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A Dirac Delta Operator

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Abstract If T is a (densely defined) self-adjoint operator acting on a complex Hilbert space $\mathcal{H}$ and I stands for the identity operator, we introduce the delta function operator $\lambda \mapsto \delta(\lambda I - T)$ at T. When T is a bounded operator, then $\delta(\lambda I - T)$ is an operator-valued distribution. If T is unbounded, $\delta(\lambda I - T)$ is a more general object that still retains some properties of distributions. We provide an explicit representation of $\delta(\lambda I - T)$ in some particular cases, derive various operative formulas involving $\delta(\lambda I - T)$ and give several applications of its usage in Spectral Theory as well as in Quantum Mechanics.

Keywords Hilbert Space, Self-adjoint Operator, Vector-valued Distribution, Spectral Measure

1 The delta function $\delta(\lambda I - T)$

The scalar delta 'function' $\lambda \mapsto \delta(\lambda - a)$ along with its derivatives were introduced by Paul Dirac in [1], and later in [2, Section 15], although its definition can be traced back to Heaviside. The rigorous treatment of this object in the context of distribution theory is due to Laurent Schwartz [6, 12]. In this paper we extend the definition of $\delta(\lambda - a)$ from real numbers to self-adjoint operators on a Hilbert space $\mathcal{H}$. We denote by $\mathcal{D}(\mathbb{R}) = \varinjlim \mathcal{D}([-n, n])$ the linear space of infinitely differentiable complex-valued functions of compact support, equipped with the inductive limit topology. As usual in physics we shall assume that the scalar product in $\mathcal{H}$ is anti-linear for the first variable.

If T is a densely defined self-adjoint operator[1] on $\mathcal{H}$ and I stands for the identity operator, we define the *delta function operator* $\lambda \mapsto \delta(\lambda I - T)$ *at* T by

$$f(T) = \int_{-\infty}^{+\infty} f(\lambda)\, \delta(\lambda I - T)\, d\lambda \tag{1.1}$$

for each $f \in \mathcal{C}(\mathbb{R})$, i. e., for each real-valued continuous function $f(\lambda)$. Here $d\lambda$ is the Lebesgue measure of $\mathbb{R}$, but the right-hand side of (1.1) is not a true integral. If T is a bounded operator, we shall see at once that $\delta(\lambda I - T)$ must be regarded as a *vector-valued distribution*, i. e., as a continuous linear map from the space $\mathcal{D}(\mathbb{R})$ into the locally convex space $\mathcal{L}(\mathcal{H})$ of the bounded linear operators (endomorphisms) on $\mathcal{H}$ equipped with the strong operator topology [10, 11], whose action on $f \in \mathcal{D}(\mathbb{R})$ we denote as an integral. If T is unbounded we shall see that $\delta(\lambda I - T)$ still retains some useful distributional-like properties. The previous equation means

$$\langle y, f(T)\, x\rangle = \int_{-\infty}^{+\infty} f(\lambda)\, \langle y, \delta(\lambda I - T)\, x\rangle\; d\lambda \tag{1.2}$$

for each $(x, y) \in D(f(T)) \times \mathcal{H}$, where $D(f(T))$ stands for the domain of the self-adjoint operator $f(T)$.

Let us recall that if T is a (densely defined) self-adjoint operator, there is a unique spectral family $\{E_\lambda : \lambda \in \mathbb{R}\}$ of self-adjoint operators defined on the whole of $\mathcal{H}$ that satisfy (i) $E_\lambda \le E_\mu$ and $E_\lambda E_\mu = E_\lambda$ for $\lambda \le \mu$, (ii) $\lim_{\epsilon \to 0^+} E_{\lambda+\epsilon} x = E_\lambda x$, and (iii) $\lim_{\lambda \to -\infty} E_\lambda x = \mathbf{0}$ and $\lim_{\lambda \to \infty} E_\lambda x = x$ in $\mathcal{H}$ for all $x \in \mathcal{H}$. The domain $D(T)$ of T consists of those $x \in \mathcal{H}$ such that

$$\int_{-\infty}^{+\infty} |\lambda|^2\; d\,\|E_\lambda x\|^2 < \infty.$$

In this case, the spectral theorem (*cf.* [8, Section 107]) and the Borel-measurable functional calculus provide a *self-adjoint* operator $f(T)$ defined by

$$f(T) = \int_{-\infty}^{+\infty} f(\lambda)\; dE_\lambda \tag{1.3}$$

[1] In what follows $\sigma(T)$ will denote the *spectrum* of T. Recall that the *residual spectrum* of a self-adjoint operator T is empty, so that $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$, where $\sigma_p(T)$ denotes the *point spectrum* (the eigenvalues) and $\sigma_c(T)$ the *continuous spectrum* of T.

for each Borel-measurable function $f(\lambda)$, whose domain

$$D(f(T)) = \left\{ x \in \mathcal{H} : \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, d\|E_\lambda x\|^2 < \infty \right\}$$

is dense in $\mathcal{H}$. Observe that if T is bounded, $f(T)$ need not be bounded. Moreover, since $\lambda \mapsto E_\lambda$ is constant on the set $\mathbb{R} \setminus \sigma(T)$ of T, an open set in $\mathbb{R}$, equation (1.3) tell us that $f(\lambda)$ need not be defined on $\mathbb{R} \setminus \sigma(T)$.

Thanks to (1.3) the definition of $\delta(\lambda I - T)$ may be extended to Borel-measurable functions by declaring that the equation (1.1) holds for $(x, y) \in D(f(T)) \times \mathcal{H}$ and each Borel function f. But, by reasons that will become clear later, we shall restrict ourselves to those Borel functions which are continuous at each point of $\sigma_p(T)$. Moreover, working with the real and complex parts, no difficulty arises if the function f involved in the equation (1.1) is complex-valued (except that $f(T)$ is no longer a self-adjoint operator whenever $Imf \neq 0$). Thus, unless otherwise stated, we shall assume that both in (1.1) and (1.3) the function f is complex-valued. Note that the complex Stieltjes measure $d\langle E_\lambda x, y\rangle$ need not be $d\lambda$-continuous. In what follows we shall denote by $\mathcal{B}_p(\mathbb{R})$ the linear space over $\mathbb{C}$ consisting of all complex-valued Borel-measurable functions of one real variable which are continuous on $\sigma_p(T)$.

If $f_n \to f$ in $\mathcal{D}(\mathbb{R})$, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and $f_n(x) \to f(x)$ at each $x \in \mathbb{R}$. So, if T is bounded on $\mathcal{H}$ (equivalently, self-adjoint on the whole of $\mathcal{H}$) it turns out that $f_n(T) \to f(T)$ in the strong operator topology [3, 10.2.8 Corollary]. Therefore, in this case $\delta(\lambda I - T)$ is an $\mathcal{L}(\mathcal{H})$-valued distribution.

