Mathematics and Statistics 9(2): 98-105, 2021 http://www.hrpub.org
DOI: 10.13189/ms.2021.090203

Numerical Solution for Fuzzy Diffusion Problem via Two Parameter Alternating Group Explicit Technique

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Received November 26, 2020; Revised January 21, 2021; Accepted March 24, 2021

Cite This Paper in the following Citation Styles

(a): *[1] A. A. Dahalan, J. Sulaiman , "Numerical Solution for Fuzzy Diffusion Problem via Two Parameter Alternating Group Explicit Technique," Mathematics and Statistics, Vol. 9, No. 2, pp. 98 - 105, 2021. DOI: 10.13189/ms.2021.090203.*

(b): *A. A. Dahalan, J. Sulaiman (2021). Numerical Solution for Fuzzy Diffusion Problem via Two Parameter Alternating Group Explicit Technique. Mathematics and Statistics, 9(2), 98 - 105. DOI: 10.13189/ms.2021.090203.*

Abstract The computational technique has become a significant area of study in physics and engineering. The first method to evaluate the problems numerically was a finite difference. In 2002, a computational approach, an explicit finite difference technique, was used to overcome the fuzzy partial differential equation (FPDE) based on the Seikkala derivative. The application of the iterative technique, in particular the Two Parameter Alternating Group Explicit (TAGE) method, is employed to resolve the finite difference approximation resulting after the fuzzy heat equation is investigated in this article. This article broadens the use of the TAGE iterative technique to solve fuzzy problems due to the reliability of the approaches. The development and execution of the TAGE technique towards the full-sweep (FS) and half-sweep (HS) techniques are also presented. The idea of using the HS scheme is to reduce the computational complexity of the iterative methods by nearly/more than half. Additionally, numerical outcomes from the solution of two experimental problems are included and compared with the Alternating Group Explicit (AGE) approaches to clarify their feasibility. In conclusion, the families of the TAGE technique have been used to overcome the linear system structure through a one-dimensional fuzzy diffusion (1D-FD) discretization using a finite difference scheme. The findings suggest that the HSTAGE approach is surpassing in terms of iteration counts, time taken, and Hausdorff distance relative to the FSTAGE and AGE approaches. It demonstrates that the number of iterations for HSTAGE approach has decreased by approximately 71.60-72.95%, whereas for the execution time, the implementation of HSTAGE method is between 74.05-86.42% better. Since TAGE is ideal for concurrent processing, this method has been seen as the key benefit as it consumes sets of independent tasks that can be performed at the same time. The ability of the suggested technique is projected to be useful for the advanced exploration in solving any multi-dimensional FPDEs.

Keywords Fuzzy Heat Equation, Implicit Scheme, Two-Stage Iteration, One-Dimensional, Finite Difference Method, Half-Sweep

1. Introduction

The computational practice has progressed significantly in the past 40 years, despite the advances made in computer technology. The numerical approach has become an essential field in the area of research in physics and engineering. The earliest method among the most crucial techniques for numerically analyzing the practical problems was a finite difference. The finite difference scheme employs divided difference expressions developed from the local Taylor series to substitute differential or partial operators. [1] was the first to introduce the notion of the fuzzy derivative. Although, the one who described and applied the extension theory was [2]. The equations of fuzzy differential and the problem of fuzzy initial value have frequently been addressed [3-6]. In [7,8], a computational approach, an explicit finite difference technique, is used to overcome a fuzzy partial differential equation (FPDE) based on the Seikkala derivative. Typically, it is exceedingly difficult to calculate the solution of FPDEs. The exact solution to this complication can only be sought in certain particular situations.

Alternating Group Explicit (AGE) approach has been demonstrated to remain an important as well as an efficient tool since it was developed. Evans and Sahimi [9] illustrates the extended AGE technique for problems involving both parabolic and hyperbolic PDEs. In this approach, however, only one parameter has been taken into account. Later, this technique has also been expanded by [10], through the implementation of the Two Parameter Alternating Group Explicit (TAGE) technique, which verified its better than the AGE scheme.

