Mathematics and Statistics 8(5): 535-541, 2020
DOI: 10.13189/ms.2020.080506 http://www.hrpub.org

$\mathbb{Z}^d$-action Induced by Shift Map on 1-Step Shift of Finite Type over Two Symbols and k-type Transitive

Nor Syahmina Kamarudin, Syahida Che Dzul-Kifli*

Faculty of Science and Technology, School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Malaysia

Received April 13, 2020; Revised June 19, 2020; Accepted July 10, 2020

(a): *[1] Nor Syahmina Kamarudin, Syahida Che Dzul-Kifli , "$\mathbb{Z}^d$ -action Induced by Shift Map on 1-Step Shift of Finite Type over Two Symbols and k-type Transitive," Mathematics and Statistics, Vol. 8, No. 5, pp. 535 - 541, 2020. DOI: 10.13189/ms.2020.080506.*

(b): *Nor Syahmina Kamarudin, Syahida Che Dzul-Kifli (2020). $\mathbb{Z}^d$ -action Induced by Shift Map on 1-Step Shift of Finite Type over Two Symbols and k-type Transitive. Mathematics and Statistics, 8(5), 535 - 541. DOI: 10.13189/ms.2020.080506.*

Abstract The dynamics of a multidimensional dynamical system may sometimes be inherited from the dynamics of its classical dynamical system. In a multidimensional case, we introduce a new map called a $\mathbb{Z}^d$ -action on space X induced by a continuous map $f : X \to X$ as $T_f : \mathbb{Z}^d \times X \to X$ such that $T_f(n,x) = f^{r(n)}(x)$, where $n \in \mathbb{Z}^d$, $x \in X$ and $r : \mathbb{Z}^d \to \mathbb{Z}$ is a map of the form $r(n) = h_1n_1 + h_2n_2 + ... + h_dn_d$. We then look at how topological transitivity of f effects the behaviour of k-type transitivity of the $\mathbb{Z}^d$ -action, T_f. To verify this, we look specifically at spaces called 1-step shifts of finite type over two symbols which are equipped with a map called the shift map, σ. We apply some topological theories to prove the $\mathbb{Z}^d$ -action on 1-step shifts of finite type over two symbols induced by the shift map, T_σ is k-type transitive for all $k \in \{1, 2, ..., 2^d\}$ whenever σ is topologically transitive. We found a counterexample which shows that not all maps T_σ are k-type transitive for all $k \in \{1, 2, ..., 2^d\}$. However, we have also found some sufficient conditions for k-type transitivity for all $k \in \{1, 2, ..., 2^d\}$. In conclusions, the map T_σ on 1-step shifts of finite type over two symbols induced by the shift map is k-type transitive for all $k \in \{1, 2, ..., 2^d\}$ whenever either the shift map is topologically transitive or satisfies the sufficient conditions. This study helps to develop the study of k-chaotic behaviours of $\mathbb{Z}^d$ -action on the multidimensional dynamical system, contributions, and its application towards symbolic dynamics.

Keywords $\mathbb{Z}^d$ -action, Shift of Finite Type, Topologically Transitive, k-type Transitive

1. Introduction

The study of $\mathbb{Z}^d$ -action has become the current interest among researchers which involves the observation of a multidimensional dynamical system. Compared to the classical dynamical system study (or $\mathbb{Z}^d$ -action), the behavior of $\mathbb{Z}^d$ -action is much more complex and almost impossible to be described. However, there are many discussions that have an approach study on the chaotic behavior of the $\mathbb{Z}^d$ -action.

The existence of $\mathbb{Z}^d$ -action on X originally came from group action as the general form. Instead, the main focus of group action was replaced with $\mathbb{Z}^d$ in the studies. There are many finding results which focus on group actions and also some other special kind of group actions. For instance, Barzanouni et al. [1] had studied group actions on metric space with expansive properties. They found several relations of expansivity in various cases such as between subgroup actions and covering maps. The study had also introduced orbit expansivity to characterize the expansive action.

