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Finite Difference Method for Pricing of Indonesian Option under a Mixed Fractional Brownian Motion

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Abstract This paper deals with an Indonesian option pricing using mixed fractional Brownian motion to model the underlying stock price. There have been researched on the Indonesian option pricing by using Brownian motion. Another research states that logarithmic returns of the Jakarta composite index have long-range dependence. Motivated by the fact that there is long-range dependence on logarithmic returns of Indonesian stock prices, we use mixed fractional Brownian motion to model on logarithmic returns of stock prices. The Indonesian option is different from other options in terms of its exercise time. The option can be exercised at maturity or at any time before maturity with profit less than ten percent of the strike price. Also, the option will be exercised automatically if the stock price hits a barrier price. Therefore, the mathematical model is unique, and we apply the method of the partial differential equation to study it. An implicit finite difference scheme has been developed to solve the partial differential equation that is used to obtain Indonesian option prices. We study the stability and convergence of the implicit finite difference scheme. We also present several examples of numerical solutions. Based on theoretical analysis and the numerical solutions, the scheme proposed in this paper is efficient and reliable.

Keywords Indonesian Option Pricing, Mixed Fractional Brownian Motion, Finite Difference

1 Introduction

The Jakarta Stock Exchange, currently called the Indonesia Stock Exchange after merging with the Surabaya Stock Exchange, launched an option on October 6, 2004. The option traded in Indonesia is different to the usual options. An Indonesia option [1] is an American option that is given a barrier, but the Indonesian option only has maximum gain of 10% of a strike price. The option price depends on the weighted moving average (WMA) price of the underlying stock price. The WMA price is a ratio of the total value of all transactions to the total volume of the stock traded in the last 30 minutes. Calculating the Indonesia option by using the WMA price is not easy due to model complexity. If the WMA price is calculated during the last 30 minutes, then the WMA price and the stock price do not differ in terms of value. This study assumed the WMA price is equal to the stock price.

In Indonesian options, if a stock price hits the barrier value, then the option will be exercised automatically with a gain of 10% of a strike price. On the contrary, if the stock price does not hit the barrier, then the option can be exercised any time before or at the maturity date. When the stock price does not hit the barrier, option buyers tend to wait until maturity. This is due to the fact that the barrier value is close enough to the strike price and the maximum duration of the contract is only 3 months. Therefore, we are interested in studying the pricing of Indonesian options that can be exercised at maturity or when the stock prices hit the barrier.

Gunardi et al. [2] introduced pricing of Indonesian options. The pricing of Indonesian options in [2, 3, 4] used Black-

Scholes and variance gamma models. The Black-Scholes model used geometric Brownian motion to model logarithmic returns of stock prices. This model assumes that logarithmic returns of stock prices ware normally and independent identically distributed (iid). However, empirical studies have shown that logarithmic returns of stock prices usually exhibit properties of self-similarity, heavy tails, and long-range dependence [5, 6, 7]. Even Cajueiro [5] and Fakhriyana [7] stated that returns of the Jakarta Composite Index have long-range dependence properties. In this situation, it is suitable to model the stock price using a fractional Brownian motion (FBM).

To use a FBM in option pricing, we must define a risk-neutral measure and the Itô formula, with analog in Brownian motion. Hu and Øksendal [8] contributed to finding the Itô formula that can be used in the FBM model. However, the determination of option prices still had an arbitrage opportunity. Cheridito [9] proposed a mixed fractional Brownian motion (MFBM) to reduce an arbitrage opportunity. In this paper, we employ the MFBM on the Indonesian option pricing to reduce the arbitrage opportunity.

In the stock market, there are many types of options traded. European and American options are standard or vanilla options. European options can be exercised at maturity, whereas American options can be exercised at any time during the contract. Pricing of European options using MFBM has been studied in [10, 11]. Chen et al. [12] investigated numerically pricing of American options under the generalization of MFBM. Options that have more complicated rules than vanilla options are called exotic options. Examples of exotic options are Asian options, rainbow options, currency options, barrier options, and also Indonesian options. Rao [13] and Zang et al. [14] discussed the pricing of Asian power options under MFBM. Wang [15] explored the pricing of Asian rainbow options under FBM. Currency options pricing under FBM and MFBM has been studied in [16, 17, 18]. Numerical solution of barrier options pricing under MFBM have been evaluated by Ballestra et al. [19].

Indonesian option is one type of barrier options. Because analytic solutions for barrier options are not easy to find [19], we determine Indonesian options using numerical solutions. One numerical solution that can be used is the finite difference method discussed in [20]. The purpose of this paper is to determine Indonesian option prices under the MFBM model using the finite difference method. In this article, we also show that the resulting finite difference scheme is stable and convergent.

2 An option pricing model by using MFBM

A mixed fractional Black Scholes market is a model consisting of two assets, one riskless asset (bank account) and one risky asset (stock). A bank account satisfies

$$dA_t = rA_t dt, \quad A_0 = 1,$$

where A_t denotes a bank account at time t, $t \in [0, T]$, with an interest rate r. Meanwhile, a stock price is modeled by using an MFBM defined in Definition A.2 (Appendix A). The stock price satisfies

$$dS_t = \mu S_t dt + \alpha\sigma S_t d\hat{B}_t + \beta\sigma S_t d\hat{B}_t^H, \quad S_0 > 0,$$

where S_t denotes a stock price at time t, $t \in [0, T]$, with an expected return μ and a volatility σ, $\hat{B}_t$ is a Brownian motion, $\hat{B}_t^H$ is an independent FBM of Hurst index H with respect to a probability measure $\hat{\mathbb{P}}^H$.

According to the fractional Girsanov theorem [21], it is known that there is a risk-neutral measure $\mathbb{P}^H$, so that if $\alpha\sigma\hat{B}_t + \beta\sigma\hat{B}_t^H = \alpha\sigma B_t + \beta\sigma B_t^H - \mu + r$ is

$$dS_t = rS_t dt + \alpha\sigma S_t dB_t + \beta\sigma S_t dB_t^H, \quad S_0 > 0. \tag{1}$$

Lemma 1. *The stochastic differential equation* (1) *admits a solution*

$$S_t = S_0 \exp\left(rt - \tfrac{1}{2}(\alpha\sigma)^2 t - \tfrac{1}{2}(\beta\sigma)^2 t^{2H} + \alpha\sigma B_t + \beta\sigma B_t^H\right). \tag{2}$$

In mathematical finance, the Black-Scholes equation is a partial differential equation (PDE) which is used to determine the price of an option based on the Black-Scholes model. The Black-Scholes type differential equation based on an MFBM is constructed in the following theorem.