The one-dimensional fuzzy diffusion (1D-FD) equation will be numerically solved using the second-order central finite difference approximation to discretize 1D-FD in linear systems based on the derivative of Seikkala [6]. The generated linear systems are then resolved by the TAGE method iteratively. Previous studies associated to the TAGE iterative technique and its modification [11-13] have demonstrated that this approach was used extensively to unravel the problems in non-fuzzy cases. This article widens the use of the TAGE iterative approach to solve fuzzy issues because of the efficiency of the methods.

The iterative technique turns out to be a natural alternative for obtaining a solution to the fuzzy numerical problem since the fuzzy linear systems exist. Finding towards the comprehension to the half-sweep iterative technique is motivated by an earlier analysis of the Explicit Decoupled Group scheme [14]. The main feature of this concept is the methodology of reducing the computational complexity. The finite grid (see Fig. 1) is used to elucidate the construction of the equations for full- and half-sweep difference approximation. Applications of these projected iterative approaches to the inner solid nodal points shall be carried out until the convergence test has been identified. In the meantime, the approximation solutions can be determined using the direct method for the remaining points [14,15].

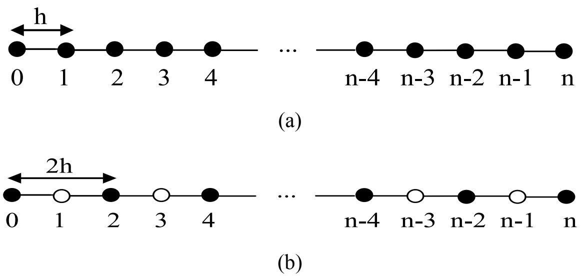

Figure 1. The arrangement of solid nodal points for full- (a) and half-sweep (b) cases

2. Methodology

For the real numbers, let $\tilde{U}$ be a fuzzy subset. As part of the membership function, the said fuzzy subset should be described as $\tilde{U}(x)$ calculated at x for a number in $[0,1]$. An α-cut of the fuzzy subset (with $\alpha =$ crisp number) is symbolized by $\tilde{U}(\alpha)$ and then defined with $I=\{(x,t)\,|\,0\le x\le 1,\ 0\le t\le T\}$. The intervals for all crisp fuzzy subsets shall be shown by $\tilde{U}(\alpha)=\left[\underline{U}(\alpha),\overline{U}(\alpha)\right]$. This is due to the fuzzy numbers α-cut are continuously closed and bounded [7]. Presuming the parametric system of the fuzzy function U is $(\underline{U},\overline{U})$.

Considering the equation of fuzzy diffusion as follows

$$\frac{\partial \tilde{U}}{\partial t}(x,t)-\lambda\frac{\partial^2 \tilde{U}}{\partial x^2}(x,t)=0,\quad 0<x<l,\ t>0. \tag{1}$$

Let $R=\{(x,t)\,|\,(x,t)\in I\}$, while S represent the boundary of R. The subsequent issues will be solved numerically:

$$\begin{aligned}&\frac{\partial \underline{U}}{\partial t}(x,t;\alpha)-\lambda\frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha)=0,\\ &\underline{U}(0,t;\alpha)=\underline{U}(l,t;\alpha)=0,\quad t>0,\\ &\underline{U}(x,0;\alpha)=\underline{f}(x;\alpha),\quad 0\le x\le l\end{aligned} \tag{2a}$$

and

$$\begin{aligned}&\frac{\partial \overline{U}}{\partial t}(x,t;\alpha)-\lambda\frac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha)=0,\\ &\overline{U}(0,t;\alpha)=\overline{U}(l,t;\alpha)=0,\quad t>0,\\ &\overline{U}(x,0;\alpha)=\overline{f}(x;\alpha),\quad 0\le x\le l\end{aligned} \tag{2b}$$

for $(x,t)\in S$. The region $[0,1]\times[0,T]$ is divided within the mesh of the $M\times N$ over dimensional step size. For x-axis, the step size of $h=\frac{1}{N}$ (with $x_i=ih$ $(i=0,1,2,\ldots,N)$). And for t-direction, the step size is $k=\frac{T}{M}$ (with $t_j=jk$ $(i=0,1,2,\ldots,M)$). Applying the implicit discretization practice to such issues would therefore yield

$$\frac{\partial \underline{U}}{\partial t}\approx\frac{\underline{U_{i,j+1}}-\underline{U_{i,j}}}{k}, \tag{3a}$$

