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Convergence Almost Everywhere of Non-convolutional Integral Operators in Lebesgue Spaces

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Abstract The case of one-dimensional and multi-dimensional non-convolutional integral operators in Lebesgue spaces is considered in this paper. The convergence in the norm and almost everywhere of non-convolution integral operators in Lebesgue spaces was insufficiently studied. The kernels $K_\varepsilon(x,y)$ of non-convolutional integral operators do not need to have a monotone majorant, so the well-known results on the convergence almost everywhere of convolutional averages are not applicable here. The kernels $K_\varepsilon(x,y)$ of non-convolutional integral operators take into account different behaviors at $|y|\to 0$ and $|y|\to\infty$ depending on $\varepsilon\to 0$ (which is important in applications) and cover the situation in the particular case of convolutional and non-convolutional integral operators. We are interested in the behavior of function $K_\varepsilon\varphi$ as $\varepsilon \to 0$. Theorems on convergence almost everywhere in the case of one-dimensional and multidimensional non-convolution integral operators in Lebesgue spaces are proved. The theorems proved are more general ones (including for convolutional integral operators) and cover a wide class of kernels.

Keywords Non-convolutional Integral Operator, Convergence Almost Everywhere, Chen-Marchaud Fractional Derivatives, Non-convolutional Averaging

1 Introduction

A modification of Liouville fractional integro-differentiation on a straight line R is considered in [6], "attached" to a certain fixed point $c\in R,\ |c|<\infty$, and convenient in a way that it can be applied to functions set on the entire straight line that can have any increment at infinity:

$$(I_c^\alpha\varphi)(x) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int\limits_c^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt, x>c,\\ \frac{1}{\Gamma(\alpha)}\int\limits_x^c \frac{\varphi(t)}{(t-x)^{1-\alpha}}\,dt, x<c,\end{cases} \tag{1}$$

where $x\in R,\ \alpha>0$. Apparently, fractional integration in this form first was mentioned in (Chen Y.W. [1]). The corresponding fractional differentiation in the Riemann-Liouville form was also considered there

$$(D_c^\alpha f)(x) := \frac{1}{\Gamma(1-\alpha)}\begin{cases} \frac{d}{dx}\int\limits_c^x \frac{f(t)}{(x-t)^{\alpha}}\,dt, & x>c,\\ -\frac{d}{dx}\int\limits_x^c \frac{f(t)}{(t-x)^{\alpha}}\,dt, & x<c,\end{cases}$$

where $x\in R,\ 0<\alpha<1$. Fractional integration

$$D_c^\alpha f = \frac{1}{\aleph(\alpha,l)}\int\limits_0^\infty \frac{(\Delta_t^l f_{c+})(x)-(\Delta_{-t}^l f_{c-})(x)}{t^{1+\alpha}}\,dt,\ l>\alpha>0$$

in the form of Marchaud was considered by A.V. Skorikov in ([11]), where it was used to describe space $L_p^\alpha(\Omega)$ (restricted on (a,b) Bessel potentials on R). A number of properties of Chen fractional integro-differentiation can be found in ([7], Section 18.5, [9]).

In [3], [6], three different ways of reduction of the Chen-Marchaud constructions for fractional differentiation of D_c^α, are considered, and these different options for reduction are used to describe and invert the fractional integrals (1) of functions from $L_p^{loc}(R)$. Three different ways of "truncating" the fractional derivative D_c^α, give different integral representations for the truncated Chen-Marchaud fractional derivatives.