Theorem 2. *Let $V(t, S)$ be an option value that depends on a time t and a stock price S. Then, under an MFBM model, $V(t, S)$ satisfies*

$$\begin{aligned}\frac{\partial V}{\partial t} &+ rS\frac{\partial V}{\partial S} + \frac{1}{2}(\alpha\sigma S)^2\frac{\partial^2 V}{\partial S^2} \\ &+ (\beta\sigma S)^2 H t^{2H-1}\frac{\partial^2 V}{\partial S^2} - rV = 0.\end{aligned} \tag{3}$$

3 A Finite Difference Method for Indonesian option pricing

An Indonesian option is an option that can be exercised at maturity or at any time before maturity but the profit does not exceed 10 percent of the strike price. The option will be exercised automatically if the stock price hits a barrier price. The barrier price in an Indonesian option is 110% of the strike price for a call option and 90% of the strike price for a put option. Because the benefits of an Indonesian option is very small, more option contract holders often choose to exercise their contracts at maturity. In other words, an Indonesian option is an option that can be exercised at maturity or when the stock hits the barrier price.

Let L is a barrier of an Indonesian option and t_L is the first time of the stock price hitting the barrier;

$$t_L = \min\{t | t \in [0, T], S_t \geq L\}. \tag{4}$$

An Indonesian call option with a strike price K can be exercised at maturity T or until the stock price of S_t hits the barrier at $L = 1.1K$. The payoff function at time T of the call option can be expressed as follows :

$$f(S_T) = \begin{cases} S_T - K & \text{if } t_L > T, \\ (L-K)e^{r(T-t_L)} & \text{if } t_L \leq T. \end{cases} \tag{5}$$

Similarly, the payoff function at time T of an Indonesian put option with barrier price $L = 0.9K$ can be expressed as follows :

$$f(S_T) = \begin{cases} K - S_T & \text{if } t_L > T, \\ (K-L)e^{r(T-t_L)} & \text{if } t_L \leq T. \end{cases} \tag{6}$$

The partial differential equation used in the Indonesian option pricing is a PDE with a final time condition. Because finite difference methods usually use an initial time condition, we make changes on variable τ i.e. $\tau = T - t$. Under this transformation, PDE (3) becomes,

$$\frac{\partial V}{\partial \tau} - rS\frac{\partial V}{\partial S} - \frac{1}{2}(\alpha\sigma S)^2\frac{\partial^2 V}{\partial S^2} - (\beta\sigma S)^2 H(T-\tau)^{2H-1}\frac{\partial^2 V}{\partial S^2} + rV = 0. \tag{7}$$

We must set up a discrete grid in this case with respect to stock prices and time to solve the PDE by finite difference methods. Suppose S_{max} is a suitably large stock price and in this case $S_{max} = L$. We need S_{max} since the domain for the PDE is unbounded with respect to stock prices, but we must bound it in some ways for computing purposes. The grid consists of points (τ_k, S_j) such that $S_j = j\Delta S$ and $\tau_k = k\Delta\tau$ with $j = 0, 1, \ldots, M$ and $k = 0, 1, \ldots, N$.

Using Taylor series expansion, we have

$$\frac{V_j^k - V_j^{k-1}}{\Delta\tau} = \frac{\partial V}{\partial \tau} + \mathcal{O}(\Delta\tau), \tag{8}$$

$$\frac{V_{j+1}^k - V_{j-1}^k}{2\Delta S} = \frac{\partial V}{\partial S} + \mathcal{O}\left((\Delta S)^2\right), \tag{9}$$

and

$$\frac{V_{j+1}^k - 2V_j^k + V_{j-1}^k}{(\Delta S)^2} = \frac{\partial^2 V}{\partial S^2} + \mathcal{O}\left((\Delta S)^2\right). \tag{10}$$

Substitution of (8), (9) and (10) in (7) yields

$$\frac{V_j^k - V_j^{k-1}}{\Delta\tau} - rj\Delta S\frac{V_{j+1}^k - V_{j-1}^k}{2\Delta S} - \frac{(\alpha\sigma)^2}{2}(j\Delta S)^2\frac{V_{j+1}^k - 2V_j^k + V_{j-1}^k}{(\Delta S)^2} - (\beta\sigma)^2(j\Delta S)^2 H(T-k\Delta\tau)^{2H-1}\frac{V_{j+1}^k - 2V_j^k + V_{j-1}^k}{(\Delta S)^2} + rV_j^k = 0, \tag{11}$$

where the local truncation error is $\mathcal{O}\left(\Delta\tau + (\Delta S)^2\right)$. Rewriting (11), we get an implicit scheme as follows

$$V_j^{k-1} = a_j V_{j-1}^k + b_j V_j^k + c_j V_{j+1}^k, \tag{12}$$

where

$$a_j = \left(-\tfrac{1}{2}(\alpha\sigma j)^2 - (\beta\sigma j)^2 H(T-k\Delta\tau)^{2H-1} + \tfrac{1}{2}rj\right)\Delta\tau, \tag{13}$$

$$b_j = \left(1 + \left((\alpha\sigma j)^2 + 2(\beta\sigma j)^2 H(T-k\Delta\tau)^{2H-1} + r\right)\Delta\tau\right), \tag{14}$$

$$c_j = \left(-\tfrac{1}{2}(\alpha\sigma j)^2 - (\beta\sigma j)^2 H(T-k\Delta\tau)^{2H-1} - \tfrac{1}{2}rj,\right)\Delta\tau. \tag{15}$$