As all integrals considered so far are over $\sigma(T)$, we have

$$\delta(\lambda I - T) = \mathbf{0} \;\; \forall \lambda \notin \sigma(T). \tag{1.4}$$

Also $\delta(-\lambda I + T) = \delta(\lambda I - T)$ for all $\lambda \in \mathbb{R}$. On the other hand, if $\mu \in \sigma_p(T)$ and y is an eigenvector corresponding to the eigenvalue μ, clearly

$$\delta(\lambda I - T)\, y = \delta(\lambda - \mu)\, y \tag{1.5}$$

for every $\lambda \in \mathbb{R}$. In the particular case when T_a is the linear operator defined on $\mathcal{H}$ by $T_a x = ax$ for a fixed $a \in \mathbb{R}$, then T_a is a self-adjoint linear operator with $\sigma(T_a) = \sigma_p(T_a) = \{a\}$. In this case $\delta(\lambda I - T_a)\, x = \delta(\lambda - a)\, x$ for every $x \in \mathcal{H}$, i. e., $\delta(\lambda I - T_a) = \delta(\lambda - a)\, I$.

Since equality $\left\langle f(T)^\dagger y, x \right\rangle = \langle y, f(T)\, x\rangle$ holds for all $x, y \in D(f(T))$ and each $f \in \mathcal{B}_p(\mathbb{R})$, we may infer that

$$\langle y, \delta(\lambda I - T)\, x\rangle = \langle \delta(\lambda I - T)\, y, x\rangle$$

holds (in a 'distributional' sense) for all $x, y \in D(T)$. This suggests that in certain sense $\delta(\lambda I - T)$ may be regarded (possibly for almost all $\lambda \in \mathbb{R}$) as a Hermitian operator on $D(T)$.

Let us also point out that as equation (1.1) holds for all $f \in \mathcal{D}(\mathbb{R})$, in a distributional sense we have

$$\frac{d}{d\lambda} \langle y, E_\lambda x\rangle = \langle y, \delta(\lambda I - T)\, x\rangle \tag{1.6}$$

If $\lambda \mapsto Y(\lambda - \mu)$ denotes the unit step function at $\mu \in \mathbb{R}$, given by $Y(\lambda - \mu) = 0$ if $\lambda < \mu$ and $Y(\lambda - \mu) = 1$ if $\lambda \geq \mu$, since $E_\lambda = Y(\lambda I - T)$ for each $\lambda \in \mathbb{R}$, formally

$$dE_\lambda / d\lambda = Y'(\lambda I - T). \tag{1.7}$$

So, from (1.6) and (1.7) we get $Y'(\lambda I - T) = \delta(\lambda I - T)$.

Proposition 1. *If T is a bounded self-adjoint operator on $\mathcal{H}$ and $f \in \mathcal{C}^1(\mathbb{R})$, then*

$$\int_{-\infty}^{+\infty} f(\lambda)\; \delta'(\lambda I - T)\; d\lambda = -f'(T).$$

The same equality holds if T is unbounded but $f \in \mathcal{D}(\mathbb{R})$.

If T is a self-adjoint operator and $f \in \mathcal{B}_p(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} |f(\lambda)|^2 \, \delta(\lambda I - T)\; d\lambda = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \; dE_\lambda.$$

where the latter equality *is the definition* of $|f(T)|^2$. So, we have the following result.

Proposition 2. *If T is self-adjoint and $f \in \mathcal{B}_p(\mathbb{R})$, then*

$$\langle f(T)\, y, f(T)\, x\rangle = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, \langle y, \delta(\lambda I - T)\, x\rangle \; d\lambda$$

for every $x, y \in D(f(T))$.

Proof. We adapt a classic argument. Indeed, for every $x, y \in D(f(T))$ we have

$$\langle f(T)\, y, f(T)\, x\rangle = \int_{-\infty}^{+\infty} \overline{f(\lambda)}\, d\langle f(T)\, y, E_\lambda x\rangle.$$

Since $E_\mu E_\lambda = E_\mu$ whenever $\mu \leq \lambda$, and $\langle E_\lambda y, x\rangle$ does not depend on μ, by splitting the integral we get

$$\int_{-\infty}^{+\infty} f(\mu)\; d\langle E_\mu y, E_\lambda x\rangle = \int_{-\infty}^{\lambda} f(\mu)\; d\langle y, E_\mu x\rangle, \tag{1.8}$$

where clearly the first integral is $\langle f(T)\, y, E_\lambda x\rangle$. Plugging $d\langle f(T)\, y, E_\lambda x\rangle$ into (1.8), we are done. □

Corollary 3. *Under the same conditions of the previous theorem, the equality*

$$\|f(T)\, x\|^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2 \, \langle x, \delta(\lambda I - T)\, x\rangle \; d\lambda \tag{1.9}$$

holds for every $x \in D(f(T))$.

Proposition 4. *If T is self-adjoint and $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence in $\mathcal{B}_p(\mathbb{R})$ such that $f_n \to f$ pointwise on $\mathbb{R}$ with $f \in \mathcal{B}_p(\mathbb{R})$, then $f_n(T)\, x \to f(T)\, x$ for every $x \in D(T)$.*

Proof. This is a straightforward consequence of preceding corollary and the Lebesgue dominated convergence theorem. □

This proposition holds in particular if $f_n \to f$ in $\mathcal{D}(\mathbb{R})$. Hence, even in the unbounded case, $\delta(\lambda I - T)$ behaves as a vector-valued distribution-like object.

Proposition 5. *Let $(\lambda,\mu) \mapsto g(\lambda,\mu)$ be a function defined on $\mathbb{R}^2$ such that $g(\lambda,\cdot) \in \mathcal{L}_1(\mathbb{R})$ for every $\lambda \in \mathbb{R}$ and $g(\cdot,\mu) \in \mathcal{B}_p(\mathbb{R})$ for every $\mu \in \mathbb{R}$. If the parametric integral*

$$f(\lambda) = \int_{-\infty}^{+\infty} g(\lambda,\mu)\, d\mu$$

is continuous on $\mathbb{R}$ and makes sense if we replace λ by a self-adjoint operator T, the value of the integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\lambda,\mu)\,\delta(\lambda I - T)\; d\mu\, d\lambda$$

does not depend on the integration ordering.