The resulting integral representations are operators that approximate the unit operator. The first integral representation falls under the influence of the well-known theorems on convolutional approximations of unity, and the operators of the second and third integral representations of a non-convolutional structure require a special study. Non-convolutional integral operators in space $L_p(R)$ were poorly studied. In this regard, one class of non-convolutional averaging is considered in [8]

$$(A_\varepsilon f)\ (x) =$$

$$= \int\limits_0^{\frac{1}{\varepsilon_1}} \cdots \int\limits_0^{\frac{1}{\varepsilon_n}} k_1(y_1)\dots k_1(y_n)\varphi(x-\varepsilon\circ y\circ(x-c))dy_1\dots dy_n, \tag{2}$$

where $c \in R^n, x - \varepsilon \circ y \circ (x-c) = (x_1-\varepsilon_1 y_1(x_1-c_1), \dots, x_n-\varepsilon_n y_n(x_n-c_n))$, $0 < \varepsilon_i < 1$, $\int\limits_0^\infty k_i(y_i)\,dy_i = 1$, $i = \overline{1,n}$ and the convergence to the function $\varphi(x)$ in norm $L_{\overline{p}}(R^n)$ (or $L_{\overline{p}}^{loc}(R^n)$) is investigated.

Non-convolutional integral representations in Lebesgue space were obtained in [13]. Operator (2) is not convolutional and kernels $k_i(t)$, $i = \overline{1,n}$, do not need to have a monotone majorant, so the well-known results on convergence of convolutional averages almost everywhere ([12], p. 77-78) are not applicable here. We will set simple conditions ensuring the convergence of construction (2) to $\varphi(x)$ almost everywhere, having obtained them as a corollary of the statement on a non-convolutional operator convergence almost everywhere.

Paper [2] argued strong convergence and $\Delta-$convergence theorems for the class of generalized non-expansive multi-valued maps in a CAT(0) space. Furthermore, Paper [10] presented different methods for bandwidth selection in bivariate kernel density estimation based on the principle of gradient method and compare the result with the biased cross-validation method. Also, the asymptotic mean integrated squared error is used as the measure of performance of the new methods.

In this paper, multi-dimensional non-convolutional integral operators are studied

$$(K_\varepsilon\varphi)(x) = \int\limits_{R^n} K_\varepsilon(x,y)\,\varphi(x-y)\,dy,\ x\in R^n \tag{3}$$

in space $L_p(R^n)$. In the case when the kernel $K_\varepsilon(x,y)$, in (3) do not depend on x, some theorems on convergence almost everywhere of similar averages with an arbitrary dependence on ε are known, (see, for example, U. Neri [5], p.16 or T Kurokawa [4]). However, even for this convolutional case, the assumptions about the kernel are restrictive in these works.

The kernel $K_\varepsilon(x,y)$ in (3) takes into account different behaviors as $|y|\to 0$ and $|y|\to\infty$ depending on $\varepsilon\to 0$ (which is important in applications) and covers the situation of operator (2). We are interested in the behavior of function $K_\varepsilon\varphi$ as $\varepsilon\to 0$.

In the proof of this theorem, we follow, to a large extent, the proof of Theorem 2.8 (T.Kurokawa [4]), where convolutional averaging was considered; our condition 3) in Theorem 2 being proved is much more general (including the convolutional operators) and covers a wide class of kernels.

In this paper, we prove the convergence almost everywhere of the non-convolutional integral operator (3) in space $L_p(R^n)$.

2 Materials and methods

The research material is the object in the case of one-dimensional and multidimensional non-convolutional integral operators. The methods of differential and integral calculus, as well as the methods of functional analysis are used in the study.

3 Auxiliary statements

Let $f(x)$ be locally integrable and $W(x,\varepsilon) = \{y : |y-x| \le \varepsilon\}$ is the ball of radius ε centered at point x. According to Lebesgue's theorem

$$\frac{1}{mesW(x,\varepsilon)} \int\limits_{W(x,\varepsilon)} f(y)\,dy \to f(x)$$

almost everywhere on R^n.

Definition 1. Point $x\ (x\in R^n)$ is called the Lebesgue function $f(x)$, if

$$\frac{1}{mesW(x,\varepsilon)} \int\limits_{W(x,\varepsilon)} |f(y)-f(x)|\,dy \to 0$$

as $\varepsilon\to 0$.

It is clear that any point of continuity of function $f(x)$ is a Lebesgue point for $f(x)$.