While Wang and Zhang [2] had focused on group actions for which the group was countable and discrete. They defined the notions of local weak mixing and Li-Yorke chaos for this kind of action to show the relation between them. Next, they had also studied the topological entropy of actions of an infinite countable amenable group and actions of an infinite countable discrete sofic group on a shift of finite type.

Then, Cairns et al. [3] had also studied on group actions and defined six notions of dynamical transitivity and mixing in the context of group actions. Interestingly, they highlighted some relations between those six notions in which they are inherited by subgroups, by taking products and when passing to the induced action on hyperspace. In addition, their discussions were also focus on semi-conjugacy and actions of abelian groups.

There are some studies which directed to the chaos of semigroup actions. Wang et al. [4] studied the action of a semigroup and abelian monoid on Polish space. They learnt about the sensitive and syndetic transitive of the system. Then, they had results on which the system was chaotic by depending on chaos in the sense of Li-Yorke and Devaney. In [5], they also studied the action of semigroup and gave a focus on periodicity and transitivity. They also learnt some chaotic changes like sensitivity to initial conditions and equicontinuous.

Our major interest to study about $\mathbb{Z}^d$ -action is mainly because of some findings from Shah and Das [6,7,8] who studied and introduced the notions of k-type transitive and some other k-type chaos notions such as k-type periodic point, k-type sensitive dependence on initial conditions, k-type Devaney chaotic and k-type mixing. All of the k-type chaos notions are defined mostly related to the chaos notions in the classical dynamical system and therefore we may see some familiarities through the definitions. The k-type chaos notions do help the research to study the behavior of $\mathbb{Z}^d$ -action on a space X with easier understanding and a better way of structure.

The study of Shah and Das in [6] was to focus on relationships between k-type Devaney chaotic of $\mathbb{Z}^d$ -action and its induced $\mathbb{Z}^d$ -action. Furthermore, they also highlight some relations especially involving k-type transitive, dense k-type periodic point, k-type sensitive dependence on the initial condition, k-type weak mixing and k-type mixing. While in [7], their major focus was on a relationship of k-type chaos notions within conjugacy, uniform conjugacy, and product spaces. They also mention the redundancy of k-type sensitive dependence on the initial condition for k-type Devaney chaotic as similar to the finding of Banks et al. in [9] for Devaney chaos on infinite metric spaces in the study of a classical dynamical system.

In [8], Shah and Das changed their focus to the notion of k-type collective sensitive and studied the relation of the new notion between uniform conjugacy and finite product. They also find the relation between k-type sensitive dependence on initial conditions and k-type collective sensitive for induced $\mathbb{Z}^d$ -action. Besides that, Kim and Li [10] had introduced and studied the notion of k-type limit set and k-type non-wandering set of $\mathbb{Z}^2$ -action. Their major purpose was to generalize the spectral decomposition theorem for k-type non-wandering points of a $\mathbb{Z}^2$ -action. On the other hand, Lima [11] had an interest of study on $\mathbb{Z}^d$ -action which is ergodic. The research had also tried to find the connection between ergodic property and positive topological entropy within $\mathbb{Z}^d$ -action.

There are also some interests of study on $\mathbb{Z}^d$ -action for symbolic dynamical systems. Many studies are interested in looking at shifts of finite type and therefore they had defined $\mathbb{Z}^d$ -action on the shift of finite type. In [12], the research had learnt about the phenomenon of transition from the classical shifts of finite type to the multidimensional shifts of finite type. The discussion is mainly about an algebraic structure called Wang Tiling which appears in certain multidimensional shifts. While the study in [13] was interested in the entropy value of multidimensional shifts of finite type. Next, Boyle and Schraudner [14] had also extended the result by finding $\mathbb{Z}^d$ shifts of finite type with positive topological entropy but cannot factor topologically onto the $\mathbb{Z}^d$ Bernoulli shift on N symbols. However, Pavlov [15] had a different approach which studied on $\mathbb{Z}^d$ -shift spaces. Its main purpose was to give conditions which guarantee a $\mathbb{Z}^d$ -shift space to be nonsofic.