Using (4) and (5), we can write an initial condition of the Indonesian call option as follows:

$$V_j^0 = \begin{cases} j\Delta S - K & \text{if } L > j\Delta S, \\ L - K & \text{if } L \leq j\Delta S, \end{cases} \tag{16}$$

and boundary conditions of the call option as follows:

$$V_0^k = 0 \quad \text{and} \quad V_M^k = (L-K)e^{-rk\Delta\tau}. \tag{17}$$

In another case, using (4) and (6), we get an initial condition and boundary conditions of the Indonesian put option shown below respectively:

$$V_j^0 = \begin{cases} K - j\Delta S & \text{if } L < j\Delta S, \\ K - L & \text{if } L \geq j\Delta S, \end{cases}$$

and

$$V_0^k = 0 \quad \text{and} \quad V_M^k = (K-L)e^{-rk\Delta\tau}.$$

We analyze the stability and convergence of the implicit finite difference scheme using Fourier analysis. Firstly, we discuss the stability of the implicit finite difference scheme. Let V_j^k be difference solution of (12) and U_j^k be another approximate solution of (12), we define a roundoff error $\varepsilon_j^k = V_j^k - U_j^k$. Next, we obtain a following roundoff error equation

$$\varepsilon_j^{k-1} = a_j\varepsilon_{j-1}^k + b_j\varepsilon_j^k + c_j\varepsilon_{j+1}^k. \tag{18}$$

Furthermore, we define a grid function as follows:

$$\varepsilon^k(S) = \begin{cases} \varepsilon_j^k & \text{if } S_j - \frac{\Delta S}{2} < S \leq S_j + \frac{\Delta S}{2}, j = 1, ..., M-1, \\ 0 & \text{if } 0 \leq S \leq \frac{\Delta S}{2} \text{ or } S_{max} - \frac{\Delta S}{2} < S \leq S_{max}. \end{cases}$$

The grid function can be expanded in a Fourier series below:

$$\varepsilon^k(S) = \sum_{l=-\infty}^{\infty} \xi^k(l)\exp\left(\frac{i2\pi lS}{S_{max}}\right), \quad k = 1, 2, \ldots, N,$$

where

$$\xi^k(l) = \frac{1}{S_{max}}\int_0^{S_{max}} \varepsilon^k(S)\exp\left(\frac{-i2\pi lS}{S_{max}}\right)dS.$$

Moreover, we let

$$\boldsymbol{\varepsilon}^k = \left[\varepsilon_1^k, \varepsilon_2^k, \ldots, \varepsilon_{N-1}^k\right]^T.$$

And we introduce a norm,

$$\left\|\boldsymbol{\varepsilon}^k\right\|_2 = \left(\sum_{j=1}^{M-1}\left|\varepsilon_j^k\right|^2\Delta S\right)^{\frac{1}{2}} = \left(\int_0^{S_{max}}\left|\varepsilon^k(S)\right|^2 dS\right)^{\frac{1}{2}}.$$

Further, by using Parseval equality,

$$\int_0^{S_{max}}\left|\varepsilon^k(S)\right|^2 dS = \sum_{l=-\infty}^{\infty}\left|\xi^k(l)\right|^2,$$

we obtain

$$\left\|\varepsilon^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} \left|\xi^k(l)\right|^2.$$

At the moment, we assume that the solution of equation (18) has the following form

$$\varepsilon_j^k = \xi^k e^{i\omega j\Delta S}, \tag{19}$$

where $\omega = \frac{2\pi l}{S_{max}}$ and $i = \sqrt{-1}$. Substituting (19) into (18), we obtain

$$\begin{aligned}\xi^{k-1}e^{i\omega j\Delta S} &= a_j\xi^k e^{i\omega(j-1)\Delta S} + b_j\xi^k e^{i\omega j\Delta S} + c_j\xi^k e^{i\omega(j+1)\Delta S}\\ &= \xi^k e^{i\omega j\Delta S}\left(a_j e^{-i\omega\Delta S} + b_j + c_j e^{i\omega\Delta S}\right). \end{aligned}\tag{20}$$

Equation (20) can be rewritten as follows,

$$\xi^{k-1} = \xi^k\left(a_j e^{-i\omega\Delta S} + b_j + c_j e^{i\omega\Delta S}\right), \tag{21}$$

$$\xi^{k-1} = \xi^k\vartheta_j, \tag{22}$$

where

$$\vartheta_j = a_j e^{-i\omega\Delta S} + b_j + c_j e^{i\omega\Delta S}. \tag{23}$$

By substituting (13), (14) and (15) into (23), we obtain

$$\begin{aligned}\vartheta_j = &\left(-(\alpha\sigma j)^2 - 2(\beta\sigma j)^2 H(T-k\Delta\tau)^{2H-1}\right)\Delta\tau\cos(\omega\Delta S)\\ &+ \left((\alpha\sigma j)^2 + 2(\beta\sigma j)^2 H(T-k\Delta\tau)^{2H-1} + r\right)\Delta\tau\\ &- rji\Delta\tau\sin(\omega\Delta S) + 1. \end{aligned}\tag{24}$$

Proposition 3. *If $\xi^k, k\in\mathbb{N}$, is a solution of* (21)*, then $|\xi^k| \le |\xi^0|$.*

Hence by (19) and Proposition 3, we have the following theorem.

Theorem 4. *The difference scheme* (12) *is unconditionally stable.*

Now we analyze the convergence of implicit finite difference scheme. Let $V(\tau_k, S_j)$ is exact solution of (7) at a point (τ_k, S_j) and

$$\begin{aligned}R_j^k = &\frac{V(\tau_k,S_j)-V(\tau_{k-1},S_j)}{\Delta\tau} - rj\Delta S\frac{V(\tau_k,S_{j+1})-V(\tau_k,S_{j-1})}{2\Delta S}\\ &-\frac{1}{2}(\alpha\sigma)^2(j\Delta S)^2\frac{V(\tau_k,S_{j+1})-2V(\tau_k,S_j)+V(\tau_k,S_{j-1})}{(\Delta S)^2}\\ &-\left((\beta\sigma)^2(j\Delta S)^2H(T-k\Delta\tau)^{2H-1}\right)\\ &\quad\times\frac{V(\tau_k,S_{j+1})-2V(\tau_k,S_j)+V(\tau_k,S_{j-1})}{(\Delta S)^2}\\ &+rV(\tau_k,S_j), \end{aligned}\tag{25}$$

where $k = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M-1$. Consequently, there is a positive constant $C_1^{k,j}$, so as

$$\left|R_j^k\right| \le C_1^{k,j}\left(\Delta\tau + (\Delta S)^2\right),$$

then, we have

$$\left|R_j^k\right| \le C_1\left(\Delta\tau + (\Delta S)^2\right), \tag{26}$$

where

$$C_1 = \max\left\{C_1^{k,j}\,\middle|\, k = 1,2,\ldots,N; j = 1,2,\ldots,M-1\right\}.$$