$$\frac{\partial \overline{U}}{\partial t}\approx\frac{\overline{U_{i,j+1}}-\overline{U_{i,j}}}{k}, \tag{3b}$$

and

$$\frac{\partial^2 \underline{U}}{\partial x^2} \approx \lambda \left[\frac{\underline{U}_{i-1,j+1} - 2\underline{U}_{i,j+1} + \underline{U}_{i+1,j+1}}{h^2} \right], \tag{4a}$$

$$\frac{\partial^2 \overline{U}}{\partial x^2} \approx \lambda \left[\frac{\overline{U}_{i-1,j+1} - 2\overline{U}_{i,j+1} + \overline{U}_{i+1,j+1}}{h^2} \right]. \tag{4b}$$

Via consuming equations (3a) and (4a), equations (2a) able to signified as

$$\left(\frac{k\lambda}{h^2}\right)\underline{U}_{i-1,j+1} - \left(2\left(\frac{k\lambda}{h^2}\right)+1\right)\underline{U}_{i,j+1} + \left(\frac{k\lambda}{h^2}\right)\underline{U}_{i+1,j+1} = -\underline{U}_{i,j}. \tag{5a}$$

Whereas, by replacing equations (3b) and (4b) towards equation (2b), presented as

$$\left(\frac{k\lambda}{h^2}\right)\overline{U}_{i-1,j+1} - \left(2\left(\frac{k\lambda}{h^2}\right)+1\right)\overline{U}_{i,j+1} + \left(\frac{k\lambda}{h^2}\right)\overline{U}_{i+1,j+1} = -\overline{U}_{i,j} \tag{5b}$$

for $i = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$. With the exception that, grounded on the α-cuts interval, both equations (5a) and (5b) possess the identical type of equation, except the variations, are found solely at the upper and lower boundary. Therefore, this may then be simplified as follows

$$-\beta U_{i-1,j+1} + (2\beta+1) U_{i,j+1} - \beta U_{i+1,j+1} = U_{i,j} \tag{6}$$

where $\beta = \left(\frac{k\lambda}{h^2}\right)$. Simplification of (6) in the form of a matrix is referred to as

$$A\tilde{U}_{j+1} = \tilde{b}_j \ . \tag{7}$$

3. Design of Two Parameter Alternating Group Explicit Scheme

Following the growth of the AGE method, the TAGE iterative method is one of the families of the AGE technique. In fact, it can be easily shown that the distinction between these methods lies entirely in the use of weighted parameter r. The AGE technique consists of a single parameter, r_1. Whereas the TAGE technique contains two parameters, r_1 and r_2, each of which is used for the calculation of the first and second sweeps, respectively. It should be emphasized that if both parameters have the same value, i.e. $r_1 = r_2$, then TAGE can be reduced as the AGE method. This makes it easier to discuss, and this section will only focus on the TAGE iterative approach.

On the basis of prior study, hypothetically [12,13,16,17] have explored how the value of parameter r can be determined. Herein, the optimal value for parameters r_1 and r_2 are measured by employing numerous computational experiment so that when the iteration number is lesser, the optimal value is found.

TAGE categories can be regarded as a two-step technique in solving a linear system. Not any of the researchers had tried to use this approach, particularly in addressing fuzzy problems initiated by the discretization of FPDE. The implementation of this scheme would resolve the fuzzy linear system as stated in equation (1). Following the partitioning of matrix A by the sum of its constitutive symmetrical and positive definite matrices [10,18], matrix A in equation (7) can be stated as

$$A = G_1 + G_2 \tag{8}$$

whereby

$$G_1 = \begin{bmatrix} g_1 & \varphi_1 & & & & & \\ \rho_2 & g_2 & & & & & \\ & & g_3 & \varphi_3 & & & \\ & & \rho_4 & g_4 & & & \\ & & & & \ddots & & \\ & & & & & g_{n-2} & \varphi_{n-2} \\ & & & & & \rho_{n-1} & g_{n-1} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} g_1 & & & & & & \\ & g_2 & \varphi_2 & & & & \\ & \rho_3 & g_3 & & & & \\ & & & \ddots & & & \\ & & & & g_{n-3} & \varphi_{n-3} & \\ & & & & \rho_{n-2} & g_{n-2} & \\ & & & & & & g_{n-1} \end{bmatrix},$$

