Mathematics and Statistics 9(1): 54-58, 2021 http://www.hrpub.org
DOI: 10.13189/ms.2021.090109

Generalized Relation between the Roots of Polynomial and Term of Recurrence Relation Sequence

Vipin Verma*, Mannu Arya

Department of Mathematics, School of Chemical Engineering and Physical Sciences Lovely Professional University, Phagwara 144411, Punjab, India

Received December 21, 2020; Revised January 18, 2021; Accepted February 24, 2021

Cite This Paper in the following Citation Styles

(a): *[1] Vipin Verma, Mannu Arya , "Generalized Relation between the Roots of Polynomial and Term of Recurrence Relation Sequence," Mathematics and Statistics, Vol. 9, No. 1, pp. 54 - 58, 2021. DOI: 10.13189/ms.2021.090109.*

(b): *Vipin Verma, Mannu Arya (2021). Generalized Relation between the Roots of Polynomial and Term of Recurrence Relation Sequence. Mathematics and Statistics, 9(1), 54 - 58. DOI: 10.13189/ms.2021.090109.*

Abstract Many researchers have been working on recurrence relation which is an important topic not only in mathematics but also in physics, economics and various applications in computer science. There are many useful results on recurrence relation sequence but there main problem to find any term of recurrence relation sequence we need to find all previous terms of recurrence relation sequence. There were many important theorems obtained on recurrence relations. In this paper we have given special identity for generalized kth order recurrence relation. These identities are very useful for finding any term of any order of recurrence relation sequence. Authors define a special formula in this paper by this we can find direct any term of a recurrence relation sequence. In this recurrence relation sequence to find any terms we need to find all previous terms so this result is very important. There is important property of a relation between coefficients of recurrence relation terms and roots of a polynomial for second order relation but in this paper, we gave this same property of recurrence relation of all higher order recurrence relation. So finally, we can say that this theorem is valid all order of recurrence relation only condition that roots are distinct. So, we can say that this paper is generalization of property of a relation between coefficients of recurrence relation terms and roots of a polynomial. **Theorem**: - Let c_1 and c_2 are arbitrary real numbers and suppose the equation $x^2 - c_1x - c_2 = 0$ (1) Has x_1 $andx_2$ are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ $n \geq 2$ (2) iff $a_n = \beta_1 x_1^n + \beta_2 x_2^n$. For n= 0, 1, 2 …where β_1 and β_2 are arbitrary constants. **Proof**: - First suppose that $< a_n >$ of type $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n$ we shall prove $< a_n >$ is a solution of recurrence relation (2). Since $x_1, x_2 and\ x_3$ are roots of equation (1) so all are satisfied equation (1) so we have $x_1^2 = c_1x_1 + c_2$, $x_2^2 = c_1x_1 + c_2$. Consider $c_1a_{n-1} + c_2a_{n-2} = c_1(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1}) + c_2(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2}) = \beta_1 x_1^{n-2}(c_1x_1 + c_2) + \beta_2 x_2^{n-2}(c_1x_2 + c_2) = \beta_1 x_1^n + \beta_2 x_2^n = a_n$. This implies $c_1a_{n-1} + c_2a_{n-2} = a_n$. So the sequence $< a_n >$ is a solution of the recurrence relation. Now we will prove the second part of theorem. Let $a_n = c_1a_{n-1} + c_2a_{n-2}$ $n \geq 2$ is a sequence with three $initial\ terms$ $a_0 = A_1, a_1 = A_2$, Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n$. So $\beta_1 + \beta_2 = A_1$ (3). $\beta_1 x_1 + \beta_2 x_2 = A_2$ (4). Multiply by x_1 to (3) and subtracts from (4). We have $\beta_2 = \frac{A_2 - A_1}{x_2 - x_1}$ similarly we can find $\beta_1 = \frac{A_2 - A_1}{x_1 - x_2}$. So we can say that values of $\beta_1 and\ \beta_2$ are defined as roots are distinct. So non- trivial values of $\beta_1 and\ \beta_2$ can find and we can say that result is valid. **Example:** Let $< a_n >$ be any sequence such that $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}, n \geq 3$ and $a_0 = 0, a_1 = 1, a_2 = 2$. Then find a_{10} for above sequence. **Solution:** The polynomial of above sequence is $x^3 - 6x^2 + 11x - 6 = 0$. Solving this equation we have roots are 1, 2, and 3 using above theorem we have $a_n = \beta_1 1^2 + \beta_2 2^2 + \beta_3 3^2$ (7). Using $a_0 = 0, a_1 = 1, a_2 = 2$ in (7) we have $\beta_1 + \beta_2 + \beta_3 = 0$ (8). $\beta_1 + 2\beta_2 + 3\beta_3 = 1$ (9). $\beta_1 + 4\beta_2 + 9\beta_3 = 2$ (10) Solving (8), (9) and (10) we have $\beta_1 = -\frac{3}{2}, \beta_2 = 2, \beta_3 = -\frac{1}{2}$. This implies $a_n = -\frac{3}{2}1^n + 22^n - \frac{1}{2}3^n$. Now put n=10 we have $a_{10} = -27478$. Recurrence relation is a very useful topic of mathematics, many problems of real life may be solved by recurrence relations, but in recurrence relation there is a major difficulty in the recurrence relation. If we want to find 100th term of sequence, then we need to find all previous 99 terms of given sequence, then we can get 100th term of sequence but above theorem is very useful if coefficients of recurrence relation of given sequence satisfies the condition of the above theorem, then we can apply above theorem and we can find direct any term of sequence without finding all previous terms.