In this paper, we may introduce a new concept of $\mathbb{Z}^d$ -action called as a $\mathbb{Z}^d$ -action on X induced by a continuous map $f: X \to X$. Then, we may focus on a specific kind of shift of finite type which is 1-step shift of finite type over two symbols. Our main purpose is to relate the transitivity of the shift map σ to the k-type transitivity of $\mathbb{Z}^d$ -action induced by the shift map on 1-step shift of finite type over two symbols.

2. $\mathbb{Z}^d$ -action and Preliminary Definitions

Let $d > 0$. We let (X, ρ) be a topological dynamical system and a $\mathbb{Z}^d$ -action on a space X was defined in most of the past studies as a continuous map $T: \mathbb{Z}^d \times X \to X$ such that

i. $T(0, x) = x$, for all $x \in X$,
ii. $T\left(n, T\left(m, x\right)\right) = T\left(n+m, x\right)$, for all $n, m \in \mathbb{Z}^d$ and for all $x \in X$.

In addition, $T^n: X \to X$ is described by $T^n(x) = T(n, x)$ for all $n \in \mathbb{Z}^d$, $x \in X$ and clearly T^n

is a homeomorphism on X [6].

In the classical system, it is said a given continuous map $f:X\to X$ is topologically transitive if for every pair of open sets U and V of X, there exists an integer $n>0$ such that $f^n(U)\cap V\neq\varnothing$.

For a $\mathbb{Z}^d$-action, we must let $k\in\{1,2,3,...,2^d\}$ and $k'\in\{0,1\}^d$ such that $k=1+\sum_{i=1}^{d}k_i' 2^{i-1}$. By letting $x=(x_1,x_2,...,x_d)\in\mathbb{Z}^d$ and $y=(y_1,y_2,...,y_d)\in\mathbb{Z}^d$, we say that $x>^k y$ if $(-1)^{k_i'}x_i>(-1)^{k_i'}y_i$ for $i=\{1,2,...,d\}$. Then, a $\mathbb{Z}^d$-action, $T:\mathbb{Z}^d\times X\to X$ is said to be k-type transitive if for every open set U and V of X, there exists $n>^k 0$ such that $T^n(U)\cap V\neq\varnothing$ where $n\in\mathbb{Z}^d$ [6].

Next, let us introduce $\mathbb{Z}^d$-action on a space X induced by a continuous map f on X into itself in the following definition.

Definition 2.1

Let $f:X\to X$ be a continuous map. $T_f:\mathbb{Z}^d\times X\to X$ is a $\mathbb{Z}^d$-action on X induced by f and is defined by

$$T_f^n(x)=T_f(n,x)=f^{r(n)}(x)$$

for $n=(n_1,n_2,...,n_d)\in\mathbb{Z}^d$, $x\in X$ and $r:\mathbb{Z}^d\to\mathbb{Z}$ is a map of the form $r(n)=h_1n_1+h_2n_2+...+h_dn_d$ where $h_i\in\mathbb{Z}$ for every $i\in\{1,2,...,d\}$.

We want to highlight a remark that the map $r:\mathbb{Z}^d\to\mathbb{Z}$ as in Definition 2.1 is a homomorphism and the map T_f has satisfied both properties of $\mathbb{Z}^d$-action.

Next, we let Full-2-shift, Σ_2 be the collection of all two sided infinite sequences of symbols 0 and 1. The elements of Σ_2 is in a form of $\mathbf{x}=...x_{-2}x_{-1}\cdot x_0x_1x_2...=(x_i)_{i\in\mathbb{Z}}$, where $x_i\in\{0,1\}$ for every $i\in\mathbb{Z}$. A finite string (block) of symbols $x_m...x_n$ is often denoted by $x_{[m,n]}$ and the Σ_2 is endowed with product topology. In this topology, two points $\mathbf{x},\mathbf{y}\in\Sigma_2$ are regarded as "close" if they agree on a large central block. That is, $x_{[-n,n]}=y_{[-n,n]}$ for some large n. The Full-2-shift, Σ_2 is a metric space which is equipped with metric

$$\rho(\mathbf{x},\mathbf{y})=\begin{cases}2^{-k}, \text{ if } \mathbf{x}\neq\mathbf{y} \text{ and } k \text{ is maximal so that } x_{[-k,k]}=y_{[-k,k]},\\ 0, \text{ if } \mathbf{x}=\mathbf{y}\end{cases}$$

Therefore, Σ_2 is a topological space induced by the metric ρ and the basic open ball is any subset of Σ_2 of the form $X_w=\{\mathbf{s}\in\Sigma_2 \mid s_{[-n,n]}=w_{[-n,n]}=w\}$ for any block w of length $2n+1$ [16].