for n is an odd number. Likewise, if n is an even number

$$G_1 = \begin{bmatrix} g_1 & \varphi_1 & & & & \\ \rho_2 & g_2 & & & & \\ & & \ddots & & & \\ & & & g_{n-3} & \varphi_{n-3} & \\ & & & \rho_{n-2} & g_{n-2} & \\ & & & & & g_{n-1} \end{bmatrix},$$

$$G_2 = \left[\begin{array}{c|cc|c|cc} g_1 & & & & & \\ \hline & g_2 & \varphi_2 & & & \\ & \rho_3 & g_3 & & & \\ \hline & & & \ddots & & \\ \hline & & & & g_{n-2} & \varphi_{n-2} \\ & & & & \rho_{n-1} & g_{n-1} \end{array}\right],$$

where $g_i = \frac{\beta_i}{2}$ $(i = 1,\ 2, \ldots,\ n-1)$. This article simply analyzes for the case n is even.

Then equation (7) becomes

$$\left(G_1 + G_2\right)\tilde{U} = b. \tag{9}$$

As a result, the explicit form of the TAGE scheme is composed as

$$\tilde{U}^{\left(k+\frac{1}{2}\right)} = \left(G_1 + r_1 I\right)^{-1}\left[b - \left(G_2 - r_1 I\right)\tilde{U}^{(k)}\right], \tag{10a}$$

$$\tilde{U}^{(k+1)} = \left(G_2 + r_2 I\right)^{-1}\left[b - \left(G_1 - r_2 I\right)\tilde{U}^{\left(k+\frac{1}{2}\right)}\right], \tag{10b}$$

where $r_1, r_2 > 0$ are the parameters of acceleration, and duo of $\left(G_1 + r_1 I\right)$ and $\left(G_2 + r_2 I\right)$ are invertible. Therefore, based on equations (10a) and (10b), the execution of the families of TAGE approaches is demonstrated as in Algorithm 1.

	Algorithm 1: Families of TAGE schemes
i.	Set $\tilde{U}^{(0)} \leftarrow 0$, $\varepsilon \leftarrow 10^{-10}$.
ii.	For $i = p, 2p, 3p, \cdots, n-p$, adjust the parameter ρ_i, β_i, φ_i, f_i, r_1, r_2, G_1 and G_2.
iii.	First sweep, calculate $\tilde{U}^{\left(k+\frac{1}{2}\right)} = \left(G_1 + r_1 I\right)^{-1}\left[b - \left(G_2 - r_1 I\right)\tilde{U}^{(k)}\right]$
iv.	Second sweep, calculate $\tilde{U}^{(k+1)} = \left(G_2 + r_2 I\right)^{-1}\left[b - \left(G_1 - r_2 I\right)\tilde{U}^{\left(k+\frac{1}{2}\right)}\right]$
v.	Convergence check. Provided that the convergence condition is fulfilled i.e. $\left\|\tilde{U}^{(k+1)} - \tilde{U}^{k}\right\|_\infty \le \varepsilon$, proceed to Phase (vi). Else return to Phase (ii).
vi.	Show approximate solutions.

4. Numerical Experiments

With the intention of verifying the efficiency of AGE, FSTAGE, and HSTAGE techniques, two problems of 1D-FD are considered. Three parameters, namely iteration counts, time taken (in seconds), and Hausdorff distance (as stated in Definition 1), are observed for comparative purposes. In Tables 1 to 5, the numerical performances of the suggested approaches for different values of α are documented on the basis of these two examples.