Keywords Generalized, Recurrence Relation, Sequence

1. Introduction

In Number Theory there are many special types of sequences. Fibonacci Sequence and Luca Sequence are special type of sequences obtained from recurrence relation with given initial terms. Recurrence relation is an equation that defines a sequence based on a method that gives the next term as relation of the previous terms [1, 2, 3]. Recurrence relations are used in mathematics as well as economics; physics and are very useful in solving real life problems. [4,5], we can calculate growth in economics by recurrence techniques. Many problems of real life can be represented in a recurrence relation and can be solved by recurrence relation method. Network marketing business is also a special type of recurrence relation and many problems of Network marketing business can be solved by recurrence methods.[6,7] In recurrence relation for finding any term of sequence we need to find all previous terms of sequence but by using the theorem given in the paper we can find direct any term of sequence. There are many identities of recurrence relation which are very important but only valid for two order recurrence relation but the result of the paper is valid for all order recurrence relation. In Number Theory there are many special types of Sequences. Both Fibonacci Sequence and Luca Sequence are special type of recurrence relation with given initial terms. Italian Mathematician Leonardo of Pisa who is also known as by his nickname Fibonacci (1170-1240) he wrote (Book of the Abacus) in 1202. He was 1st European mathematician who works on Indian and Arabian mathematics. He gave a special type of sequence

$$F_n = F_{n-1} + F_{n-2} n \geq 2 \qquad (1.1)$$

with initial Term $F_0 = 0$ and $F_1 = 1$

Edouard Lucas dominated the field recursive series during the period 1878-1891 he was 1st mathematician who applied Fibonacci's name for sequence (1.1) and it has been known as Fibonacci sequence since then. Lucas sequence defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} n \geq 2 \qquad (1.2)$$

With initial term, $L_0 = 2 \qquad L_1 = 1$

Terms of the Lucas sequence are called Lucas numbers. Binet forms of nth Fibonacci and nth Lucas numbers were given by Bernoulli (1724) and Euler (1726) respectively

2. Third Order Recurrence Relation

In the third order recurrence relation new term depends on three previous terms with three initial terms are given.

For example

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}, n \geq 3$$

with the initial terms $a_0 = 0, a_1 = 1, a_2 = 2$

Theorem3.1: - Let c_1, c_2 and c_3 are arbitrary real numbers and suppose the equation

$$x^3 - c_1x^2 - c_2x - c_3 = 0 \qquad (3.1)$$

has $x_1, x_2 and x_3$ are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} n \geq 3 \qquad (3.2)$$

If and only if $a_n = \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n$ for $n = 0, 1, 2, ...$ where β_1, β_2 and β_3 are arbitrary constants.

Proof: - First suppose that $< a_n >$ of type $a_n = \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n$ we shall prove $< a_n >$ is a solution of recurrence relation (3.2). Since x_1, x_2 and x_3 are roots of equation (1) so all are satisfied equation (3.1) so we have

$$x_1^3 = c_1x_1^2 + c_2x_1 + c_3$$

$$x_2^3 = c_1x_2^2 + c_2x_2 + c_3$$

$$x_3^3 = c_1x_3^2 + c_2x_3 + c_3$$

Consider

$$\begin{aligned} c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} \\ = c_1(\beta_1x_1^{n-1} + \beta_2x_2^{n-1} + \beta_3x_3^{n-1}) \\ + c_2(\beta_1x_1^{n-2} + \beta_2x_2^{n-2} + \beta_3x_3^{n-2}) \\ + c_3(\beta_1x_1^{n-3} + \beta_2x_2^{n-3} + \beta_3x_3^{n-3}) \end{aligned}$$

$$\begin{aligned} &= \beta_1x_1^{n-3}(c_1x_1^2 + c_2x_1 + c_3) \\ &\quad + \beta_2x_2^{n-3}(c_1x_2^2 + c_2x_2 + c_3) \\ &\quad + \beta_3x_3^{n-3}(c_1x_3^2 + c_2x_3 + c_3) \\ &= \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n = a_n \end{aligned}$$

This implies

$$c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} = a_n$$

So, the sequence $< a_n >$ is a solution of the recurrence relation(3.2).