A continuous map on Σ_2, the shift map $\sigma:\Sigma_2\to\Sigma_2$ is defined by $(\sigma\mathbf{x})_i=x_{i+1}$ that shifts all sequences to the left. While σ^{-1} is the inverse operation which shifts the sequence to the right. Hence, σ is one-to-one and onto. The composition of σ with itself $k>0$ times $\sigma^k=\sigma\circ...\circ\sigma$ shifts sequences k places to the left, while $\sigma^{-k}=(\sigma^{-1})^k$ shifts the same amount to the right [16].

A shift space is a closed, shift-invariant subset of Σ_2. Equivalently, let F be any set of blocks (or later will be called as set of forbidden blocks), the set $X=X_F$ of sequences that do not contain any element of F is a shift space. If the set F is finite, then it is called as a shift of finite type. A shift of finite type is an M-step if the set of the forbidden block F contains all blocks which have length $M+1$. Therefore, 1-step shift of finite type over two symbols is a shift space in which its forbidden block, F contains blocks of length 2.

3. Results and Discussion

3.1. Shift Map on 1-Step Shifts of Finite Type over Two Symbols

In this subsection, we will discuss the shift map on 1-step shifts of finite type over two symbols. With only two symbols, we have four possible different blocks of length two i.e. 00, 01, 10, and 11, then we have 16 sets of forbidden blocks.

$F_1=\varnothing$	$F_2=\{00\}$	$F_3=\{01\}$	$F_4=\{10\}$
$F_5=\{11\}$	$F_6=\{00,01\}$	$F_7=\{00,10\}$	$F_8=\{00,11\}$
$F_9=\{01,10\}$	$F_{10}=\{01,11\}$	$F_{11}=\{10,11\}$	$F_{12}=\{00,01,10\}$
$F_{13}=\{00,01,11\}$	$F_{14}=\{00,10,11\}$	$F_{15}=\{01,10,11\}$	$F_{16}=\{00,01,10,11\}$

For each $i\in\{1,2,...,16\}$, $X_i\subset\Sigma_2$ is the 1-step shift of finite type with a set of forbidden blocks F_i. However, there are some of them which are singletons, empty set or the whole Σ_2. They are not in our interest of study due to their trivial dynamics. While there are also some of them which are either equal or topologically conjugate.

Firstly, it is clear that $X_1=\Sigma_2$ and $X_{13}=X_{14}=X_{16}=\varnothing$. While $X_6,X_7,X_{10},X_{11},X_{12}$ and X_{15} are singletons. One can show that X_3 and X_4 are topologically conjugate. Then, X_2 and X_5 are topologically conjugate. While X_8 and X_9 are the sets which contain only two elements. From here, we will only look at four different 1-step shifts of finite type which are X_2, X_3, X_8 and X_9.

$$X_2 = \{(x_i)_{i\in\mathbb{Z}} \mid \text{if for every } x_j = 0, \text{then } x_{j-1} = x_{j+1} = 1\}$$

$$X_3 = \{(x_i)_{i\in\mathbb{Z}} \mid \text{if for every } x_i = 1 \text{ and } x_{i+1} = 0, \text{ then } x_j = 1 \text{ for all } j < i \text{ and } x_k = 0 \text{ for all } k > i+1\}$$

$$X_8 = \{...\overline{01}\cdot\overline{01}...,...\overline{10}\cdot\overline{10}...\}$$

$$X_9 = \{...\overline{00}\cdot\overline{00}...,...\overline{11}\cdot\overline{11}...\}$$

It is very trivial to proof the shift map σ on each of the four 1-step shifts of finite type is topologically transitive or not.