Proof. Since $g(\cdot,\mu) \in \mathcal{B}_p(\mathbb{R})$ for every $\mu \in \mathbb{R}$., one has

$$g(T,\mu) = \int_{-\infty}^{+\infty} g(\lambda,\mu)\,\delta(\lambda I - T)\, d\lambda,$$

which implies

$$f(T) = \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} g(\lambda,\mu)\,\delta(\lambda I - T)\, d\lambda\right\} d\mu.$$

On the other hand, by the definition of $\delta(\lambda I - T)$ we have

$$f(T) = \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} g(\lambda,\mu)\, d\mu\right\}\delta(\lambda I - T)\; d\lambda,$$

for $(x,y) \in D(T) \times \mathcal{H}$. So, the proposition follows. □

Theorem 6. *If T is a self-adjoint operator on $\mathcal{H}$, then*

$$\int_{0}^{+\infty} f(\lambda)\,\delta\left(\lambda I - T^2\right) d\lambda = \qquad (1.10)$$

$$\int_{0}^{+\infty} \frac{1}{2\sqrt{\lambda}}\left\{\delta\left(\sqrt{\lambda}I - T\right) - \delta\left(\sqrt{\lambda}I + T\right)\right\} f(\lambda)\, d\lambda$$

if $\lambda > 0$ and $f \in \mathcal{B}_p(\mathbb{R})$, both members acting on $D\left(T^2\right)$.

Proof. First note that $T^2 \geq 0$. Hence $\sigma\left(T^2\right) \subseteq [0,+\infty)$, which implies that $\delta\left(\lambda I - T^2\right) = \mathbf{0}$ if $\lambda < 0$. Since T^2 is a self-adjoint operator, for $f \in \mathcal{B}_p(\mathbb{R})$ we have

$$\int_{0}^{+\infty} f(\lambda)\,\delta\left(\lambda I - T^2\right) d\lambda = f\left(T^2\right)$$

On the other hand, it is clear that

$$\int_{0}^{+\infty} \frac{f(\lambda)}{2\sqrt{\lambda}}\,\delta\left(\sqrt{\lambda}I - T\right) d\lambda = \int_{0}^{+\infty} f\left(\mu^2\right)\delta(\mu I - T)\, d\mu$$

whereas, using that $\delta(-\mu I + T) = \delta(\mu I - T)$, we have

$$-\int_{0}^{+\infty} \frac{f(\lambda)}{2\sqrt{\lambda}}\,\delta\left(\sqrt{\lambda}I + T\right) d\lambda = \int_{-\infty}^{0} f\left(\mu^2\right)\delta(\mu I - T)\, d\mu$$

So, the right-hand side of (1.10) coincides with

$$\int_{-\infty}^{+\infty} f\left(\mu^2\right)\delta(\mu I - T)\, d\mu = f\left(T^2\right)$$

since $\mu \mapsto f\left(\mu^2\right)$ is a Borel function. □

If we denote by $L(\mathcal{H})$ the linear space of all linear endomorphisms on $\mathcal{H}$, the next theorem summarize some previous results.

Theorem 7. *If T is a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$, there is an $L(\mathcal{H})$-valued linear map δ_T on $\mathcal{B}_p(\mathbb{R})$, whose action on $f \in \mathcal{B}_p(\mathbb{R})$ we denote by*

$$\langle \delta_T, f\rangle = \int_{-\infty}^{+\infty} f(\lambda)\;\delta(\lambda I - T)\, d\lambda,$$

such that $\langle \delta_T, f\rangle = f(T)$. If $\{f_n\} \subseteq \mathcal{B}_p(\mathbb{R})$ is uniformly bounded and $f_n(t) \to f(t)$, with $f \in \mathcal{B}_p(\mathbb{R})$, for all $t \in \mathbb{R}$ then $\langle \delta_T, f_n\rangle x \to \langle \delta_T, f\rangle x$ for all $x \in \mathcal{H}$. If T is bounded, δ_T is an $\mathcal{L}(\mathcal{H})$-valued distribution, so $\langle \delta_T, f\rangle$ is a bounded operator on $\mathcal{H}$. In addition $\delta(\lambda I - T) = \mathbf{0}$ if $\lambda \notin \sigma(T)$ and $\langle y, \delta(\lambda I - T)x\rangle = \langle \delta(\lambda I - T)y, x\rangle$ for $x, y \in D(T)$.

2 Explicit form of $\delta(\lambda I - T)$

If Q is a vector-valued distribution, the *Fourier transform* of Q is defined as the vector valued distribution $\mathcal{F}Q$ on $\mathcal{S}(\mathbb{R})$ such that $\langle \mathcal{F}Q, f\rangle = \langle Q, \mathcal{F}f\rangle$. As usual, we denote by $\mathcal{F}^{-1}$ the inverse Fourier transform.

Theorem 8. *If T is a self-adjoint operator, the identity*

$$\delta(\lambda I - T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it(\lambda I - T)} dt \qquad (2.1)$$

holds for every $\lambda \in \mathbb{R}$, and the action $f(T)$ of $\delta(\lambda I - T)$ on $f \in \mathcal{S}(\mathbb{R})$ is given by

$$f(T) = \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} \frac{f(\lambda)}{2\pi}\, e^{it(\lambda I - T)}\, d\lambda\right\} dt.$$

Proof. Setting $\delta_T(\lambda) = \delta(\lambda I - T)$ observe that

$$(\mathcal{F}\delta_T)(t) = \frac{1}{\sqrt{2\pi}}\, e^{-itT}.$$

Indeed, if $f \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}\langle \mathcal{F}\delta_T, f\rangle &= \langle \delta_T, \mathcal{F}f\rangle = \int_{-\infty}^{+\infty} (\mathcal{F}f)(\lambda)\,\delta(\lambda I - T)\, d\lambda \\ &= (\mathcal{F}f)(T) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\, e^{-itT} dt.\end{aligned}$$

Consequently

$$\delta_T = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\, e^{-itT}\right\}. \qquad (2.2)$$

Functionally, the action of δ_T on $f \in S(\mathbb{R})$ by means of equation (2.2) becomes

$$\langle \delta_T, f\rangle = \left\langle \frac{1}{\sqrt{2\pi}}\, e^{-itT}, \left(\mathcal{F}^{-1}f\right)(t)\right\rangle \qquad (2.3)$$

Consequently, we have

$$\langle \delta_T, f\rangle = \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} \frac{f(\mu)}{2\pi}\, e^{it(\mu I - T)}\, d\mu\right\} dt$$

with the order of the integration as stated. □

Corollary 9. *If $e^{-itT}x = x(t)$, for $x \in D(T)$ one has*

$$\delta(\lambda I - T)\, x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda t} x(t)\ dt$$

and if $x \in D(f(T))$ and $f \in \mathcal{S}(\mathbb{R})$, then

$$f(T)\, x = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\lambda)\, e^{i\lambda t} d\lambda\right\} x(t)\ dt.$$