Now we will prove the second part of theorem

Let $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} n \geq 3$ is a sequence with three initial terms $a_0 = A_1, a_1 = A_2, a_2 = A_3$

Let $a_n = \beta_1x_1^n + \beta_2x_2^n + \beta_3x_3^n$

So

$$\beta_1 + \beta_2 + \beta_3 = A_1 \quad (3.3)$$

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = A_2 \quad (3.4)$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 = A_3 \quad (3.5)$$

The system of linear equations has a non-trivial solution if and only

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \neq 0$$

We know that

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \neq 0 \quad (3.6)$$

Equation (3.6)is always non-zero as roots are distinct. So non-trivial values of β_1, $\beta_2$and $\beta_3$can find and we can say that result is valid.

3. Fourth Order Recurrence Relation

In the fourth order recurrence relation new term depends on four previous terms and four initial terms are given.

For example

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-2}a_{n-3}, n \geq 4$$

with the initial terms $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$

Theorem 4.1: -Let c_1, $c_2, c_3 and c_4$ are arbitrary real numbers and suppose the equation

$$x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4 = 0 \quad (4.1)$$

Has x_1, x_2, x_3 and x_4 are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} n \geq 4 \quad (4.2)$$

if and only if $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n$ for $n = 0, 1, 2, \ldots.$ where $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary constants.

Proof: - First suppose that $< a_n >$ of type $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n$ we shall prove $< a_n >$ is a solution of recurrence relation (4.2) . Since $x_1, x_2, x_3 and x_4$ are roots of equation (4.1) so all are satisfied equation (4.1) so we have

$$x_1^4 = c_1 x_1^3 + c_2 x_1^2 + c_3 x_1 + c_4$$

$$x_2^4 = c_1 x_2^3 + c_2 x_2^2 + c_3 x_2 + c_4$$

$$x_3^4 = c_1 x_3^3 + c_2 x_3^2 + c_3 x_3 + c_4$$

$$x_4^4 = c_1 x_4^3 + c_2 x_4^2 + c_3 x_4 + c_4$$

Consider

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} &= c_1(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1} + \beta_3 x_3^{n-1} \\ &+ \beta_4 x_4^{n-1}) \\ &+ c_2(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2} + \beta_3 x_3^{n-2} \\ &+ \beta_4 x_4^{n-2}) \\ &+ c_3(\beta_1 x_1^{n-3} + \beta_2 x_2^{n-3} + \beta_3 x_3^{n-3} \\ &+ \beta_4 x_4^{n-3}) + c_4(\beta_1 x_1^{n-4} + \beta_2 x_2^{n-4} \\ &+ \beta_3 x_3^{n-4} + \beta_4 x_4^{n-4}) \end{aligned}$$

$$\begin{aligned} &= \beta_1 x_1^{n-4}(c_1 x_1^3 + c_2 x_1^2 + c_3 x_1 + c_4) \\ &+ \beta_2 x_2^{n-4}(c_1 x_2^3 + c_2 x_2^2 + c_3 x_2 + c_4) \\ &+ \beta_3 x_3^{n-4}(c_1 x_4^3 + c_2 x_4^2 + c_3 x_4 + c_4) \\ &+ \beta_4 x_2^{n-4}(c_1 x_4^3 + c_2 x_4^2 + c_3 x_4 + c_4) \end{aligned}$$

$$= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n = a_n$$

This implies

$$c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} = a_n$$

So, the sequence $< a_n >$ is a solution of the recurrence relation.

Now we will prove another part of theorem

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} n \geq 3$ is a sequence with three initial term

$$a_0 = A_1, a_1 = A_2, a_2 = A_3, a_3 = A_4$$

Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n$

So

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = A_1$$

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = A_2$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2 = A_3$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_4^2 + \beta_4 x_4^3 = A_4$$

The system of linear equations has a non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} \neq 0$$

We know that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_1) \neq 0 \qquad (4.7)$$

equation (4.7)is always non-zero as roots are distinct. So non-trivial values of β_1, β_2 $\beta_3$and $\beta_4$can find and we can say that result is valid.