Table 1. The performance of three terms amongst AGE, FSTAGE, and HSTAGE approaches at $\alpha = 0.00$

		Methods	n				
			512	1024	2048	4096	8192
Example 1	Iteration counts	AGE	5199	18926	67947	239878	829284
		FSTAGE	5001	18324	65958	235952	829284
		HSTAGE	1357	5001	18324	65958	235952
	Time taken	AGE	9.80	71.25	511.46	3622.74	25049.26
		FSTAGE	9.45	68.93	495.66	3549.95	24977.48
		HSTAGE	1.31	9.36	68.32	498.01	3716.55
	Hausdorff distance	AGE	4.7013e-04	4.6910e-04	4.6513e-04	4.4929e-04	3.8569e-04
		FSTAGE	4.7034e-04	4.6968e-04	4.6736e-04	4.5317e-04	3.8569e-04
		HSTAGE	4.7060e-04	4.7034e-04	4.6968e-04	4.6736e-04	4.5317e-04
Example 2	Iteration counts	AGE	1749	6462	23816	87009	314488
		FSTAGE	1747	6458	23816	87009	314488
		HSTAGE	484	1747	6458	23814	87009
	Time taken	AGE	14.74	67.42	338.32	2180.24	13353.35
		FSTAGE	14.72	66.27	334.78	2058.80	13245.96
		HSTAGE	3.69	14.77	66.63	336.86	2127.83
	Hausdorff distance	AGE	3.7252e-03	3.7232e-03	3.7198e-03	3.7071e-03	3.6561e-03
		FSTAGE	3.7253e-03	3.7234e-03	3.7198e-03	3.7071e-03	3.6561e-03
		HSTAGE	3.7301e-03	3.7253e-03	3.7234e-03	3.7197e-03	3.7071e-03

Table 2. The performance of three terms amongst AGE, FSTAGE, and HSTAGE approaches at $\alpha = 0.25$.

		Methods	*n*				
			512	1024	2048	4096	8192
Example 1	Iteration counts	AGE	5224	19026	68348	241494	835853
		FSTAGE	5024	18418	66337	237521	835853
		HSTAGE	1363	5024	18418	66337	237521
	Time taken	AGE	9.86	71.60	515.27	3644.49	25184.00
		FSTAGE	9.47	69.24	498.36	3574.98	25171.99
		HSTAGE	1.31	9.41	68.73	500.84	3583.61
	Hausdorff distance	AGE	4.3092e-04	4.2990e-04	4.2593e-04	4.1009e-04	3.4650e-04
		FSTAGE	4.3113e-04	4.3048e-04	4.2814e-04	4.1394e-04	3.4650e-04
		HSTAGE	4.3139e-04	4.3113e-04	4.3048e-04	4.2814e-04	4.1394e-04
Example 2	Iteration counts	AGE	1756	6488	23920	87430	316172
		FSTAGE	1754	6484	23920	87430	316172
		HSTAGE	486	1754	6484	23919	87430
	Time taken	AGE	14.77	67.42	339.34	2097.65	13897.57
		FSTAGE	14.77	66.57	337.35	2075.37	13603.57
		HSTAGE	3.77	14.77	66.80	339.32	2144.81
	Hausdorff distance	AGE	3.4147e-03	3.4129e-03	3.4095e-03	3.3967e-03	3.3457e-03
		FSTAGE	3.4148e-03	3.4130e-03	3.4095e-03	3.3967e-03	3.3457e-03
		HSTAGE	3.4193e-03	3.4148e-03	3.4130e-03	3.4094e-03	3.3967e-03

Table 3. The performance of three terms amongst AGE, FSTAGE, and HSTAGE approaches at $\alpha = 0.50$.

		Methods	*n*				
			512	1024	2048	4096	8192
Example 1	Iteration counts	AGE	5240	19090	68605	242527	840044
		FSTAGE	5039	18479	66579	238524	840044
		HSTAGE	1369	5039	18479	66579	238524
	Time taken	AGE	9.89	71.87	516.64	3662.89	25503.54
		FSTAGE	9.50	69.53	500.39	3587.95	25307.30
		HSTAGE	1.32	9.44	68.98	503.22	3594.38
	Hausdorff distance	AGE	3.9172e-04	3.9070e-04	3.8673e-04	3.7089e-04	3.0730e-04
		FSTAGE	3.9193e-04	3.9127e-04	3.8892e-04	3.7471e-04	3.0730e-04
		HSTAGE	3.9217e-04	3.9193e-04	3.9127e-04	3.8892e-04	3.7471e-04
Example 2	Iteration counts	AGE	1760	6505	23987	87698	317249
		FSTAGE	1758	6500	23987	87698	317249
		HSTAGE	487	1758	6500	23986	87698
	Time taken	AGE	14.79	68.09	341.25	2064.06	13454.90
		FSTAGE	14.72	66.69	337.88	2058.49	13448.08
		HSTAGE	3.82	14.85	66.98	339.70	2130.18
	Hausdorff distance	AGE	3.1043e-03	3.1025e-03	3.0991e-03	3.0864e-03	3.0354e-03
		FSTAGE	3.1044e-03	3.1026e-03	3.0991e-03	3.0864e-03	3.0354e-03
		HSTAGE	3.1084e-03	3.1044e-03	3.1026e-03	3.0991e-03	3.0864e-03