Remark 10. Consider the one-parameter unitary group $\{U(t) : t \in \mathbb{R}\}$ generated by the self-adjoint operator T, that is, $U(t) = \exp(-itT)$ for every $t \in \mathbb{R}$. If $\mathcal{F}$ denotes the Fourier transform, equation (2.2) can be written as

$$\delta(\lambda I - T) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}^{-1}(U)(\lambda). \tag{2.4}$$

So, equation (2.3) reads as

$$f(T) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\mathcal{F}^{-1} f\right)(t)\, U(t)\ dt. \tag{2.5}$$

In what follows we shall compute the spectral family $\{E_\lambda : \lambda \in \mathbb{R}\}$ for some useful self-adjoint operators of Quantum Mechanics by means of the delta $\delta(\lambda I - T)$. Nonetheless, although $E_\lambda = Y(\lambda I - T)$, the identification

$$Y(\lambda I - T) = \int_{-\infty}^{+\infty} Y(\lambda - \mu)\,\delta(\mu I - T)\, d\mu$$

might be not well-defined because $\mu \mapsto Y(\lambda - \mu)$ has a jump discontinuity at $\mu = \lambda$. Indeed, if $\lambda \in \sigma_p(T)$ and x is an eigenvector corresponding to λ, then

$$\int_{-\infty}^{+\infty} Y(\lambda - \mu)\,\delta(\mu I - T)\, x\, d\mu = \left\{\int_{-\infty}^{\lambda} \delta(\mu - \lambda)\ d\mu\right\} x$$

and the right-hand integral makes no sense (see [4] for a useful discussion). If $\lambda \notin \sigma_p(T)$ we define

$$E_\lambda = \int_{-\infty}^{+\infty} Y(\lambda - \mu)\,\delta(\mu I - T)\, d\mu \tag{2.6}$$

If λ belongs to $\sigma_p(T)$, then $(\mu \mapsto Y(\lambda - \mu)) \notin \mathcal{B}_p(\mathbb{R})$. In order to define E_λ we enlarge a little the interval of integration by considering the integral

$$\int_{-\infty}^{\lambda+\epsilon} \delta(\mu - \lambda)\ d\mu$$

for small $\epsilon > 0$. So, if $\lambda \in \sigma_p(T)$ we define

$$E_\lambda = \lim_{\epsilon \to 0^+}\int_{-\infty}^{+\infty} Y(\lambda + \epsilon - \mu)\,\delta(\mu I - T)\, d\mu. \tag{2.7}$$

The limit is well-defined since $\lim_{\epsilon\to 0^+} E_{\lambda+\epsilon} = E_\lambda$ pointwise on $\mathcal{H}$. In the particular case when λ belongs to $\sigma_d(T)$, the discrete part of $\sigma_p(T)$, λ is isolated in $\sigma_p(T)$.

Example 11. *The spectral family of the (up to a sign) one-dimensional Quantum Mechanics momentum operator of the free particle $P = iD$, where $D\varphi = \varphi'$, acting on the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$ is given by*

$$(E_\lambda \varphi)(x) = \frac{1}{2}\,\varphi(x) + \frac{1}{2\pi i}\ \text{p.v.}\int_{-\infty}^{+\infty}\frac{e^{i\lambda(s-x)}}{s-x}\,\varphi(s)\, ds$$

for every regular compactly supported $\varphi \in D(P)$.

Proof. As is well-known P is a self-adjoint operator with $D(P) = H^{2,1}(\mathbb{R})$ and $\sigma_c(P) = \mathbb{R}$. Since

$$\left(e^{-itP}\varphi\right)(x) = \left(e^{tD}\varphi\right)(x) = \varphi(x+t)$$

for a regular enough $\varphi \in D(P)$, by Corollary 9 we have

$$\{\delta(\mu I - P)\,\varphi\}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\mu t}\varphi(x+t)\, dt.$$

Note that the integral of the right-hand side does exist because φ has compact support.

According to the definition of E_λ for the continuous spectrum and keeping in mind the order of integration as indicated in Corollary 9, one has

$$\{E_\lambda \varphi\}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Y(\lambda - \mu)\, e^{i\mu t}\varphi(x+t)\ d\mu\, dt.$$

So, since

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Y(\lambda - \mu)\, e^{i\mu t} d\mu = \mathcal{F}(Y)(t)\cdot e^{i\lambda t},$$

bearing in mind the distributional relation

$$\mathcal{F}(Y)(t) = \sqrt{\frac{\pi}{2}}\left(\delta(t) + \frac{1}{i\pi}\ \text{p.v.}\frac{1}{t}\right), \tag{2.8}$$

we get

$$\{E_\lambda \varphi\}(x) = \frac{1}{2}\,\varphi(x) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda(s-x)}}{s-x}\,\varphi(s)\, ds$$

where the last integral must be understood in Cauchy's principal value sense. □

Example 12. *The spectral family of the one-dimensional Quantum Mechanics kinetic energy term of the free particle, corresponding to the Laplace operator $T = -D^2$ on $\mathcal{H} = L_2(\mathbb{R})$, where $D^2\varphi = \varphi''$, is given by*

$$(E_\lambda \varphi)(x) = \frac{1}{i\pi}\ \text{p.v.}\int_{-\infty}^{+\infty}\frac{\cos(\lambda(s-x)) - 1}{s-x}\,\varphi(s)\, ds$$

for $\lambda > 0$ and $E_\lambda = \mathbf{0}$ whenever $\lambda < 0$, where φ is a regular function with compact support belonging to $D(T)$.