From (12), (13), (14), (15) and definition R_j^k in (25), we have

$$\begin{aligned}V(\tau_{k-1}, S_j) = &a_jV(\tau_k, S_{j-1}) + b_jV(\tau_k, S_j)\\ &+ c_jV(\tau_k, S_{j+1}) - \Delta\tau R_j^k. \end{aligned}\tag{27}$$

By subtracting (12) from (27), we obtain

$$\epsilon_j^{k-1} = a_j\epsilon_{j-1}^k + b_j\epsilon_j^k + c_j\epsilon_{j+1}^k - \Delta\tau R_j^k, \tag{28}$$

where an error $\epsilon_j^k = V(\tau_k, S_j) - V_j^k$. The error equation satisfies a boundary conditions,

$$\epsilon_0^k = \epsilon_M^k = 0, \quad k = 1, 2, ..., N,$$

and an initial condition,

$$\epsilon_j^0 = 0, \quad j = 1, 2, ..., M. \tag{29}$$

Next, we define the following grid functions,

$$\epsilon^k(S) = \begin{cases}\epsilon_j^k & \text{if } S_j - \frac{\Delta S}{2} < S \le S_j + \frac{\Delta S}{2}, j = 1, ..., M-1,\\ 0 & \text{if } 0 \le S \le \frac{\Delta S}{2} \text{ or } S_{max} - \frac{\Delta S}{2} < S \le S_{max},\end{cases}$$

and

$$R^k(S) = \begin{cases}R_j^k & \text{if } S_j - \frac{\Delta S}{2} < S \le S_j + \frac{\Delta S}{2}, j = 1, ..., M-1,\\ 0 & \text{if } 0 \le S \le \frac{\Delta S}{2} \text{ or } S_{max} - \frac{\Delta S}{2} < S \le S_{max}.\end{cases}$$

The grid functions can be expanded in a Fourier series respectively as follows

$$\epsilon^k(S) = \sum_{l=-\infty}^{\infty}\varrho^k(l)\exp\left(\frac{i2\pi lS}{S_{max}}\right), \quad k = 1, 2, \ldots, N,$$

and

$$R^k(S) = \sum_{l=-\infty}^{\infty}\rho^k(l)\exp\left(\frac{i2\pi lS}{S_{max}}\right), \quad k = 1, 2, \ldots, N,$$

where

$$\varrho^k(l) = \frac{1}{S_{max}}\int_0^{S_{max}}\epsilon^k(S)\exp\left(\frac{-i2\pi lS}{S_{max}}\right)dS,$$

and

$$\rho^k(l) = \frac{1}{S_{max}}\int_0^{S_{max}}R^k(S)\exp\left(\frac{-i2\pi lS}{S_{max}}\right)dS.$$

Thus, we let

$$\boldsymbol{\epsilon}^k = \left[\epsilon_1^k, \epsilon_2^k, \ldots, \epsilon_{N-1}^k\right]^T$$

and

$$\boldsymbol{R}^k = \left[R_1^k, R_2^k, \ldots, R_{N-1}^k\right]^T,$$

and we define their corresponding norms

$$\left\|\boldsymbol{\epsilon}^k\right\|_2 = \left(\sum_{j=1}^{M-1}\left|\epsilon_j^k\right|^2\Delta S\right)^{\frac{1}{2}} = \left(\int_0^{S_{max}}\left|\epsilon^k(S)\right|^2 dS\right)^{\frac{1}{2}},$$

and

$$\left\|\boldsymbol{R}^k\right\|_2=\left(\sum_{j=1}^{M-1}\left|R_j^k\right|^2\Delta S\right)^{\frac{1}{2}}=\left(\int_0^{S_{max}}\left|R^k(S)\right|^2dS\right)^{\frac{1}{2}}, \tag{30}$$

respectively. By using Parseval equality, we get

$$\int_0^{S_{max}}\left|\epsilon^k(S)\right|^2dS=\sum_{l=-\infty}^{\infty}\left|\varrho^k(l)\right|^2$$

and

$$\int_0^{S_{max}}\left|R^k(S)\right|^2dS=\sum_{l=-\infty}^{\infty}\left|\rho^k(l)\right|^2,$$

respectively. As a consequence, we can show that

$$\left\|\boldsymbol{\epsilon}^k\right\|_2^2=\sum_{l=-\infty}^{\infty}\left|\varrho^k(l)\right|^2 \tag{31}$$

and

$$\left\|\boldsymbol{R}^k\right\|_2^2=\sum_{l=-\infty}^{\infty}\left|\rho^k(l)\right|^2. \tag{32}$$

Further, we assume that the solution of (28) has the following form

$$\epsilon_j^k=\varrho^k e^{i\omega j\Delta S} \tag{33}$$

and

$$R_j^k=\rho^k e^{i\omega j\Delta S}. \tag{34}$$

Substituting (33) and (34) into (28), we obtain

$$\varrho^{k-1}e^{i\omega j\Delta S}=e^{i\omega j\Delta S}\left(\varrho^k\left(a_je^{-i\omega\Delta S}+b_j+c_je^{i\omega\Delta S}\right)-\Delta\tau\rho^k\right). \tag{35}$$

Equation (35) can be simply rewritten as follows

$$\varrho^{k-1}=\varrho^k\left(a_je^{-i\omega\Delta S}+b_j+c_je^{i\omega\Delta S}\right)-\Delta\tau\rho^k. \tag{36}$$

By using equations (13), (14), (15) and (36), we obtain

$$\begin{aligned}\varrho^{k-1}=&\left[\left(-(\alpha\sigma j)^2-2(\beta\sigma j)^2H(T-k\Delta\tau)^{2H-1}\right)\Delta\tau\cos(\omega\Delta S)\right.\\&+\left((\alpha\sigma j)^2+2(\beta\sigma j)^2H(T-k\Delta\tau)^{2H-1}+r\right)\Delta\tau\\&\left.-rji\Delta\tau\sin\left(\omega\Delta S\right)+1\right]\varrho^k-\Delta\tau\rho^k.\end{aligned} \tag{37}$$

Equation (37) can be effectively expressed as follows

$$\varrho^k=\tfrac{1}{\vartheta_j}\varrho^{k-1}+\tfrac{1}{\vartheta_j}\Delta\tau\rho^k, \tag{38}$$

where ϑ_j is defined in (24).