Remark 1. Note that Definition 1 implies that if $x\ (x\in R^n)$ is a Lebesgue point, then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int\limits_{|y|\le\varepsilon} |f(x-y)-f(x)|\,dy = 0. \tag{4}$$

4 Main results

4.1 The case of one-dimensional non-convolutional integral operators

Consider the non-convolution integral operator

$$(K_\varepsilon\varphi)(x) = \int\limits_{-\infty}^{\infty} k_\varepsilon(x,y)\,\varphi(x-y)\,dy,\ x\in R.$$

Theorem 1. Let $\varphi(x)\in L_p\left(R^1\right)$, $1\le p\le\infty$ and let for any sufficiently small $\mu>0$, the following conditions hold:

1) $\int\limits_{|y|<\mu} k_\varepsilon(x,y)\,dy \to 1$ as $\varepsilon \to 0$ for almost all x;

2) $\int\limits_{|y|>\mu} |k_\varepsilon(x,y)|^{p'}\,dy \to 0$, if $p>1$ and

$\sup\limits_{|y|>\mu} |k_\varepsilon(x,y)| \to 0$, if $p=1$, as $\varepsilon \to 0$ for almost all x;

3) There are numbers $\alpha>0,\ \beta>0$ and almost everywhere finite functions are $a(x)>0,\ c(x)>0$, such that at $|y|<\mu$

$$|k_\varepsilon(x,y)| \le \begin{cases} c(x)\,|y|^{\alpha-1}\cdot\varepsilon^{-\alpha},\ at\ \ |y| \le a(x)\cdot\varepsilon, \\ c(x)\,|y|^{-\beta-1}\,\varepsilon^{\beta},\ at\ \ |y| > a(x)\cdot\varepsilon. \end{cases}$$

Then

$$\lim_{\varepsilon\to 0}\,(K_\varepsilon\varphi)(x) = \varphi(x) \tag{5}$$

holds for almost all x (functions $\varphi(x)$ at Lebesgue points, if functions $a(x)$, $c(x)$ are finite at these points).

If $k_\varepsilon(x,y) \equiv 0$ at $|y| \ge N = N(x)$ for any sufficiently small ε, then (5) is true for any function $\varphi(x) \in L_p^{loc}(R^1)$, $1 \le p \le \infty$.

Proof. Fix a point x, considering it the Lebesgue point of function $\varphi(x)$. Let h be an arbitrary small number $h>0$. Let us show that

$$|(K_\varepsilon\varphi)(x) - \varphi(x)| < ch \tag{6}$$

by choosing $\varepsilon = \varepsilon(h)$ where c does not depend on h and ε (but may depend on$x$).

For simplicity of further notation, consider the case when $k_\varepsilon(x,y) = 0$ for $y<0$, based on the fact that the case when $k_\varepsilon(x,y) \equiv 0$ for $y>0$ is considered completely similar due to the "evenness" of the theorem conditions with respect to y.

We have

$$|(K_\varepsilon\varphi)(x) - \varphi(x)| \le I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^\mu |k_\varepsilon(x,y)|\,|\varphi(x-y) - \varphi(x)|\,dy,$$

$$I_2 = \left|\int_0^\mu k_\varepsilon(x,y)\,dy - 1)\right| |\varphi(x)|\,,$$

$$I_3 = \int_\mu^\infty |k_\varepsilon(x,y)|\,|\varphi(x-y)|\,dy,$$

μ while $\mu>0$ is arbitrary. Since x is the Lebesgue point, we can indicate such $\mu>0$ that

$$\frac{1}{y}\int_0^y |\varphi(x-\tau) - \varphi(x)|\,d\tau < h,\ 0<y\le\mu. \tag{7}$$

Assume that μ is chosen so small that (7) and condition 3) of Theorem 1 hold. Now let's fix $\mu = \mu(h) = \mu(h,x)$. Using condition 1) of Theorem 1, we obtain

$$I_2 \le c_2 h \tag{8}$$

by choosing $\varepsilon = \varepsilon(h,\mu)$, for $c_2 = |\varphi(x)|$. The application of Holder's inequality in I_3 gives

$$I_3 :\le \left(\int_\mu^\infty |k_\varepsilon(x,y)|^{p'}\,dy\right)^{\frac{1}{p'}} \times$$

$$\times \left(\int_{|y|>\mu} |\varphi(x-y)|^p\,dy\right)^{\frac{1}{p}} \le C_3 h \tag{9}$$

with condition 2) of Theorem 1 (also by choosing $\varepsilon = \varepsilon(h,\mu)$), $C_3 = \|\varphi(x)\|_{L_p(R)}$.