Theorem 3.1

The 1-step shift of finite type (X_2,σ) which has set of forbidden blocks $F_2 = \{00\}$ is topologically transitive.

Proof

Let $w = w_{-l}...w_{-1}w_0w_1...w_l$ and $v = v_{-k}...v_{-1}v_0v_1...v_k$ be two allowed blocks in X_2. Then, X_w and X_v are the two nonempty basic open subsets of X_2. With all possible allowable blocks of w and v in X_2, $w1v$ or $v1w$ is also allowable. Let

$$\mathbf{x} = ...w_{-l}...w_{-1}\cdot w_0w_1...w_l1v_{-k}...v_{-1}v_0v_1...v_k... \in X_w.$$

Then,

$$\sigma^{l+k+2}(\mathbf{x}) = ...w_{-l}...w_{-1}w_0w_1...w_l1v_{-k}...v_{-1}\cdot v_0v_1...v_k... \in X_v.$$

Therefore, $\sigma^{l+k+2}(X_w)\cap X_v \neq \varnothing$. Hence, σ is topologically transitive.

Theorem 3.2

The 1-step shift of finite type (X_3,σ) which has set of forbidden blocks $F_3 = \{01\}$ is not topologically transitive.

Proof

Suppose by contradiction that σ is topologically transitive. Then, for every pair of two nonempty basic open subsets of X_3, U and V, there exists $n > 0$ such that $\sigma^n(U)\cap V \neq \varnothing$. Let $w = 000$ and $v = 111$. Then, X_w and X_v are two nonempty basic open subsets of X_3. Since σ is transitive, then there exists $m > 0$ such that $\sigma^m(X_w)\cap X_v \neq \varnothing$. Then, there exists $\mathbf{x}\in X_w$ such that $\sigma^m(\mathbf{x})\in X_v$. Then, the sequence $\mathbf{x}$ should be $\mathbf{x} = ...0\cdot\underbrace{00...1}_{m}11...$. However, $\mathbf{x}\notin X_3$ and this is a contradiction since all sequences in X_3 are in the form $...\overline{1100}...$. Therefore, $\sigma^s(X_w)\cap X_v = \varnothing$ for all $s > 0$. Hence, σ is not topologically transitive.

Theorem 3.3

The 1-step shift of finite type (X_8,σ) which has set of forbidden blocks $F_8 = \{00,11\}$ is topologically transitive.

Proof.

Since X_8 contains only two elements, then there are four possible open subsets of X_8. The subsets are $X_8 = \{...\overline{01}\cdot\overline{01}...,...\overline{10}\cdot\overline{10}...\}$, $U = \{...\overline{01}\cdot\overline{01}...\}$, $V = \{...\overline{10}\cdot\overline{10}...\}$ and $W = \{\varnothing\}$. Let A and B be any pair of the subsets. Then, clearly that $\sigma(A)\cap B \neq \varnothing$. Therefore, σ is topologically transitive.

Theorem 3.4

The 1-step shift of finite type (X_9,σ) which has set of forbidden block $F_9 = \{01,10\}$ is not topologically transitive.

Proof

Since X_9 contains only two elements, then there are four possible open subsets of X_9. The subsets are $X_9 = \{...\overline{00}\cdot\overline{00}...,...\overline{11}\cdot\overline{11}...\}$, $U = \{...\overline{00}\cdot\overline{00}...\}$, $V = \{...\overline{11}\cdot\overline{11}...\}$ and $W = \{\varnothing\}$. However, $\sigma^m(U)\cap V = \varnothing$ for all $m > 0$. Therefore, σ is not topologically transitive.

3.2. $\mathbb{Z}^d$ -action Induced by Shift Map on 1-Step Shifts of Finite Type over Two Symbols

Our main objective is to study the behavior of $\mathbb{Z}^d$ -action induced by shift map on 1-step shifts of finite type over two symbols. Firstly, a $\mathbb{Z}^d$ -action induced by shift map on 1-step shifts of finite type over two symbols is given by the following definition.