Proof. In this case T is a self-adjoint operator with $\sigma(T) = [0, +\infty)$. Since $T = (iD)^2$, according to (1.10) we have

$$\delta(\lambda I - T) = \frac{1}{2\sqrt{\lambda}}\left\{\delta\left(\sqrt{\lambda} I - iD\right) - \delta\left(\sqrt{\lambda} I + iD\right)\right\}$$

regarded as a functional on $\mathcal{S}(\mathbb{R})$ through $d\lambda$-integration over $[0,+\infty)$. Plugging

$$(\delta(\mu I \mp iD)\varphi)(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\mu t}\varphi(x \pm t)\,dt$$

into the previous expression and keeping in mind the correct order of integration, we see that

$$\int_0^{\infty} f(\lambda)(\delta(\lambda I - T)\varphi)(x)\,d\lambda =$$
$$\frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(\lambda)\frac{e^{i\sqrt{\lambda}t}}{\sqrt{\lambda}}\left[\varphi(x+t)-\varphi(x-t)\right]d\lambda\,dt$$

for every $f \in \mathcal{S}(\mathbb{R})$. By the definition of E_λ if $\lambda > 0$ and the fact that $\delta(\mu I - T) = \mathbf{0}$ whenever $\mu < 0$, we have

$$(E_\lambda\varphi)(x) = \int_0^{+\infty} Y(\lambda-\mu)(\delta(\mu I - T)\varphi)(x)\,d\mu.$$

Working out the penultimate integral with μ instead of λ and $f(\mu) = Y(\lambda - \mu)$, we obtain

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} Y(\lambda-\mu)\frac{e^{i\sqrt{\mu}t}}{\sqrt{\mu}}\left[\varphi(x+t)-\varphi(x-t)\right]d\mu\,dt$$
$$= \int_{-\infty}^{+\infty}\left\{\int_0^{\lambda}\frac{e^{i\sqrt{\mu}t}}{\sqrt{\mu}}\,d\mu\right\}\left[\varphi(x+t)-\varphi(x-t)\right]dt$$

for $\lambda > 0$. So, by setting $u = \sqrt{\mu}$ we get

$$(E_\lambda\varphi)(x) = \int_{-\infty}^{+\infty}\frac{dt}{2\pi}\left[\varphi(x+t)-\varphi(x-t)\right]\int_0^{\lambda} e^{iut}du.$$

Now we have

$$\frac{1}{\sqrt{2\pi}}\int_0^{\lambda} e^{iut}du = \left(1-e^{i\lambda t}\right)\mathcal{F}^{-1}(Y)(t),$$

so, using that $\mathcal{F}^{-1}(Y(v)) = \mathcal{F}(1-Y(v))(t)$ as well as equation (2.8), we get

$$\frac{1}{\sqrt{2\pi}}\int_0^{\lambda} e^{iut}du = \left(1-e^{i\lambda t}\right)\sqrt{\frac{\pi}{2}}\left(\delta(t) - \frac{1}{i\pi}\,\text{p.v.}\frac{1}{t}\right)$$

which implies

$$(E_\lambda\varphi)(x) =$$
$$-\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(s)}{s-x}\,ds + \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\cos(\lambda(s-x))}{s-x}\varphi(s)\,ds$$

where the integrals are understood in Cauchy's principal value sense. □

Example 13. *Spectral family of the (up to a sign) one-dimensional Quantum Mechanics momentum operator S for a bounded particle on $\mathcal{H} = L_2[-\pi,\pi]$ with domain*

$$\{\varphi \in L_2[-\pi,\pi] : \varphi' \in L_2[-\pi,\pi],\, \varphi(-\pi) = \varphi(\pi)\}$$

As is well-known this is a self-adjoint operator with discrete spectrum $\sigma(S) = \mathbb{Z}$ whose eigenfunction system $\{\varphi_n : n \in \mathbb{Z}\}$, with $\varphi_n(x) = (2\pi)^{-1/2}e^{-inx}$, are the solutions of the eigenvalue problem $i\varphi' = \lambda\varphi$ with $\varphi(-\pi) = \varphi(\pi)$. So, for $\varphi \in D(S)$ we have $\varphi \overset{L_2}{=} \sum_{n\in\mathbb{Z}} c_n\varphi_n$ with

$$c_n = \langle\varphi,\varphi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)e^{inx}dx$$

for every $n \in \mathbb{Z}$. Since $\sigma(S) = \sigma_d(S)$, recalling the definition of the operator E_λ for $\lambda \in \sigma_d(S)$, clearly we have

$$(E_\lambda\varphi)(x) = \lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} Y(\lambda+\epsilon-\mu)(\delta(\mu I - S)\varphi)(x)\,d\mu$$

for every $\lambda \in \mathbb{R}$. So, the fact that E_λ is a bounded operator yields

$$E_\lambda\varphi = \sum_{n\in\mathbb{Z}} c_n E_\lambda\varphi_n$$

Using that $\delta(\mu I - S)e^{-inx} = \delta(\mu - n)e^{-inx}$ and that $Y(\lambda + 0 - n) = Y(\lambda - n)$, we get

$$(E_\lambda\varphi)(x) = \sum_{n\in\mathbb{Z}}\frac{c_n}{\sqrt{2\pi}}Y(\lambda-n)e^{-inx} = \sum_{n\in\mathbb{Z},\, n\le[\lambda]}\frac{e^{-inx}}{\sqrt{2\pi}}.$$

Remark 14. Since in the previous example S is bounded on $\mathcal{H} = L_2[-\pi,\pi]$, the delta operator $\delta(\lambda I - S)$ should be regarded as a continuous endomorphism as well. In this case

$$\delta(\lambda I - S)\varphi = \sum_{n\in\mathbb{Z}} c_n\,\delta(\lambda-n)\varphi_n.$$

Example 15. *The one-dimensional Quantum Mechanics position operator on $L_2(\mathbb{R})$.* This operator is defined on $\mathcal{H} = L_2(\mathbb{R})$ by $(Q\varphi)(x) = x\varphi(x)$ for every $x \in \mathbb{R}$. Clearly $\sigma_c(Q) = \mathbb{R}$ and $\varphi \in D(Q)$ if $(x \mapsto x\varphi(x)) \in \mathcal{L}_2(\mathbb{R})$. Moreover, it is clear that

$$\{\exp(it(\lambda I - Q))\varphi\}(x) = e^{i(\lambda-x)t}\varphi(x).$$

So we have

$$(\delta(\lambda I - Q)\varphi)(x) = \delta(\lambda - x)\varphi(x).$$

Hence, in this case we can write

$$\{E_\lambda\varphi\}(x) = \int_{-\infty}^{+\infty} Y(\lambda-\mu)\,\delta(\mu-x)\varphi(x)\,d\mu$$

Therefore, if $\lambda \neq x$ we get

$$\{E_\lambda\varphi\}(x) = Y(\lambda-x)\varphi(x).$$

Example 16. *Explicit form of $\delta(\lambda I - M)$ for the Hermitian matrix of $\mathcal{H} = \mathbb{C}^3$*

$$M = \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}.$$

Proof. In this case $M = PJ_MP^{-1}$ with $\sigma(M) = \{-1, 2\}$ and

$$J_M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Using (2.1) we get

$$\delta(\lambda I - M) = P \begin{bmatrix} \delta(\lambda+1) & 0 & 0 \\ 0 & \delta(\lambda+1) & 0 \\ 0 & 0 & \delta(\lambda-2) \end{bmatrix} P^{-1}$$