It remains to determine I_1. We have:

$$I_1 := \int_0^{a(x)\varepsilon} |k_\varepsilon(x,y)|\,|\varphi(x-y) - \varphi(x)|\,dy +$$

$$+ \int_{a(x)\varepsilon}^{\mu} |k_\varepsilon(x,y)|\,|\varphi(x-y) - \varphi(x)|\,dy$$

assuming that ε is chosen so small that $0 < a(x)\,\varepsilon < \mu$. Using assumption 3) of Theorem 1, we obtain:

$$I_1 \le \frac{c(x)}{\varepsilon^\alpha}\int_0^{a(x)\cdot\varepsilon} |y|^{\alpha-1}\,|\varphi(x-y) - \varphi(x)|\,dy +$$

$$+ c(x)\cdot\varepsilon^\beta \int_{a(x)\cdot\varepsilon}^{\infty} |y|^{-\beta-1}\,|\varphi(x-y) - \varphi(x)|\,dy = I_1' + I_1''.$$

Denote by $G(y) = G_x(y) := \int\limits_0^y |\varphi(x-\tau) - \varphi(x)|\,d\tau$, so that

$$G(y) \le hy,\ 0<y<\mu \tag{10}$$

according to (7). Integrating by parts we have:

$$I_1' = \frac{c(x)}{\varepsilon^\alpha}\{y^{\alpha-1}\,G(y)\,\Big|_{y=0}^{y=a(x)\,\varepsilon} -$$

$$-(\alpha-1)\int_0^{a(x)\cdot\varepsilon} y^{\alpha-2}\,G(y)\,dy\}. \tag{11}$$

With (10) we obtain $y^{\alpha-1}\,G(y)\Big|_{y=0} = 0$. Using (10) with (11) considering that $a(x)\cdot\varepsilon < \mu$ we have

$$I_1' \le c_1' h, \tag{12}$$

where $c_1' = \left[1 + \frac{|\alpha-1|}{\alpha}\right] c(x)\,a(x)^\alpha$. Similar integration by parts in I_1'' gives:

$$I_1'' = c(x)\cdot\varepsilon^\beta\{y^{-\beta-1}\,G(y)\,\Big|_{y=a(x)\,\varepsilon}^{y=\mu} +$$

$$+ (\beta+1)\, h \int\limits_{a(x)\cdot\varepsilon}^{\mu} y^{-\beta-2} G(y)\, dy\}.$$

Using (10) again, we obtain:

$$I_1'' \le \frac{1+2\beta}{\beta} c(x)\, h \left[a(x)^{-\beta} + \left(\frac{\varepsilon}{\mu}\right)^{\beta} \right].$$

Since $\varepsilon < \frac{\mu}{a(x)}$, hence

$$I_1^{''} \le c_1^{''} h, \tag{13}$$

where $c_1^{''} = \frac{2}{\beta}(1+2\beta)\frac{c(x)}{a(x)^{\beta}}$. Collecting inequalities (8), (9), (12), and (13), we obtain the required estimate (6).

It remains to add that the intersection of the set of Lebesgue points with the set of points where $a(x)$ and $c(x)$ are finite, has full measure.

Finally, if the kernel $k_\varepsilon(x,y)$is finite in terms of y, then estimate (9) is the only one where the information that $\varphi(x) \in L_p(R)$-function $\varphi(x)$ is integrated over a finite segment was used, so it is suffice that $\varphi(x) \in L_p^{loc}(R)$.