Proposition 5. *Assuming that $\varrho^k(k=1,2,\dots,N)$ is a solution of* (37)*, then there exist a positive constant C_2, so that*

$$|\varrho^k|\leq C_2k\Delta\tau|\rho^1|.$$

The following theorem gives convergence of the different scheme (12).

Theorem 6. *The difference scheme* (12) *is L_2-convergent, and the convergence order is $\mathcal{O}(\Delta\tau+(\Delta S)^2)$.*

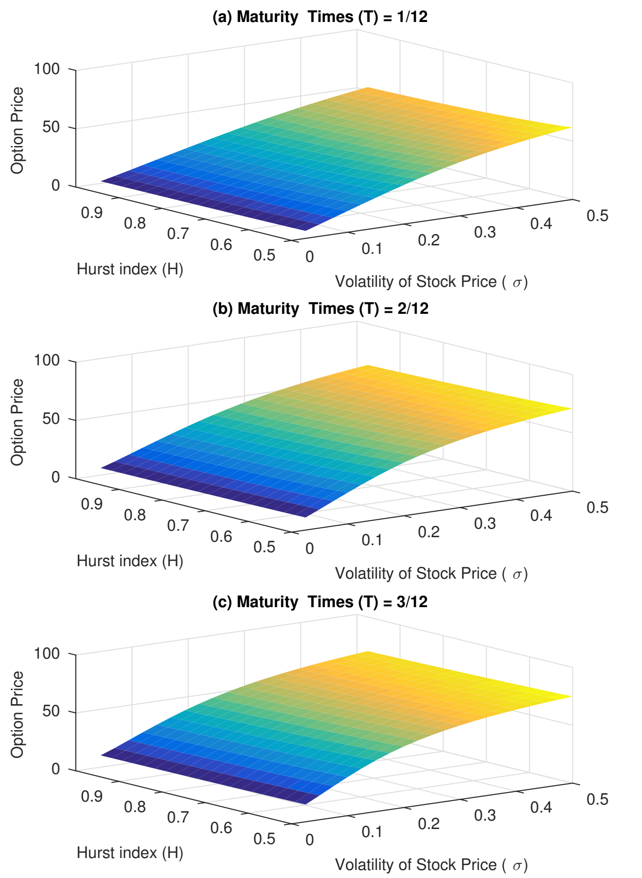

Figure 1. Indonesian option prices on H and σ for values $T=\frac{1}{12}, T=\frac{2}{12}$ and $T=\frac{3}{12}$.

4 Numerical examples and discussions

An Indonesian option pricing based on an MFBM has been studied. An implicit difference scheme of (7) is given in (12) and initial and boundary conditions of an Indonesian call option is given in (16) and (17), respectively. We provide several numerical results that illustrate the stability and convergence of the finite difference method in calculating an Indonesian call option price using Matlab in this section. In Examples 1, 2 and 3, we show that the scheme is stable. We also show that the scheme is convergent in Example 4. Furthermore, Example 5 compares the option price generated by the scheme with the exact solution in [2] when $\alpha=0$, $\beta=1$, $H=\frac{1}{2}$.

Example 1. An Indonesian call option pricing model is based on (12) where $\alpha=\beta=1$, an initial condition (16) and boundary conditions (17) under the following parameters,

$$\Delta S=1, \Delta\tau=0.0001, r=0.05, S_0=1000, K=1000,$$

and various values of parameters,

$$H\in(0.5,1),\ \sigma\in(0,0.5),\ T\in\left\{\tfrac{1}{12},\tfrac{2}{12},\tfrac{3}{12}\right\}$$

Figure 1 exhibits the price surface of an Indonesian call option with a change of the Hurst index (H) and a change of

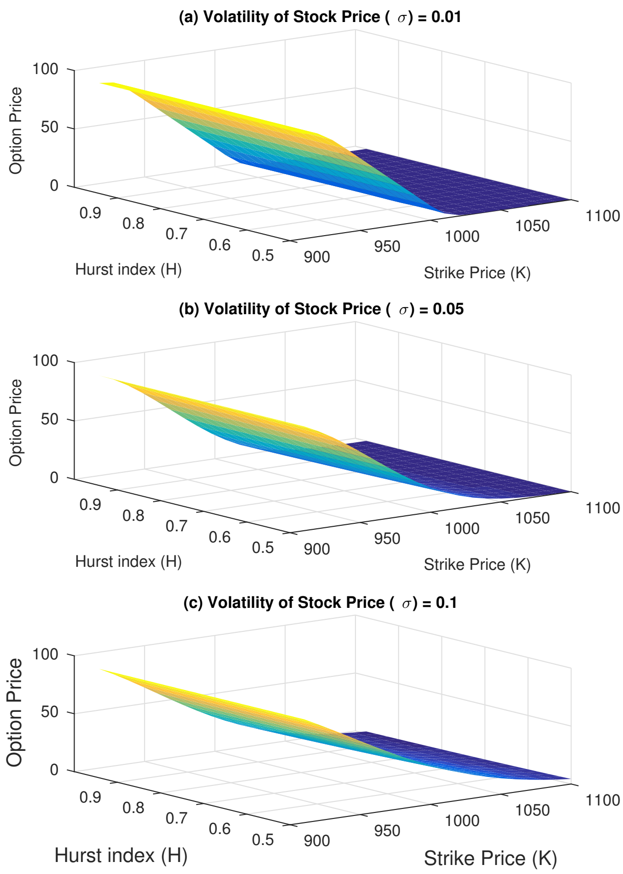

Figure 2. Indonesian option prices on H and K for values $\sigma = 0.01$, $\sigma = 0.05$ and $\sigma = 0.1$.

