are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2,$ at the point $(0,1,0)$ are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2\varepsilon,$ at the point $(0,0,1)$ are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4\varepsilon,$, at a_4 are $\lambda = 1$ at the point a_5 are

$$\lambda_1 = 1, \lambda_2 = \frac{-i\sqrt{64\varepsilon^4 - 144\varepsilon^3 + 103\varepsilon^2 - 30\varepsilon + 3} + 8\varepsilon^2 - 13\varepsilon + 3}{8\varepsilon^2 - 12\varepsilon + 3},$$

$$\lambda_3 = \frac{i\sqrt{64\varepsilon^4 - 144\varepsilon^3 + 103\varepsilon^2 - 30\varepsilon + 3} + 8\varepsilon^2 - 13\varepsilon + 3}{8\varepsilon^2 - 12\varepsilon + 3}.$$

It is obvious that the fixed points a_1, a_2, a_4, a_5 are unstable points and a_3 is a stable point. Therefore, the transformation $f(x_2) = x_2(1 - x_1 + (1-4\varepsilon)x_3)$ is a decreasing transformation and $x_2 = 0$ is a fixed point. Then, $\lim_{n\to\infty} x_2{}^n = 0,$, so that, $\lim_{n\to\infty} x_1{}^n = 0,$ and $\lim_{n\to\infty} x_3{}^n = 1,$. Therefore, we have proved the following theorem.

Theorem 7: If the parameter r_6 satisfies the inequalities $\frac{1-\sqrt{5}}{2} < r_6 < \frac{1-\sqrt{13}}{6}$ or $\frac{1+\sqrt{13}}{6} < r_6 < \frac{1-\sqrt{5}}{2}$. Then, the quadratic stochastic operator W (26) is a regular QSO and the limiting point of such operator converges to the fixed point $(0,0,1)$.

3.4. Case Four: For $0 < \varepsilon < \frac{1}{4}$

For the case $0 < \varepsilon < \frac{1}{4}$, we introduce the following domains:

$$\begin{aligned} G_1 &= \{(x,y,z) \in S^2 : x \ge y \ge z\}, \\ G_2 &= \{(x,y,z) \in S^2 : x \ge z \ge y\}, \\ G_3 &= \{(x,y,z) \in S^2 : z \ge x \ge y\}, \\ G_4 &= \{(x,y,z) \in S^2 : z \ge y \ge x\}, \\ G_5 &= \{(x,y,z) \in S^2 : y \ge z \ge x\}, \\ G_6 &= \{(x,y,z) \in S^2 : y \ge x \ge z\}. \end{aligned} \tag{31}$$

For any point $a=(x_1,x_2,x_3)\in G_1$ where $a\neq a_5$, then $x_1\geq x_2\geq x_3$.

It is easy to see that $x_1^{'}\geq x_1; x_2^{'}\leq x_2; x_3^{'}\geq x_3$. Then, after some iterations, the iterated sequence moves from G_1 to G_2

For any point $a=(x_1,x_2,x_3)\in G_2$ then $x_1\geq x_3\geq x_2$.

Assume $x_1^{'}\geq x_1$, then $x_1(x_2-x_3)+2\varepsilon x_2x_3\geq 0$ which implies that $\varepsilon\leq\frac{1-x_2}{2x_3}-\frac{1-x_3}{2x_2}$. For example, if $x_2=\frac{1}{8}, x_3=\frac{1}{4}$, then, $\varepsilon\leq-\frac{5}{4}$ which contradicts to $0<\varepsilon<\frac{1}{4}$. Hence, $x_1^{'}\leq x_1$. It is obvious that $x_2^{'}\leq x_2, x_3^{'}\geq x_3$ after some iterations the iterated sequence will move from G_2 to G_3.

Also, for any point $a=(x_1,x_2,x_3)\in G_3$ then $x_3\geq x_1\geq x_2$.

Similarly, one can investigate that for any point $a=(x_1,x_2,x_3)\in G_i$, the iterated sequence will move from G_i to G_{i+1} for $i=3,4,5$ and move from G_6 to G_1. Collecting all the trajectories of the QSO W, we conclude the following theorem.

Theorem 9: A quadratic stochastic operator W in (15) is a non-regular QSO for the parameter $0<\varepsilon<\frac{1}{4}$ and for any point into simplex, the QSO W has the route $G_1\to G_2\to G_3\to G_4\to G_5\to G_6\to G_1$..

Remark: for $0<\varepsilon<\frac{1}{4}$ leads to the parameter r_6 satisfies the inequality $0<r_6<\frac{1+\sqrt{13}}{6}$. Then, the QSO is a nonregular transformation.

Remark: It is easy to see a Jacobian matrix shows instability of the fixed points.

4. Results

In this research, we constructed a new type of quadratic stochastic operators which is called mixing Geometric QSO.

A mixing Geometric quadratic stochastic operator $W(15)$ has the following properties:

1. regular when $\frac{1}{4}\leq\varepsilon\leq\frac{1}{2}$ and the behavior trajectory converges to the fixed point $(0,0,1)$.
2. non-regular when $0<\varepsilon<\frac{1}{4}$ and for any point into simplex, the QSO $0<\varepsilon<\frac{1}{4}$ has the route $G_1\to G_2\to G_3\to G_4\to G_5\to G_6\to G_1$.

5. Conclusion

In this paper, we construct mixing geometric QSO. For a special case of parameters P_{ijk}, of the QSO, study of the regularity of such operators. Comparing with previous studies for example in [5] the authors proved every quadratic stochastic operator generated by geometric distribution is a regular transformation. Moreover in [6] the authors also proved every quadratic stochastic operator generated by geometric distribution and three partitions with a different parameters is a regular transformation. Also, the existence of regularity and ergodicity of QSO W on 3-dimension simplex can be generalized to higher dimension.