4.2 The case of multidimensional non-convolutional integral operators

Theorem 2. Let $\varphi(x) \in L_p(R^n)$, $1 \le p \le \infty$ and let the following conditions be met for any sufficiently small $\mu > 0$:

1) $\int\limits_{|y|<\mu} K_\varepsilon(x,y)\, dy \to 1$ as $\varepsilon \to 0$ for almost all x;

2) $\int\limits_{|y|>\mu} |K_\varepsilon(x,y)|^{p'} dy \to 0$, if $p > 1$ and $\sup\limits_{|y|>\mu} |K_\varepsilon(x,y)| \to 0$, if $p = 1$, as $\varepsilon \to 0$ for almost all x;

3) There are numbers $\alpha > 0$, $\beta > 0$ and finite function almost everywhere $a(x) > 0$, $c(x) > 0$, such that $|y| < \mu$

$$|K_\varepsilon(x,y)| \le \begin{cases} c(x)\, |y|^{\alpha-1} \cdot \varepsilon^{-\alpha}, & |y| \le a(x)\cdot\varepsilon, \\ c(x)\, |y|^{-\beta-1} \varepsilon^{\beta}, & |y| > a(x)\cdot\varepsilon. \end{cases}$$

Then

$$\lim_{\varepsilon\to 0} (K_\varepsilon\varphi)(x) = \varphi(x) \tag{14}$$

is true for almost all x.

If $K(x,y) \equiv 0$ for $|y| \ge N = N(x)$ for any sufficiently small ε, then (14) is true for all $\varphi(x) \in L_p^{loc}(R^n)$.

Proof. Let x $(x \in R^n)$ be the Lebesgue point of function $f(x)$. We have

$$|(K_\varepsilon\varphi)(x) - \varphi(x)| \le J_1 + J_2 + J_3,$$

where

$$J_1 := \left| \int\limits_{|y|<\mu} K_\varepsilon(x,y)\, dy - 1) \right| |\varphi(x)|,$$

$$J_2 := \int\limits_{|y|<\mu} |K_\varepsilon(x,y)|\, |\varphi(x-y) - \varphi(x)|\, dy,$$

$$J_3 := \int\limits_{|y|>\mu} |K_\varepsilon(x,y)|\, |\varphi(x-y)|\, dy.$$

By condition 1) of the Theorem 2 we get that $J_1 \le C_1 h$ by choosing $J_1 \le C_1 h$, here $C_1 = |\varphi(x)|$.

Application of Holders inequality in J_3 gives:

$$J_3 :\le \left(\int\limits_{|y|>\mu} |K_\varepsilon(x,y)|^{p'} dy \right)^{\frac{1}{p'}} \left(\int\limits_{|y|>\mu} |\varphi(x-y)|^{p} dy \right)^{\frac{1}{p}} \le$$
$$\le C_3 h$$

by condition 2) of Theorem 2 (and by choosing $\varepsilon = \varepsilon(h,\mu)$), $C_3 = \|\varphi(x)\|_{L_p(R^n)}$.

It remains to estimate J_2. Using assumption 3) of the Theorem 2, we get

$$J_2 \le I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{c(x)}{\varepsilon^\alpha} \int\limits_{|y|<a(x)\cdot\varepsilon} |y|^{\alpha-1} |\varphi(x-y) - \varphi(x)|\, dy,$$

$$I_2 = c(x)\cdot\varepsilon^\beta \int\limits_{a(x)\cdot\varepsilon<|y|<\mu} |y|^{-\beta-1} |\varphi(x-y) - \varphi(x)|\, dy,$$

$$I_3 = c(x)\cdot\varepsilon^\beta \int\limits_{|y|>\mu} |y|^{-\beta-1} |\varphi(x-y) - \varphi(x)|\, dy.$$

Introduce auxiliary function

$$G(\rho) = G_x(\rho) = \int\limits_{|y|\le\rho} |\varphi(x-y) - \varphi(x)|\, dy. \tag{15}$$