Let us compute the spectral family and the projection operator onto the eigenspace $\ker(M + I)$. Clearly

$$E_\lambda = P \begin{bmatrix} Y(\lambda+1) & 0 & 0 \\ 0 & Y(\lambda+1) & 0 \\ 0 & 0 & Y(\lambda-2) \end{bmatrix} P^{-1}$$

for every $\lambda \in \mathbb{R}$. If $\lambda_1 = -1$, the orthogonal projection P_{λ_1} onto $\ker(I + M)$ is

$$P_{\lambda_1} = \frac{1}{3} P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

since $P_{\lambda_1} = E_{\lambda_1} - E_{\lambda_1 - 0} = E_{\lambda_1}$. □

Example 17. *Consider a compact self-adjoint operator K acting on a separable Hilbert space $\mathcal{H}$ which does not admit the eigenvalue zero. Let $\{u_i : i \in \mathbb{N}\}$ be a Hilbert basis of $\mathcal{H}$ with its corresponding sequence of real eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$, where $|\lambda_{i+1}| \leq |\lambda_i|$ for every $i \in \mathbb{N}$. Let us compute the action of the operator $(\lambda I - K)^{-1}$ on any $x \in \mathcal{H}$ and the operator $\delta(\lambda I - T)$.*

Proof. If $x \in \mathcal{H}$, we can write $x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$. Since $(\lambda I - K)^{-1}$ is a bounded operator whenever $\lambda \notin \sigma(K)$, we have

$$(\lambda I - K)^{-1} x = \sum_{i=1}^{\infty} \langle x, u_i \rangle \int_{-\infty}^{+\infty} \frac{1}{\lambda - \mu} \delta(\mu I - K) u_i \, d\mu$$

so we obtain the classic series

$$(\lambda I - K)^{-1} x = \sum_{i=1}^{\infty} \frac{1}{\lambda - \mu_i} \langle x, u_i \rangle u_i.$$

For the solution of the equation $(I - zK)x = y$ with $z \in \mathbb{C}$ we get the Schmidt series

$$x = (I - zK)^{-1} y = \sum_{i=1}^{\infty} \frac{1}{1 - z\mu_i} \langle y, u_i \rangle u_i$$

whenever $z^{-1} \notin \sigma(T)$. On the other hand, since $\delta(\lambda I - K)$ acts on $\mathcal{H}$ as a continuous endomorphism, equation

$$\delta(\lambda I - K) x = \sum_{i=1}^{\infty} \langle x, u_i \rangle \delta(\lambda - \mu_i) u_i.$$

holds for every $x \in \mathcal{H}$. □

If T is an unbounded self-adjoint operator then $D(T) \neq \mathcal{H}$ and $D(T^n)$ becomes smaller as n grows. So, the following result, makes sense only if the operator T is bounded.

Theorem 18. *In general, if T is a bounded self-adjoint operator, one has*

$$\delta(\lambda I - T) = \sum_{n=0}^{\infty} (-1)^n \frac{\delta^{(n)}(\lambda)}{n!} T^n \tag{2.9}$$

which is the Taylor series of $\delta(\lambda I - T)$ at λI.

Proof. Developing the operator function $\exp(itT)$, which is well-defined by the spectral theorem, we get

$$\delta(\lambda I - T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda t} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} T^n dt,$$

so that, formally interchanging the sum and the integral, we may write

$$\delta(\lambda I - T) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\mathcal{F}\{(it)^n\}(\lambda)}{n!} T^n.$$

Using the fact that

$$\mathcal{F}\{(it)^n\}(\lambda) = (-1)^n \sqrt{2\pi}\, \delta^{(n)}(\lambda)$$

for every $n \in \mathbb{N}$, we obtain (2.9). □

3 The resolvent operator and $\delta(\lambda I - T)$

Recall that the spectrum $\sigma(T)$ of a (densely defined) self-adjoint operator on a complex Hilbert space $\mathcal{H}$ is a closed subset of $\mathbb{C}$ contained in $\mathbb{R}$ (see for instance [9, 3.2]). If $z \in \mathbb{C} \setminus \sigma(T)$, i. e., if z is a regular point of T, and

$$\mathcal{R}(z, T) = (zI - T)^{-1}$$

denotes the *resolvent* operator of T at z (see [7, Definition 8.2]), the function $\lambda \mapsto (z - \lambda)^{-1}$ is continuous on $\sigma(T)$. The resolvent is well-defined over $\mathcal{H}$, so it is a bounded normal operator. If $z \in \mathbb{R} \setminus \sigma(T)$ then $\mathcal{R}(z, T)$ is even self-adjoint. From (1.1) it follows that

$$\mathcal{R}(z, T) = \int_{-\infty}^{+\infty} \frac{1}{z - \lambda} \delta(\lambda I - T) \, d\lambda$$

which is the *integral form of the resolvent* of T. So, by considering the complex-valued function $f(\lambda) = (z - \lambda)^{-1}$ with $z \in \mathbb{C} \setminus \sigma(T)$ and using the fact that

$$\mathcal{F}^{-1}\left(\frac{1}{\lambda - z}\right)(t) = \sqrt{2\pi}\, i e^{izt} Y(t)$$

then, according to (2.5), for $Im z > 0$ we have

$$(zI - T)^{-1} = -i \int_0^{\infty} e^{izt} U(t) \, dt.$$

From here, it follows that

$$\mathcal{R}(z, iT) = i\mathcal{R}(iz, -T) = \left(\mathcal{L}U^{-1}\right)(z)$$

if $Imz > 0$, where $\mathcal{L}$ is the Laplace transform. This is *the Hille-Yosida theorem* which relates the resolvent with the one-parameter group of unitary transformations $\{U(t) : t \in \mathbb{R}\}$ generated by the self-adjoint operator T.