Definition 3.1

Let $X = X_F \subset \Sigma_2$ be 1-step shift of finite type over two symbols and $\sigma: X\to X$ be a shift map. $T_\sigma: \mathbb{Z}^d\times X\to X$ is a $\mathbb{Z}^d$ -action on X induced by σ and is defined by

$$T_\sigma^n(\mathbf{x}) = T_\sigma(n,\mathbf{x}) = \sigma^{r(n)}(\mathbf{x})$$

for $n=(n_1,n_2,...,n_d)\in\mathbb{Z}^d$, a sequence $\mathbf{x}\in X$ and $r:\mathbb{Z}^d\to\mathbb{Z}$ is a map of the form $r(n)=h_1n_1+h_2n_2+...+h_dn_d$ where $h_i\in\mathbb{Z}$ for every $i\in\{1,2,...,d\}$.

We will consider four different 1-step shifts of finite type over two symbols which are X_2, X_3, X_8 and X_9. We have already seen that the shift map σ on X_2 and X_8 is topologically transitive while it is not on both X_3 and X_9. Now, we want to show the $\mathbb{Z}^d$-action induced by shift map, T_σ on each of the four 1-step shifts of finite type is k-type transitive for all $k\in\{1,2,...,2^d\}$ or not.

Theorem 3.5

The $\mathbb{Z}^d$-action on X_2 induced by σ, $T_\sigma:\mathbb{Z}^d\times X_2\to X_2$ is k-type transitive for all $k\in\{1,2,...,2^d\}$.

Proof

Let w and v be two allowed blocks in X_2 with length $2l+1$ and $2j+1$ respectively. That is, $w=w_{-l}...w_{-1}w_0w_1...w_l$ and $v=v_{-j}...v_{-1}v_0v_1...v_j$. Then, X_w and X_v are two nonempty open subsets of X_2. For X_2 with only forbidden block $\{00\}$, $w\underbrace{11...11}_{s}v$ or $v\underbrace{11...11}_{s}w$ for all $s\in\mathbb{N}$ is always allowable. Let $k\in\{1,2,...,2^d\}$. The first case is if $h_1n_1+h_2n_2+...+h_dn_d>0$ for some $n=(n_1,n_2,...,n_d)>^k 0$. Take $m=(m_1,m_2,...,m_d)>^k 0$ such that $h_1m_1+h_2m_2+...+h_dm_d>l+j+1$. Let $L=h_1m_1+h_2m_2+...+h_dm_d-l-j-1$. Then, take $\mathbf{x}=...w_{-l}...w_{-1}\cdot w_0w_1...w_l\underbrace{11...11}_{L}v_{-j}...v_{-1}v_0v_1...v_j...\in X_w$.

Then,

$$T_\sigma^m(\mathbf{x})=T_\sigma(m,\mathbf{x})=\sigma^{r(m)}(\mathbf{x})=\sigma^{h_1m_1+h_2m_2+...+h_dm_d}(\mathbf{x})=\sigma^{L+l+j+1}(\mathbf{x})$$
$$=...w_{-l}...w_{-1}w_0w_1...w_l\underbrace{11...11}_{L}v_{-j}...v_{-1}\cdot v_0v_1...v_j...\in X_v.$$

So, $T_\sigma^m(X_w)\cap X_v\neq\varnothing$ for $m>^k 0$. The other case is if $h_1n_1+h_2n_2+...+h_dn_d<0$ for all $n=(n_1,n_2,...,n_d)>^k 0$. Take $m=(m_1,m_2,...,m_d)>^k 0$ such that $h_1m_1+h_2m_2+...+h_dm_d<-l-j-1$. Let $L=-l-j-1-(h_1m_1+h_2m_2+...+h_dm_d)$. Then, take $\mathbf{x}=...v_{-j}...v_{-1}v_0v_1...v_j\underbrace{11...11}_{L}w_{-l}...w_{-1}\cdot w_0w_1...w_l...\in X_w$.