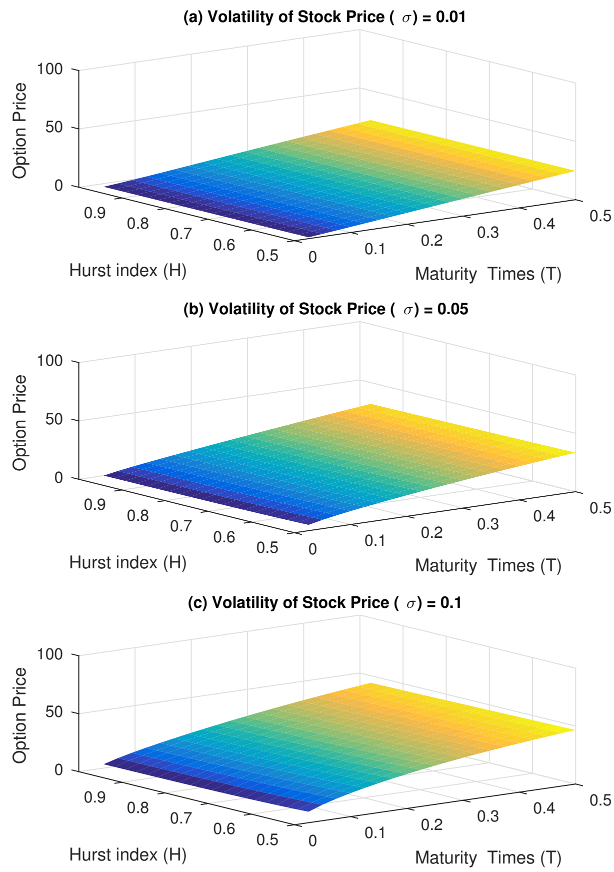

Figure 3. Indonesian option prices with H and T for values of $\sigma = 0.01$, $\sigma = 0.05$ and $\sigma = 0.1$

stock price volatility (σ) for difference maturity time (T). The Hurst Index, stock price volatility and maturity time affect option prices. As the Hurst index decreases and the stock price volatility and maturity time increase, we see that the price of Indonesian options increase.

Example 2. Consider an Indonesian call option pricing at (12), (16) and (17) with $\alpha = \beta = 1$ and parameters,

$$\Delta S = 1, \Delta\tau = 0.0001, r = 0.05, S_0 = 1000, T = \tfrac{3}{12},$$

and various values of parameters,

$$H \in (0.5, 1),\ K \in (900, 1100),\ \sigma \in \{0.01, 0.05, 0.1\}$$

Figure 2 shows the price surface of an Indonesian call option with a change of Hurst index (H) and a change of strike price (K) for various volatility values of the stock price (σ). As the stock price volatility increases, the Hurst index and strike price decrease, we see that the price of Indonesian options increase.

Example 3. Consider an Indonesian call option pricing problem (12), (16) and (17) with $\alpha = \beta = 1$ and parameters,

$$\Delta S = 1, \Delta t = 0.0001, r = 0.05, S_0 = 1000, K = 1000,$$

and various values of parameters

$$H \in (0.5, 1),\ T \in (0, 0.5),\ \sigma \in \{0.01, 0.05, 0.1\}.$$

Figure 3 shows the price surface of an Indonesian call option with a change of the Hurst index (H) and a change of maturity time (T) for various values of stock price volatility (σ). Similar to the result obtained in Example 1, we see that the price of Indonesian options increase when the stock price volatility and maturity time increase while the Hurst index decreases.

Example 4. Consider an Indonesian call option pricing at (12), (16) and (17) with $\alpha = \beta = 1$ and parameters,

$$r = 0.05, \sigma = 0.1, T = 0.25, S_0 = 1000, K = 1000, H = 0.7.$$

This example will show the convergence of the scheme (12). The convergence is demonstrated by the difference between consecutive approximation processes in Table 1. The numerical results from Table 1 confirm the results of the theoretical analysis (B.8) in Theorem 6.

Example 5. Let Indonesian call option pricing at (12), (16)

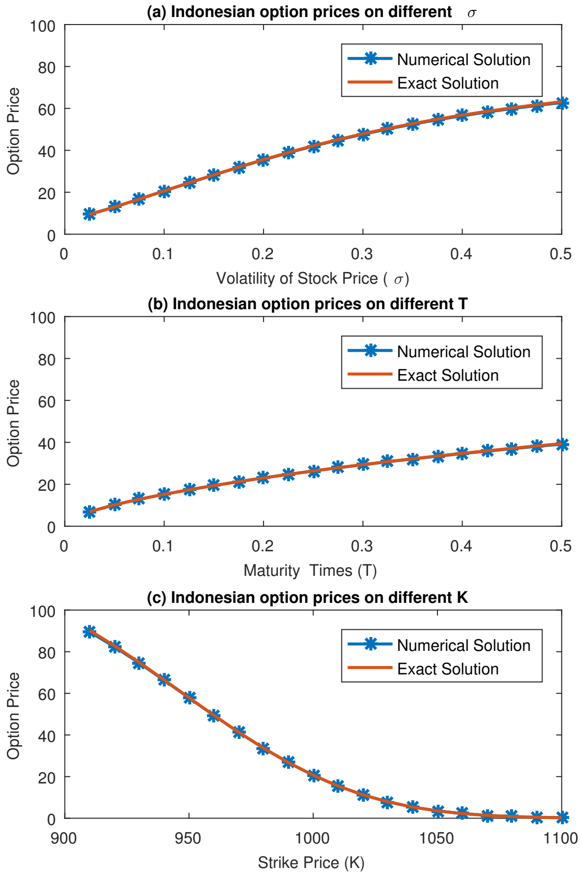

Figure 4. The price of Indonesian options uses the exact and numerical solution for $H = \frac{1}{2}$.

and (17) with $\alpha = 0, \beta = 1, H = \frac{1}{2}$ and parameters,

$$\Delta S = 1, \Delta\tau = 0.0001, r = 0.05, \sigma = 0.1, T = \frac{2}{12},$$
$$S_0 = 1000, K = 1000.$$

Equation (7) with $\alpha = 0, \beta = 1$ and $H = \frac{1}{2}$ is a stock price model under a Brownian motion. Figure 4 shows the comparison of numerical and exact solutions of Indonesian option prices for stock prices modeled by Brownian motion. The exact solution for determining Indonesian option prices is obtained by a formula in [2]. Whereas, the numerical solution is obtained by the implicit finite difference method (12) with $\alpha = 0, \beta = 1$ and $H = \frac{1}{2}$.