4. kth Order Recurrence Relation Sequence

Theorem 2.1: - Let $c_1, c_2, c_3, \dots, c_k$ are arbitrary real numbers and suppose the equation

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} \dots - c_k = 0 \qquad (2.1)$$

Has $x_1, x_2, x_3 \dots, x_k$ are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots \dots + c_k a_{n-k}, n > k \qquad (2.2)$$

if and only if $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n$for $n = 0, 1, 2 \dots$ where $\beta_1, \beta_2, \beta_3 \dots \beta_k$ are arbitrary constants.

Proof: - Let us suppose that the sequence $< a_n >$ as$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n$ now we shall prove $< a_n >$ is a solution of recurrence relation (2.2). Since $x_1, x_2, x_3, \dots, x_k$ are roots of equation (2.1) so all satisfy equation (2.1) so we have

$$\begin{aligned} x_1^k &= c_1 x_1^{k-1} + c_2 x_1^{k-2} + \dots + c_k \\ x_2^k &= c_1 x_2^{k-1} + c_2 x_2^{k-2} + \dots + c_k \\ &\vdots \\ x_k^k &= c_1 x_k^{k-1} + c_2 x_k^{k-2} + \dots + c_k \end{aligned}$$

Consider

$$\begin{aligned} & c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \\ &= c_1(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1} + \beta_3 x_3^{n-1} + \cdots + \beta_k x_k^{n-1}) \\ &+ c_2(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2} + \beta_3 x_3^{n-2} + \cdots + \beta_k x_k^{n-2}) \\ &+ c_3(\beta_1 x_1^{n-3} + \beta_2 x_2^{n-3} + \beta_3 x_3^{n-3} + \dots + \beta_k x_k^{n-3}) \\ &+ c_4(\beta_1 x_1^{n-4} + \beta_2 x_2^{n-4} + \beta_3 x_3^{n-4} + \dots + \beta_k x_k^{n-4}) \\ &= \beta_1 x_1^{n-k}(c_1 x_1^{k-1} + c_2 x_1^{k-2} + \dots + c_3 x_1 + c_k) \\ &+ \beta_2 x_2^{n-k}(c_1 x_2^{k-1} + c_2 x_2^{k-2} + \dots + c_3 x_2 + c_k) \\ &+ \beta_3 x_3^{n-k}(c_1 x_3^{k-1} + c_2 x_3^{k-2} + \dots + c_3 x_3 + c_k) + \cdots \\ &+ \beta_k x_k^{n-k}(c_1 x_k^{k-1} + c_2 x_k^{k-2} + \dots + c_3 x_k + c_k) \\ &= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n \\ &= a_n \end{aligned}$$

This implies

$$c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} = a_n$$

So the sequence $< a_n >$ is a solution of the recurrence relation.

Now we will prove another part of theorem

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} n > k$ is a sequence k with initial terms $a_0 = A_0, a_1 = A_1, a_2 = A_2, \dots, a_k = A_k$

Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \dots + \beta_k x_k^n$

So

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_k &= A_0 \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots + \beta_k x_k &= A_1 \\ \beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \cdots + \beta_k x_k^2 &= A_2 \\ &\vdots \\ \beta_1 x_1^k + \beta_2 x_2^k + \beta_3 x_3^k + \cdots + \beta_k x_k^k &= A_k \end{aligned}$$

The system of linear equations has a non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 & \dots 1 \\ x_1 & x_2 & \dots x_k \\ \vdots & \vdots & \ddots \vdots \\ x_1^k & x_2^k \dots & x_k^k \end{vmatrix} \neq 0$$

$$\begin{vmatrix} 1 & 1 & \dots 1 \\ x_1 & x_2 & \dots x_k \\ \vdots & \vdots & \ddots \vdots \\ x_1^k & x_2^k \dots & x_k^k \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)\dots(x_k - x_1) = 0 \quad (3.3)$$

Equation (3.3)is always non-zero as roots are distinct. So non-trivial values of $\beta_1, \beta_2\, \beta_3, \dots, \beta_k$ can find and we can say that result is valid.

5. Conclusions

There is important property of a relation between coefficients of recurrence relation terms and roots of a polynomial for second order relation but in this paper, we gave this same property of recurrence relation of all higher order recurrence relation. So finally, we can say that this theorem is valid all order of recurrence relation only condition that roots are distinct. So, we can say that this paper is generalization of property of a relation between coefficients of recurrence relation terms and roots of a polynomial.

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