If T is a bounded self-adjoint operator, γ is a closed Jordan contour that encloses $\sigma(T)$ and $f(z)$ is holomorphic inside the connected region surrounded by the path γ, the Dunford integral formula asserts that

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,\mathcal{R}(z,T)\;dz = f(T).$$

In [13] is pointed out that $(2\pi i)^{-1}\mathcal{R}(z,T)$ can be considered as the indicatrix of a vector-valued distribution with values in $\mathcal{L}(\mathcal{H})$. Dunford integral formula is easily obtained by using the $\delta(\lambda I - T)$ operator since, if we apply the Proposition 5 with $g(\lambda,\mu) = f(z(\mu))(z(\mu)-\lambda)^{-1}$, where $z(\mu) = \gamma(\mu)$ and $0 \le \mu \le 1$, then

$$\begin{aligned}\int_{\gamma} f(z)\,\mathcal{R}(z,T)\,dz &= \int_{-\infty}^{+\infty}\left\{\int_{\gamma}\frac{f(z)}{z-\lambda}\,dz\right\}\delta(\lambda I - T)\,d\lambda\\ &= 2\pi i\int_{-\infty}^{+\infty} f(\lambda)\,\delta(\lambda I - T)\;d\lambda = 2\pi i f(T).\end{aligned}$$

Example 19. *Derivation of the orthogonal projection operator onto* $\ker(M+I)$ *of the Hermitian matrix* M *of the Example 16 by the resolvent technique.* We must compute

$$P_{\lambda_1} = \frac{1}{2\pi i}\int_{|z+1|=1}\mathcal{R}(z,M)\;dz.$$

Clearly, we have

$$\mathcal{R}(z,M) = \frac{1}{z^2-z-2}\begin{bmatrix} z-1 & 1 & 1\\ 1 & z-1 & 1\\ 1 & 1 & z-1\end{bmatrix}.$$

Using that

$$\int_{|z+1|=1}\frac{\{1, z-1\}}{(z+1)(z-2)}\,dz = \left\{-\frac{2\pi i}{3}, \frac{4\pi i}{3}\right\}$$

we reproduce the result we got earlier.

4 The $\delta(\lambda I - T)$ operator as a limit

As $\mu \mapsto (\lambda \pm i\epsilon - \mu)^{-1}$ is continuous, for self-adjoint T

$$\begin{aligned}&((\lambda - i\epsilon)I - T)^{-1} - ((\lambda + i\epsilon)I - T)^{-1}\\ &= \int_{-\infty}^{+\infty}\left(\frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu}\right)\delta(\mu I - T)\;d\mu.\end{aligned}$$

If $f \in \mathcal{D}(\mathbb{R})$, Proposition 5 yields

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{f(\lambda)}{2\pi i}\left\{((\lambda - i\epsilon)I - T)^{-1} - ((\lambda + i\epsilon)I - T)^{-1}\right\}d\lambda\\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\lambda)\frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\;d\lambda\right\}\delta(\mu I - T)\,d\mu.\end{aligned}$$

Since in the sense of distributions

$$\frac{1}{2\pi i}\left(\frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu}\right) \to \delta(\lambda - \mu)$$

as $\epsilon \to 0^+$, we have

$$\frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\;d\lambda \to \int_{\mathbb{R}} f(\lambda)\,\delta(\lambda-\mu)\,d\lambda$$

as $\epsilon \to 0^+$. Hence, if g_n is defined by the left-hand side μ-parametric integral with $\epsilon = 1/n$, then $g_n \to f$ pointwise on $\mathbb{R}$. So, if $f \in \mathcal{D}(\mathbb{R})$ and T is bounded (hence with $\sigma(T)$ compact), as can be easily checked $\{g_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of continuous functions, with $\sup_{n\in\mathbb{N}}\|g_n\|_\infty \le \|f\|_\infty$, that converges pointwise on $\mathbb{R}$ to f. Thus, by [3, 10.2.8 Corollary] one has $g_n(T) \to f(T)$ in the strong operator topology, that is

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\lambda)\frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\;d\lambda\right\}\delta(\mu I - T)\,d\mu\\ &\qquad\to \int_{-\infty}^{+\infty} f(\mu)\,\delta(\mu I - T)\,d\mu\end{aligned}$$

as $\epsilon \to 0^+$ in the strong operator topology of $\mathcal{L}(\mathcal{H})$. Therefore, if T is bounded and $f \in \mathcal{D}(T)$ then

$$\int_{-\infty}^{+\infty}\frac{f(\lambda)}{2\pi i}\left\{((\lambda - i\epsilon)I - T)^{-1} - ((\lambda + i\epsilon)I - T)^{-1}\right\}d\lambda$$

goes to $f(T)$ as $\epsilon \to 0^+$. This proves that for bounded T

$$\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}((\lambda - i\epsilon)I - T)^{-1} - ((\lambda + i\epsilon)I - T)^{-1}$$

coincides with $\delta(\lambda I - T)$ as an $\mathcal{L}(\mathcal{H})$-valued distribution.

5 Unitary equivalence of $\delta(\lambda I - T)$

Theorem 20. *If T is a self-adjoint operator defined on the whole of $\mathcal{H}$, there exist a finite measure μ on the Borel sets of the compact space $\sigma(T)$ and a linear isometry U from $L_2(\sigma(T),\mu)$ onto $\mathcal{H}$ such that*

$$U^{-1}\delta(\lambda I - T)\,U = \delta(\lambda I - Q)$$

where $(Q\varphi)(x) = x\,\varphi(x)$ is the position operator.

Proof. According to [5] there exist a finite measure μ on the Borel sets of the compact space $\sigma(T)$ and a linear isometry U from $L_2(\sigma(T),\mu)$ onto $\mathcal{H}$ such that

$$\left(U^{-1}TU\right)\varphi = Q\varphi$$

for every $\varphi \in \mathcal{L}_2(\sigma(T),\mu)$. So, since $U^{-1}TU$ is a self-adjoint operator on $L_2(\sigma(T),\mu)$, we have

$$U^{-1}\delta(\lambda I - T)\,U = \delta\left(\lambda I - U^{-1}TU\right) = \delta(\lambda I - Q)$$

as stated. □

Remark 21. *For such linear isometry U the equation*

$$\left(U^{-1}\delta(\lambda I - T)\,U\,\varphi\right)(x) = \delta(\lambda - x)\,\varphi(x)$$

holds for every $\varphi \in \mathcal{L}_2(\sigma(T),\mu)$.

6 Commutation relations

Let S and T be two self-adjoint operators defined on the whole of $\mathcal{H}$ for which equations $[S,[S,T]] = [T,[S,T]] = \mathbf{0}$ hold. In this case

$$[-itS,[-itS,-isT]] = it^2 s\,[S,[S,T]] = \mathbf{0}$$

and the Baker-Campbell-Hausdorff formula yields

$$\delta(\lambda I - S)\,\delta(\mu I - T) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} e^{i(t\lambda+s\mu)}e^{-itS}e^{-isT}\,dt\,ds = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} e^{i(t\lambda+s\mu)}e^{\frac{-st}{2}[S,T]}e^{-i(tS+sT)}\,dt\,ds.$$

Likewise, since $[T,[T,S]] = [S,[T,S]] = \mathbf{0}$ one has

$$\delta(\mu I - T)\,\delta(\lambda I - S) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} e^{i(t\lambda+s\mu)}e^{\frac{-st}{2}[T,S]}e^{-i(tT+sS)}\,dt\,ds = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} e^{i(t\lambda+s\mu)}e^{\frac{st}{2}[S,T]}e^{-i(tS+sT)}\,dt\,ds.$$