Then,

$$T_\sigma^m(\mathbf{x})=T_\sigma(m,\mathbf{x})=\sigma^{r(m)}(\mathbf{x})=\sigma^{h_1m_1+h_2m_2+...+h_dm_d}(\mathbf{x})=\sigma^{-l-j-1-L}(\mathbf{x})$$
$$=...v_{-j}...v_{-1}\cdot v_0v_1...v_j\underbrace{11...11}_{L}w_{-l}...w_{-1}w_0w_1...w_l...\in X_v.$$

So, $T_\sigma^m(X_w)\cap X_v\neq\varnothing$ for $m>^k 0$. By the both cases, T_σ is k-type transitive for all $k\in\{1,2,...,2^d\}$.

Theorem 3.6

The $\mathbb{Z}^d$-action on X_3 and X_9 induced by σ is not k-type transitive for all $k\in\{1,2,...,2^d\}$.

Proof

The proof which is trivial also use the similar reasons as in the Theorem 3.2 and 3.4.

It is complicated to say $\mathbb{Z}^d$-action on X_8 induced by σ is k-type transitive for all $k\in\{1,2,...,2^d\}$ or not. Here we illustrate an example of $\mathbb{Z}^2$-action on X_8 induced by σ which is not k-type transitive for all $k\in\{1,2,3,4\}$.

Example 3.1

Let $\mathbb{Z}^2$-action on X_8 induced by σ be $T_\sigma(n,\mathbf{x})=\sigma^{2n_1+4n_2}(\mathbf{x})$ for $n=(n_1,n_2)\in\mathbb{Z}^2$ and $\mathbf{x}\in X_8$. Then, T_σ is not k-type transitive for all $k\in\{1,2,3,4\}$.

Proof

Let $k\in\{1,2,3,4\}$. Let $U=\{...\overline{01}\cdot\overline{01}...\}$ and $V=\{...\overline{10}\cdot\overline{10}...\}$. Then, $\sigma^m(U)\cap V\neq\varnothing$ for only all odd integer m. However, $2n_1+4n_2=2(n_1+2n_2)$ is always even for all entries of $n=(n_1,n_2)>^k 0$. Therefore, $T_\sigma^n(U)\cap V=\varnothing$ for all $n>^k 0$. Hence, T_σ is not k-type transitive for all $k\in\{1,2,3,4\}$.

Therefore, there is a sufficient condition for homomorphism $r(n)$ to prove the k-type transitivity of T_σ on X_8. We have a supporting lemma before proving $\mathbb{Z}^d$-action on X_8 induced by σ is k-type transitive for all $k\in\{1,2,...,2^d\}$.

Lemma 3.1

If $r(n)=h_1n_1+h_2n_2+...+h_dn_d$ with at least one h_i is odd integer for some $i\in\{1,2,...,d\}$, then for every $k\in\{1,2,...,2^d\}$, there exists $m>^k 0$ such that $r(m)$ is an odd integer.

Proof

Let $r(n)=h_1n_1+h_2n_2+...+h_dn_d$ where $h_i\in\mathbb{Z}$ for

$i \in \{1,2,...,d\}$ and $n = (n_1, n_2, ..., n_d) \in \mathbb{Z}^d$. Suppose that at least one h_i is odd integer for some $i \in \{1,2,...,d\}$. The first case is if h_i is odd integer for some $i \in \{1,2,...,d\}$. Without loss of generality, let $h_1, h_2, ..., h_e, h_{e+1}, ..., h_d \in \mathbb{Z}^+$ for $1 < e < d$ such that $h_1, h_2, ..., h_e$ are odd integers and $h_{e+1}, h_{e+2}, ..., h_d$ are even integers. Then, $h_i = 2l_i + 1$ for $i \in \{1,2,...,e\}$ and $h_t = 2j_t$ for $t \in \{e+1, e+2, ..., d\}$ for some $l_1, l_2, ..., l_e, j_{e+1}, j_{e+2}, ..., j_d \in \mathbb{N}$. Then,

$$r(n) = h_1n_1 + h_2n_2 + ... + h_dn_d = (2l_1+1)n_1 + (2l_2+1)n_2 + ... + (2l_e+1)n_e + 2j_{e+1}n_{e+1} + 2j_{e+2}n_{e+2} + ... + 2j_dn_d = 2(l_1n_1 + ... + l_e n_e + j_{e+1}n_{e+1} + ... + j_dn_d) + n_1 + n_2 + ... + n_e.$$