Since x is the Lebesgue point, then according to (4), we have

$$\lim_{\rho\to 0} \rho^{-n} G(\rho) = 0.$$

So, setting arbitrary number $h > 0$, we can determine $\mu > 0$, so that $G(\rho) \le h\rho^n$, for all $\rho \le \mu$. Now estimate I_1, I_2, passing to spherical coordinates $|y| = \rho$, $\frac{y}{|y|} = y'$, $dy = \rho^{n-1} d r d y'$, y' is the point on a unit sphere Σ, dy' is the element of the surface area of a unit sphere. We have

$$I_1 \le \frac{c(x)}{\varepsilon^\alpha} \int\limits_{\Sigma} dy' \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha+n-2} |\varphi(x-y) - \varphi(x)|\, d\rho =$$

$$= \frac{c(x)}{\varepsilon^\alpha} \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-1} d\rho \int\limits_{\Sigma} \rho^{n-1} |\varphi(x-y) - \varphi(x)|\, dy' =$$

$$= \frac{c(x)}{\varepsilon^\alpha} \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-1} d\left(\int\limits_{0}^{\rho} \frac{d\rho}{r^{-n+1}} \int\limits_{\Sigma} |\varphi(x-y) - \varphi(x)|\, dy' \right) =$$

$$= \frac{c(x)}{\varepsilon^\alpha} \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-1} d\left(\int\limits_{|y|\le\rho} |\varphi(x-y) - \varphi(x)|\, dy \right) =$$

$$= \frac{c(x)}{\varepsilon^{\alpha}} \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-1}\, dG(\rho),$$

$$I_2 \le c(x)\cdot\varepsilon^{\beta} \int\limits_{\Sigma} dy' \int\limits_{a(x)\cdot\varepsilon}^{\mu} \rho^{-\beta+n-2} \left|\varphi(x-y)-\varphi(x)\right| d\rho =$$

$$= c(x)\cdot\varepsilon^{\beta} \int\limits_{a(x)\cdot\varepsilon}^{\mu} \frac{d\rho}{\rho^{\beta+1}} \int_{\Sigma} \rho^{n-1} \left|\varphi(x-y)-\varphi(x)\right| dy' =$$

$$= \int\limits_{a(x)\cdot\varepsilon}^{\mu} \frac{c(x)\,\varepsilon^{\beta}}{\rho^{\beta+1}}\, d\left(\int\limits_{0}^{\rho} \frac{d\rho}{r^{-n+1}} \int\limits_{\Sigma} \left|\varphi(x-y)-\varphi(x)\right| dy'\right) =$$

$$= \int\limits_{a(x)\cdot\varepsilon}^{\mu} \frac{c(x)\cdot\varepsilon^{\beta}}{\rho^{\beta+1}}\, d\left(\int\limits_{|y|\le\rho} \left|\varphi(x-y)-\varphi(x)\right| dy\right) =$$

$$= c(x)\cdot\varepsilon^{\beta} \int\limits_{a(x)\cdot\varepsilon}^{\mu} \rho^{-\beta-1}\, dG(\rho).$$

Thus, the following equalities are true

$$I_1 = c(x)\cdot\varepsilon^{-\alpha} \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-1}\, dG(\rho), \tag{16}$$

$$I_2 = c(x)\cdot\varepsilon^{\beta} \int\limits_{a(x)\cdot\varepsilon}^{\mu} \rho^{-\beta-1}\, dG(\rho), \tag{17}$$

where $G(\rho)$ is taken from (15). Integrating (16) and (17) by parts we determine

$$I_1 = \frac{c(x)}{\varepsilon^{\alpha}} \{\rho^{\alpha-1} G(\rho) \,\Big|_{\rho=0}^{\rho=a(x)\cdot\varepsilon} -$$

$$-(\alpha-1) \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha-2} G(\rho)\, d\rho\}, \tag{18}$$

$$I_2 = c(x)\cdot\varepsilon^{\beta} \{\rho^{-\beta-1}\cdot G(\rho) \,\Big|_{\rho=a(x)\cdot\varepsilon}^{\rho=\mu} +$$

$$+(\beta+1) \int\limits_{a(x)\cdot\varepsilon}^{\mu} \rho^{-\beta-2} G(\rho)\, d\rho\}. \tag{19}$$