So, using that

$$\exp\left(\frac{ist}{2}i\,[S,T]\right) - \exp\left(-\frac{ist}{2}i\,[S,T]\right) = 2i\sin\left(\frac{ist}{2}\,[S,T]\right)$$

we have

$$[\delta(\lambda I - S),\delta(\mu I - T)] = \frac{i}{2\pi^2}\int\int_{\mathbb{R}^2}\sin\left(\frac{ist}{2}\,[S,T]\right)e^{i(t\lambda+s\mu)}e^{-i(tS+sT)}\,dt\,ds.$$

For position Q and momentum P of a one-dimensional particle, one has $\mathcal{H} = L_2(\mathbb{R})$ and $[Q,P] = i\hbar I$. Therefore $[Q,[Q,P]] = [P,[Q,P]] = \mathbf{0}$ and

$$[\delta(\lambda I - Q),\delta(\mu I - P)] = \frac{i}{2\pi^2}\int\int_{\mathbb{R}^2}\sin\left(-\frac{st}{2}\hbar\right)e^{i(t\lambda+s\mu)}e^{-i(tQ+sP)}\,dt\,ds.$$

According to Theorem 8, if $[\delta(\lambda I - S),\delta(\mu I - T)]$ acts on $f(\lambda) = \lambda$, formally we have

$$\int_{-\infty}^{+\infty}\lambda\,[\delta(\lambda I - S),\delta(\mu I - T)]\,d\lambda = \frac{i}{2\pi^2}\int\int_{\mathbb{R}^2}\sin\left(\frac{ist}{2}\,[S,T]\right)\left\{\int_{-\infty}^{+\infty}\lambda e^{it\lambda}\,d\lambda\right\}e^{is\mu}e^{-i(tS+sT)}\,dt\,ds.$$

So, using the distributional equality

$$\int_{-\infty}^{+\infty}\lambda e^{it\lambda}d\lambda = \frac{2\pi}{i}\,\delta'(t) \tag{6.1}$$

and integrating by parts, it follows that

$$\int_{-\infty}^{+\infty}\lambda\,[\delta(\lambda I - S),\delta(\mu I - T)]\,d\lambda = -\frac{i}{2\pi}\,[S,T]\int_{-\infty}^{+\infty}se^{is(\mu I - T)}ds.$$

Observe that a second application of equation (6.1) and a second integration by parts yield

$$\int_{-\infty}^{+\infty}\lambda\mu\,[\delta(\lambda I - S),\delta(\mu I - T)]\,d\lambda\,d\mu = -\frac{i}{2\pi}\,[S,T]\int_{-\infty}^{+\infty}se^{-isT}\left\{\int_{-\infty}^{+\infty}\mu e^{is\mu}d\mu\right\}ds = [S,T]\int_{-\infty}^{+\infty}\delta(s)\{1 - isT\}e^{-isT}ds = [S,T]$$

as expected.

7 A remark on the Stone formula

Let T be a self-adjoint operator densely defined on a Hilbert space $\mathcal{H}$. If A is a Borel set in $\sigma(T)$, defining

$$E(A) := \int_{-\infty}^{+\infty}\chi_A(\lambda)\,dE_\lambda \tag{7.1}$$

where χ_A stands for the characteristic function of A (which is a bounded Borel function), then E is an $\mathcal{L}(\mathcal{H})$-valued finitely additive and pointwise countably additive measure (i.e., countable additivity under the strong operator topology of $\mathcal{L}(\mathcal{H})$) on the σ-algebra $\mathcal{A}$ of Borel subsets of $\sigma(T)$. So, if the characteristic function χ_A of A with respect to $\mathbb{R}$ is continuous on $\sigma_p(T)$ then

$$E(A) = \int_{-\infty}^{+\infty}\chi_A(\lambda)\,\delta(\lambda I - T)\,d\lambda.$$

For $-\infty < a < b < \infty$ and $\epsilon > 0$, we have

$$\int_a^b\left\{\int_{-\infty}^{+\infty}\left(\frac{1}{\lambda - i\epsilon - \mu} - \frac{1}{\lambda + i\epsilon - \mu}\right)\delta(\mu I - T)\,d\mu\right\}d\lambda = \int_{-\infty}^{+\infty}\left\{\int_a^b\frac{2i\epsilon\,d\lambda}{(\lambda-\mu)^2+\epsilon^2}\right\}\delta(\mu I - T)\,d\mu = 2i\int_{\mathbb{R}}\left\{\arg\tan\left(\frac{b-\mu}{\epsilon}\right) - \arg\tan\left(\frac{a-\mu}{\epsilon}\right)\right\}\delta(\mu I - T)\,d\mu.$$

If the limit as $\epsilon \to 0^+$ the bracketed function is equal to 0 if $\mu \in \mathbb{R}\setminus[a,b]$, equal to π if $a < \mu < b$ and equal to $\pi/2$ if $\mu \in \{a,b\}$. So, if $a,b \notin \sigma_p(T)$ so that $\chi_{(a,b)}$ and $\chi_{[a,b]}$ both belong to $\mathcal{B}_p(\mathbb{R})$, setting

$$g_n(\mu) := \frac{1}{\pi}\int_a^b\frac{2in^{-1}\,d\lambda}{(\lambda-\mu)^2+n^{-2}}$$

for each $n \in \mathbb{N}$ and

$$f(\mu) := \chi_{(a,b)}(\mu) + \chi_{[a,b]}(\mu),$$

then $g_n(\mu) \to f(\mu)$ for every $\mu \in \mathbb{R}$ and $\sup_{n\in\mathbb{N}} \|g_n\|_\infty \leq 1$ which, according to Proposition 4, implies that $g_n(T)x \to f(T)x$ for every $x \in D(T)$. In other words

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_a^b \left((\lambda - i\epsilon - T)^{-1} - (\lambda + i\epsilon - T)^{-1}\right) d\lambda = \frac{1}{2}\int_{-\infty}^{+\infty} \left(\chi_{(a,b)} + \chi_{[a,b]}\right)\delta(\mu I - T)\, d\mu,$$

holds pointwise on the domain $D(T)$ of T. Hence, by virtue of (7.1) we get

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_a^b \left((\lambda - i\epsilon - T)^{-1} - (\lambda + i\epsilon - T)^{-1}\right) d\lambda = \frac{1}{2}E((a,b)) + \frac{1}{2}E([a,b]) = E(a,b) + \frac{1}{2}E(a) + \frac{1}{2}E(b)$$

which is Stone's formula.

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