Moreover, if we set $\alpha = 1$ and $\beta = 0$ in (12), then we get a similar trend of option prices as shown in Figure 4. As can be seen, both solutions overlap each other. In other words, the numerical solution is similar to the analytical solution.

In Examples 1, 2, 3 and 4, we choose small ΔS and $\Delta\tau$ values. The implicit finite difference scheme can still produce Indonesian option prices using these values. In other words, even though the values chosen are very small, it still produces option prices. We need to mention here that the calculation process takes a longer time. In addition, we can see that trends

Table 1. Convergence results of the scheme (12)

ΔS	$\Delta\tau$	Value	Difference	Ratio
10.00000	0.001000000	30.7251		
5.00000	0.000500000	30.8103	0.0852	
2.50000	0.000250000	30.8352	0.0249	3.4217
1.25000	0.000125000	30.8433	0.0081	3.0741
0.62500	0.000062500	30.8463	0.0030	2.7000
0.31250	0.000031250	30.8475	0.0012	2.5000
0.15625	0.000015625	30.8480	0.0005	2.4000

and visible shapes of option price solutions of the proposed scheme are similar to the option price solutions in [2] (Example 5). Therefore, it can be concluded that the implicit finite difference scheme used to determine Indonesian option prices is stable and convergent.

5 Conclusions

In this paper, we apply an implicit finite difference method to solve Indonesian option pricing problems. Given that Jakarta Composite Index is long-range dependent, an MFBM is used to model the stock returns. The implicit finite difference scheme has been developed to solve a partial differential equation that is used to determine Indonesian option prices. We study the stability and convergence of the implicit finite difference scheme for Indonesian option pricing. We also present several examples of numerical solutions for Indonesian option pricing. Based on theoretical analysis and numerical solutions, the scheme proposed in this paper is efficient and reliable.

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Appendix

A Review of a mixed fractional Brownian motion

In Appendix A, we recall several definitions and lemma which are used in this paper.

Definition A.1. [21] Let $H \in (0, 1)$ be given. A fractional Brownian motion $B^H = (B_t^H)_{t\geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E\left[B_t^H, B_u^H\right] = \frac{1}{2}\left(|t|^{2H} + |u|^{2H} - |t-u|^{2H}\right),$$

for all $t, u > 0$.

A FBM is a generalization of the standard Brownian motion. To see this take $H = \frac{1}{2}$ in the Definition A.1. Standard Brownian motion has been employed to model stock prices in the Black-Scholes model. However, it cannot model time series with long-range dependence (long memory). It is known that a FBM is able to model time series with long-range dependence for $\frac{1}{2} < H < 1$.

One main problem of using a FBM in financial models is that it exhibits arbitrage which is usually excluded in the modeling. To avoid the possibility of arbitrage, Cheridito [22] introduced an MFBM.

Definition A.2. [22, 23] A mixed fractional Brownian motion of parameters α, β and H is a process $M^H = \left(M_t^{H,\alpha,\beta}\right)_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^H)$ by

$$M_t^{H,\alpha,\beta} = \alpha B_t + \beta B_t^H, \quad t \geq 0,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion and $(B_t^H)_{t\geq 0}$ is an independent FBM of Hurst index H.

We rewrite the following lemma which is derived from the Ito formula [21, 24] and properties of an MFBM. The lemma will be used later in option pricing based on stock price modeled by an MFBM.

Lemma A.3. *[25] Let $f = f(t, S_t)$ is a differentiable function. Let $(S_t)_{t\geq 0}$ be a stochastic process given by*

$$dS_t = \mu S_t dt + \sigma_1 S_t dB_t + \sigma_2 S_t dB_t^H,$$

where B_t is a Brownian motion, B_t^H is a FBM, and assume that B_t and B_t^H are independent, then we have

$$df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{\sigma_1^2 S_t^2}{2}\frac{\partial^2 f}{\partial S_t^2} + H\sigma_2^2 S_t^2 t^{2H-1}\frac{\partial^2 f}{\partial S_t^2}\right) dt + \sigma_1 S_t \frac{\partial f}{\partial S_t} dB_t + \sigma_2 S_t \frac{\partial f}{\partial S_t} dB_t^H.$$

B Proofs

Proof of Lemma 1

Proof. Using Lemma A.3 with $\mu = r$, $\sigma_1 = \alpha\sigma$ and $\sigma_2 = \beta\sigma$ and taking $f(S_t) = \ln(S_t)$, be obtained:

$$d\ln(S_t) = \left(r - \tfrac{1}{2}(\alpha\sigma)^2 - (\beta\sigma)^2 H t^{2H-1}\right) dt + \alpha\sigma dB_t + \beta\sigma dB_t^H,$$

and hence,

$$\ln\left(\frac{S_t}{S_0}\right) = rt - \tfrac{1}{2}(\alpha\sigma)^2 t - \tfrac{1}{2}(\beta\sigma)^2 t^{2H} + \alpha\sigma B_t + \beta\sigma B_t^H,$$

which can be related as (2). $\square$

Proof of Theorem 2

Proof. To prove the statement, a portfolio consisting an option $V(t,S)$ and a quantity q of stock, will be first set, i.e.