Table 4. The performance of three terms amongst AGE, FSTAGE, and HSTAGE approaches at $\alpha = 0.75$.

		Methods	n				
			512	1024	2048	4096	8192
Example 1	Iteration counts	AGE	5248	19125	68749	243106	842391
		FSTAGE	5047	18512	66715	239086	842391
		HSTAGE	1371	5047	18512	66715	239086
	Time taken	AGE	9.91	71.98	517.61	3673.02	25372.88
		FSTAGE	9.52	69.59	501.06	3599.82	25371.09
		HSTAGE	1.33	9.47	69.04	504.17	3598.52
	Hausdorff distance	AGE	3.5251e-04	3.5150e-04	3.4753e-04	3.3169e-04	2.6810e-04
		FSTAGE	3.5272e-04	3.5206e-04	3.4970e-04	3.3548e-04	2.6810e-04
		HSTAGE	3.5295e-04	3.5272e-04	3.5206e-04	3.4970e-04	3.3548e-04
Example 2	Iteration counts	AGE	1762	6514	24025	87849	317854
		FSTAGE	1760	6509	24025	87849	317854
		HSTAGE	487	1760	6509	24023	87849
	Time taken	AGE	15.02	67.86	352.58	2132.63	14071.04
		FSTAGE	14.87	66.67	338.05	2076.88	13480.32
		HSTAGE	3.74	14.90	67.33	339.94	2134.84
	Hausdorff distance	AGE	2.7938e-03	2.7921e-03	2.7888e-03	2.7761e-03	2.7251e-03
		FSTAGE	2.7939e-03	2.7923e-03	2.7888e-03	2.7761e-03	2.7251e-03
		HSTAGE	2.7976e-03	2.7939e-03	2.7923e-03	2.7887e-03	2.7761e-03

Table 5. The performance of three terms amongst AGE, FSTAGE, and HSTAGE approaches at $\alpha = 1.00$

		Methods	n				
			512	1024	2048	4096	8192
Example 1	Iteration counts	AGE	5251	19137	68795	243293	843148
		FSTAGE	5050	18523	66759	239267	843148
		HSTAGE	1372	5052	18523	66759	239267
	Time taken	AGE	9.91	72.07	518.22	3677.87	25393.35
		FSTAGE	9.53	69.66	501.69	3600.26	25392.15
		HSTAGE	1.31	9.45	69.10	504.65	3602.47
	Hausdorff distance	AGE	3.1331e-04	3.1230e-04	3.0833e-04	2.9249e-04	2.2890e-04
		FSTAGE	3.1351e-04	3.1285e-04	3.1047e-04	2.9624e-04	2.2890e-04
		HSTAGE	3.1373e-04	3.1349e-04	3.1285e-04	3.1047e-04	2.9625e-04
Example 2	Iteration counts	AGE	1763	6517	24037	87897	318049
		FSTAGE	1761	6512	24037	87897	318049
		HSTAGE	487	1761	6512	24036	87897
	Time taken	AGE	15.06	67.94	357.77	2132.47	13916.13
		FSTAGE	14.84	66.83	338.23	2091.35	13624.88
		HSTAGE	3.78	14.79	67.30	340.42	2110.91
	Hausdorff distance	AGE	2.4834e-03	2.4818e-03	2.4785e-03	2.4658e-03	2.4148e-03
		FSTAGE	2.4834e-03	2.4819e-03	2.4785e-03	2.4878e-03	2.4148e-03
		HSTAGE	2.4867e-03	2.4834e-03	2.4819e-03	2.4841e-03	2.4658e-03

Definition 1 [19]