Acknowledgment

This research was partially funded by fundamental research grant scheme from ministry of education Malaysia project code FRGS/1/2019/STG06/UIAM/03/1 with project ID (FRGS19-122-0731).

REFERENCES

[1] Bernstein S.N., Solution of a Mathematical Problem Connected with the Theory of Heredity. The Annals of Mathematical Statistics, 13(1), 53–61. (1942).

[2] Ganikhodzhaev R., Mukhamedov F., Rozikov U., Quadratic Stochastic Operators and Processes: Results and Open Problems. Infinite Dimensional Analysis, Quantum Probability and Related Topics, 14(02), (2011). doi.org/10.1142/S0219025711004365

[3] Nagylaki T., Evolution of a Finite Population under Gene Conversion. Proceedings of the National Academy of Sciences of the United States of America, 80(20), 6278–6281. (1983) DOI: 10.1073/pnas.80.20.6278

[4] Nagylaki T., Evolution of a Large Population under Gene Conversion. Proceedings of the National Academy of Sciences of the United States of America, 80(October), 5941–5945. (1983). DOI: 10.1073/pnas.80.19.5941

[5] Ganikhodjaev N., Hamzah N. Z. A., "Geometric quadratic stochastic operator on countable infinite set". AIP

Conference Proceedings, 1643(1), pp. 706-712 (2015). DOI.org/10.1063/1.4907516

[6] Karim S.N, Hamzah N. Z. A., Ganikhodjaev N., A class of geometric quadratic stochastic operator on countable state space and its regularity Malaysian Journal of Fundamental and Applied Sciences (MJFAS), 15 (6). pp. 872-877. (2019). DOI: org/10.11113/mjfas.v15n6.1423

[7] Ganikhodjaev N., Hamzah N. Z. A., On Poisson Nonlinear Transformations, The Scientific World Journal, (832861). 0 pp. 1-7. (2014). DOI: org/10.1155/2014/832861

[8] Hamzah N. Z. A., Ganikhodjaev N., Nonhomogeneous Poisson Nonlinear Transformations on Countable Infinite Set, Malaysian Journal of Mathematical Sciences, 10 (Special Issue). 0 pp. 143-155. (2016). URL: http://einspem.upm.edu.my/journal

[9] Ganikhodjaev N., Hamzah N. Z. A., On Gaussian nonlinear transformations. AIP Conference Proceedings, 1682. pp. 040009-1. 1551-7616 (O) (2015). doi.org/10.1063/1.49324 82

[10] Ganikhodjaev N., Hamzah N. Z. A., On (3,3)-Gaussian Quadratic Stochastic Operators. J. Phys.: Conference Series. 819 012007(2017). DOI:org/10.1088/1742-6596/819/1/012 007

[11] Hamzah N. Z. A., Ganikhodjaev N., On non-ergodic Gaussian quadratic stochastic operators. Proceedings of the 25th National Symposium on Mathematical Sciences: Mathematical Sciences as the Core of Intellectual Excellence (SKSM 2017) AIP Conference Proceedings, 1974. 0 pp. 030021-1-030021-7. (2018).

[12] Ganikhodjaev N., on stochastic processes generated by quadratic operators, Journal of Theoretical Probability. 4, 639-653. (1991). DOI: org/10.1007/BF01259547

[13] Ganikhodjaev N., Hamzah N. Z. A., Quadratic Stochastic Operators on Segment [0,1] and Their Limit Behavior Indian Journal of Science and Technology, 8 (30). 0 pp. 1-12. (2015). DOI: 10.17485/ijst/2015/v8i30/86780

[14] Ganikhodjaev N., Ftameh K., on classification of the genetic and evolution Rock-Paper-Scissor algebras AIP Conference Proceeding, 2184(1), 20004. (2019) DOI: org/10.1063/1.51 36358

[15] Ganikhodjaev N., Daoud J. I., Usmanova M., Linear and nonlinear models of heredity for blood groups and Rhesus linear and nonlinear models of heredity for blood groups and Rhesus factor, journal of applied sciences 10(16), 1748-1754. (2010). DOI: 10.3923/jas.2010.1748.1754

[16] Akin E., Losert V., Dynamics of Games and Genes: Discrete Versus Continuous Time. Journal of Mathematical Biology, 17(2), 241–251. (1983) doi.org/10.1007/BF00305762

[17] Akin E., Losert V., Evolutionary Dynamics of Zero-Sum Games. Journal of Mathematical Biology, 20(3), 231–258. (1984). DOI: org/10.1007/BF00275987

[18] Ganikhodjaev N., & Rozikov, U. (2006). On Quadratic Stochastic Operators Generated by Gibbs Distributions. Regular and Chaotic Dynamics, 11(4), 467–473. DOI: 10.1070/RD2006v011n04ABEH000364

[19] Ganikhodjaev N., (2012). Lattice Gas and Thermodynamics in Models of Heredity. International Journal of Modern Physics: Conference Series, 09(01), 157–162. DOI: org/10.1142/S2010194512005211

[20] Y. I. Lyubich, E. Akin, Mathematical Structures in Population Genetics, Biomathematics, 22, (Springer-Verlag 1992).

[21] Zakharevich M. I., On the Behaviour of Trajectories and the Ergodic Hypothesis for Quadratic Mappings of a Simplex. Russian Mathematical Surveys, 33(6), (2007). DOI: org/10.1070/RM1978v033n06ABEH003890