$$\Pi = V(t,S) - qS. \tag{B.1}$$

Thus, changes in portfolio value in a short time can be written as

$$d\Pi = dV(t,S) - qdS. \tag{B.2}$$

Now, applying Lemma A.3 and $f(t, S_t) = V(t,S)$, we obtain

$$dV = \left(\tfrac{\partial V}{\partial t} + rS\tfrac{\partial V}{\partial S} + \tfrac{1}{2}(\alpha\sigma S)^2\tfrac{\partial^2 V}{\partial S^2} + (\beta\sigma S)^2 H t^{2H-1}\tfrac{\partial^2 V}{\partial S^2}\right) dt + \alpha\sigma S\tfrac{\partial V}{\partial S} dB_t + \beta\sigma S\tfrac{\partial V}{\partial S} dB_t^H. \tag{B.3}$$

Substituting (B.3) and (1) into (B.2), we have

$$d\Pi = \Big(\tfrac{\partial V}{\partial t} + rS\left(\tfrac{\partial V}{\partial S} - q\right) + \tfrac{1}{2}(\alpha\sigma S)^2\tfrac{\partial^2 V}{\partial S^2} + (\beta\sigma S)^2 H t^{2H-1}\tfrac{\partial^2 V}{\partial S^2}\Big) dt + \alpha\sigma S\left(\tfrac{\partial V}{\partial S} - q\right) dB_t + \beta\sigma S\left(\tfrac{\partial V}{\partial S} - q\right) dB_t^H.$$

Further, we choose $q = \frac{\partial V}{\partial S}$ to eliminate the random noise. Then we get

$$d\Pi = \left(\tfrac{\partial V}{\partial t} + \tfrac{1}{2}(\alpha\sigma S)^2\tfrac{\partial^2 V}{\partial S^2} + (\beta\sigma S)^2 H t^{2H-1}\tfrac{\partial^2 V}{\partial S^2}\right) dt. \tag{B.4}$$

On the other hand, the portfolio becomes riskless if the portfolio yield is only determined by the risk-free interest rate r, which satisfies $d\Pi = r\Pi dt$. From (B.1), we have

$$r\Pi dt = r(V - qS)dt = \left(rV - rS\tfrac{\partial V}{\partial S}\right) dt, \tag{B.5}$$

and also from (B.4) and (B.5), we get

$$\left(\tfrac{\partial V}{\partial t} + \tfrac{1}{2}(\alpha\sigma S)^2\tfrac{\partial^2 V}{\partial S^2} + (\beta\sigma S)^2 H t^{2H-1}\tfrac{\partial^2 V}{\partial S^2}\right) dt = \left(rV - rS\tfrac{\partial V}{\partial S}\right) dt,$$

which yields (3). $\square$

Proof of Proposition 3

Proof. Since $|\vartheta_j| \geq 1$ and using (22) for $k = 1$, we have

$$\left|\xi^1\right| = \frac{1}{|\vartheta_j|}\left|\xi^0\right| \leq \left|\xi^0\right|.$$

If $|\xi^{k-1}| \leq |\xi^0|$, then using (22), we obtain

$$\left|\xi^k\right| = \frac{1}{|\vartheta_j|}\left|\xi^{k-1}\right| \leq \frac{1}{|\vartheta_j|}\left|\xi^0\right| \leq \left|\xi^0\right|.$$

This completes the proof. $\square$

Proof of Theorem 4

Proof. Using Proposition 3 and (19), we obtain

$$\left\|\boldsymbol{\varepsilon}^k\right\|_2 \leq \left\|\boldsymbol{\varepsilon}^0\right\|_2, \quad k = 1, 2, ..., N,$$

which means that the difference scheme (12) is unconditionally stable. $\square$

Proof of Proposition 5

Proof. From (26) and (30), we have

$$\begin{aligned}\left\|\boldsymbol{R}^k\right\|_2 &\leq \left(\sum_{j=1}^{M-1} C_1\left(\Delta\tau + (\Delta S)^2\right)^2 \Delta S\right)^{\frac{1}{2}} \\ &\leq C_1\left(\Delta\tau + (\Delta S)^2\right)\sqrt{M\Delta S} \\ &\leq C_1\sqrt{S_{max}}\left(\Delta\tau + (\Delta S)^2\right)\end{aligned} \tag{B.6}$$

where $k = 1, 2, \dots, N$. If the series of the right hand side of (32) convergent, then there is a positive constant C_2^k, such that

$$|\rho^k| \equiv |\rho^k(l)| \le C_2^k |\rho^1| \equiv C_2^k |\rho^1(l)|$$

Then, we have

$$|\rho^k| \le C_2 |\rho^1|, \tag{B.7}$$

where $C_2 = \max\left\{ C_2^k \middle| k = 1, 2, \dots, N \right\}$. By using (29) and (31), we have $\varrho^0 = 0$. For $k = 1$, from (38) and (B.7), we get

$$|\varrho^1| = \Delta\tau |\rho^1| \le C_2 \Delta\tau |\rho^1|$$

Suppose now that $|\varrho^n| \le C_2 n \Delta\tau |\rho^1|, n = 1, 2, \dots, k-1$, then by using 38 and B.7, we obtain

$$\begin{aligned} |\varrho^k| &\le \tfrac{1}{|\vartheta_j|} C_2 (k-1) \Delta\tau |\rho^1| + \tfrac{1}{|\vartheta_j|} C_2 \Delta\tau |\rho^1| \\ &\le \left(\tfrac{(k-1)}{k|\vartheta_j|} + \tfrac{1}{k|\vartheta_j|} \right) C_2 k \Delta\tau |\rho^1| \\ &\le C_2 k \Delta\tau |\rho^1| \end{aligned}$$

This completes the proof. □

Proof of Theorem 6

Proof. By using Proposition and (31), (32) and (B.6), we obtain

$$\begin{aligned} \left\| \boldsymbol{\epsilon}^k \right\|_2 &\le C_2 k \Delta\tau \left\| \boldsymbol{R}^1 \right\|_2 \\ &\le C_1 C_2 k \Delta\tau \sqrt{S_{max}} (\Delta\tau + (\Delta S)^2) \end{aligned}$$

Because $k\Delta\tau \le T$, we have

$$\left\| \boldsymbol{\epsilon}^k \right\|_2 \le C(\Delta\tau + (\Delta S)^2) \tag{B.8}$$

where $C = C_1 C_2 T \sqrt{S_{max}}$ □

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