Given two minimum bounding rectangles P and Q, a lower bound of the Hausdorff distance from the elements confined by P to the elements confined by Q is defined as

$$HausDistLB(P,Q) = Max\begin{Bmatrix} MinDist(f_\alpha, Q): \\ f_\alpha \in FacesOf(P) \end{Bmatrix}$$

Example 1 [20]

$$\frac{\partial \tilde{U}}{\partial t}(x,t) = 4\frac{\partial^2 \tilde{U}}{\partial x^2}(x,t), \quad 0 < x < 1,\ t > 0 \tag{11}$$

with $\tilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [0.50\alpha + 0.50, 1.50 - 0.50\alpha]$ and subject to the boundary conditions and initial condition

$$\tilde{U}(0,t) = \tilde{U}(1,t) = 0, \quad t > 0$$

$$\tilde{U}(x,0) = \tilde{f}(x) = \frac{2}{\pi}\tilde{K}\sin \pi x$$

The exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,t;\alpha) = 4\frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha), \tag{12a}$$

$$\frac{\partial \overline{U}}{\partial t}(x,t;a) = 4\frac{\partial^2 \overline{U}}{\partial x^2}(x,t;a), \tag{12b}$$

are

$$\underline{U}(x,t;\alpha) = \frac{2}{\pi}\underline{k}(\alpha)e^{-4\pi^2 t}\sin \pi x \quad , \tag{13a}$$

$$\overline{U}(x,t;\alpha) = \frac{2}{\pi}\overline{k}(\alpha)e^{-4\pi^2 t}\sin \pi x \quad , \tag{13b}$$

respectively.

Example 2 [7]

$$\frac{\partial \tilde{U}}{\partial t}(x,t) = \frac{\partial^2 \tilde{U}}{\partial x^2}(x,t), \quad 0 < x < 1,\ t > 0 \tag{14}$$

where $\tilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [0.50\alpha + 0.50, 1.50 - 0.50\alpha]$ and subject to the boundary conditions and initial condition

$$\tilde{U}(0,t) = \tilde{U}(l,t) = 0, \quad t > 0$$

$$\tilde{U}(x,0) = \tilde{f}(x) = \tilde{K}\sin \pi x, \quad 0 \le x \le 1$$

The exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,t;\alpha) = \frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha), \tag{15a}$$

$$\frac{\partial \overline{U}}{\partial t}(x,t;a) = \frac{\partial^2 \overline{U}}{\partial x^2}(x,t;a), \tag{15b}$$

are

$$\underline{U}(x,t;\alpha) = \underline{k}(\alpha)e^{-\pi^2 t}\sin \pi x, \tag{16a}$$

$$\overline{U}(x,t;\alpha) = \overline{k}(\alpha)e^{-\pi^2 t}\sin \pi x, \tag{16b}$$

respectively.

From the analyses of all experimental findings by imposing the iterative methods of AGE, FSTAGE, and HSTAGE, it is apparent that the number of iterations for HSTAGE approach has decreased by approximately 71.60-72.95% relative to the FSTAGE and AGE methods. Additionally, with regard to execution time, the implementation of HSTAGE method is between 74.05-86.42% better than the FSTAGE and AGE methods. This implies that the HSTAGE technique requires the least amount of iteration counts and computing time as relative to the FSTAGE and AGE iterative approaches.

5. Conclusions

Herein, the families of the TAGE technique were utilized in order to resolve linear system structure by discretizing 1D-FD using a finite difference scheme. The outcomes indicate that HSTAGE technique is much better compared with FSTAGE and AGE methods, concerning iteration counts, time taken (in seconds), and Hausdorff distance. Ever since TAGE is fit adapted for parallel computation, this approach has been seen as the main advantage as it consumes sets of independent tasks that can be executed at the same time. The potential of the recommended technique is expected to be useful for the advanced exploration in deciphering any multi-dimensional FPDEs [20,21]. The outcomes of this work can basically be categorized as one of the half-sweep procedure. In order to accelerate the convergence rate of the ordinary suggested iterative approaches, a further examination on quarter-sweep techniques [22-25] may as well be taken into account in addition to the theory of the half-sweep practice.

Acknowledgements

The National Defence University of Malaysia has sponsored this research study.

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