By virtue of $G(\rho) \le h\cdot\rho^{n}$ we obtain $\rho^{\alpha-1} G(\rho)\Big|_{\rho=0} = 0$. Using $G(\rho) \le h\rho^{n}$ in (18), (19) considering that $a(x)\cdot\varepsilon < \mu$, we have

$$I_1 \le \frac{c(x)}{\varepsilon^{\alpha}} \{ \rho^{\alpha+n-1} h\Big|^{\rho=a(x)\cdot\varepsilon} -$$

$$-|\alpha-1|\cdot h \int\limits_{0}^{a(x)\cdot\varepsilon} \rho^{\alpha+n-2}\, d\rho \} \le$$

$$\le c(x)\cdot\varepsilon^{n-1} (a(x))^{\alpha+n-1} [1 + \frac{|\alpha-1|}{\alpha+n-1}]h \le C_2' h,$$

$$I_2 = c(x)\cdot\varepsilon^{\beta} \{\rho^{-\beta+n-1}\cdot h \,\Big|_{\rho=a(x)\cdot\varepsilon}^{\rho=\mu} +$$

$$+(\beta+1)h \int\limits_{a(x)\cdot\varepsilon}^{\mu} \rho^{-\beta+n-2}\, d\rho\} \le$$

$$\le c(x)\cdot h\{\varepsilon^{\beta}\mu^{-\beta+n-1} - \varepsilon^{n-1} (a(x))^{-\beta+n-1}\}\times$$

$$\times[1 + \frac{\beta+1}{\beta-n+1}] \le C_2'' h.$$

Assume that $p = 1$ (for $p > 1$, the Hölder's inequality should be used in the estimation of integral $\int\limits_{|y|>\mu} |y|^{-\beta-1} \left|\varphi(x-y)\right| dy$)

$$I_3 = c(x)\cdot\varepsilon^{\beta} \int\limits_{|y|>\mu} |y|^{-\beta-1} \left|\varphi(x-y)-\varphi(x)\right| dy \le$$

$$\le c(x)\cdot\varepsilon^{\beta}\{ \int\limits_{|y|>\mu} |y|^{-\beta-1} \left|\varphi(x-y)\right| dy +$$

$$+ \left|\varphi(x)\right| \int\limits_{|y|>\mu} |y|^{-\beta-1}\, dy \}\} \le$$

$$\le c(x)\,\varepsilon^{\beta} \left\{ \frac{1}{\mu^{\beta+1}} \left\|\varphi\right\|_{L_1} + \frac{mes\Sigma}{\beta+1-n} \frac{1}{\mu^{\beta+1-n}} \right\} \le C_2''' h. \tag{20}$$

Note that in (20) μ is fixed. Since $\varphi(x)$ is locally integrable, $\varphi(x)$ is finite almost everywhere. Therefore, for any Lebesgue point, except maybe a set of measure zero, $|\varphi(x)|$ is a finite value. But then the right-hand side in (20) tends to zero as $\varepsilon \to 0$, i.e. $I_3 \to 0$ as $\varepsilon \to 0$.

It remains to add that the intersection of the set of Lebesgue points with the set of points, where $a(x)$ and $c(x)$ are finite values, has full measure.

Finally, if the kernel $K_{\varepsilon}(x, y)$ is finite with respect to y, then in (3), at fixed x, only the local integrability of the function is used.

5 Conclusions

The importance of studying generalizations of non-convolutional integral operators is due to their application in the theory of fractional integro-differentiation. The results obtained in this paper are sufficient conditions for the kernel $\varepsilon \to 0$ in non-convolutional integral operators (3), that consider different behavior at $|y| \to 0$ and $|y| \to \infty$ depending on $\varepsilon \to 0$, and cover the situation in the particular case of convolutional and non-convolutional integral operators.

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