For $k \in \{1,2,...,2^d\}$, take $m = (m_1, m_2, ..., m_d) >^k 0$ such that $m_1 = (-1)^{k_1^b}(2a_1+1)$, $m_i = (-1)^{k_i^b}(2b_i)$ for $i \in \{2,3,...,e\}$ and any arbitrary $m_t \in \mathbb{Z}^+$ for $t \in \{e+1, e+2, ..., d\}$ for some $a_1, b_2, b_3, ..., b_e \in \mathbb{N}$ where $(k_1^b, k_2^b, ..., k_e^b, ..., k_d^b) \in \{0,1\}^d$ such that $k = 1 + \sum_{i=1}^{d} k_i^b 2^{i-1}$. Then, $r(m)$ is odd integer. While the second case is if h_i is odd integer for all $i \in \{1,2,...,d\}$. Without loss of generality, let $h_i \in \mathbb{Z}^+$ such that h_i are odd integers for all $i \in \{1,2,...,d\}$. Then, $h_i = 2l_i + 1$ for some $l_i \in \mathbb{N}$ and $i \in \{1,2,...,d\}$. Then,

$$r(n) = h_1n_1 + h_2n_2 + ... + h_dn_d = (2l_1+1)n_1 + (2l_2+1)n_2 + ... + (2l_d+1)n_d = 2(l_1n_1 + l_2n_2 + ... + l_dn_d) + n_1 + n_2 + ... + n_d.$$

For $k \in \{1,2,...,2^d\}$, take $m = (m_1, m_2, ..., m_d) >^k 0$ such that $m_1 = (-1)^{k_1^b}(2a_1+1)$ and $m_i = (-1)^{k_i^b}(2b_i)$ for $i \in \{2,3,...,d\}$ for some $a_1, b_2, b_3, ..., b_d \in \mathbb{N}$ where $(k_1^b, k_2^b, ..., k_d^b) \in \{0,1\}^d$ such that $k = 1 + \sum_{i=1}^{d} k_i^b 2^{i-1}$. Then, $r(m)$ is odd integer. Based on the both cases, there exists $m >^k 0$ such that $r(m)$ is an odd integer for every $k \in \{1,2,...,2^d\}$.

Theorem 3.7

Let $\mathbb{Z}^d$ -action on X_8 induced by σ be $T_\sigma(n, \mathbf{x}) = \sigma^{r(n)}(\mathbf{x})$ for $n = (n_1, n_2, ..., n_d) \in \mathbb{Z}^d$ and $\mathbf{x} \in X_8$. If $r(n) = h_1n_1 + h_2n_2 + ... + h_dn_d$ with at least one h_i is odd integer for some $i \in \{1,2,...,d\}$, then T_σ is k -type transitive for all $k \in \{1,2,...,2^d\}$.

Proof

The four possible open subsets are $X_8 = \{...\overline{01}\cdot\overline{01}...,...\overline{10}\cdot\overline{10}...\}$, $U = \{...\overline{01}\cdot\overline{01}...\}$, $V = \{...\overline{10}\cdot\overline{10}...\}$ and $W = \{\varnothing\}$. Suppose that $r(n) = h_1n_1 + h_2n_2 + ... + h_dn_d$ with at least one h_i is odd integer for some $i \in \{1,2,...,d\}$. By Lemma 3.1, for every $k \in \{1,2,...,2^d\}$, there exists $m >^k 0$ such that $r(m) = L$ where L is an odd integer. Note that $\sigma^n(...\overline{01}\cdot\overline{01}...) = ...\overline{10}\cdot\overline{10}...$ and $\sigma^n(...\overline{10}\cdot\overline{10}...) = ...\overline{01}\cdot\overline{01}...$ for every odd integer n. Let A and B be any pair of the subsets. Then,

$$T_\sigma^m(A) \cap B = T_\sigma(m, A) \cap B = \sigma^{r(m)}(A) \cap B = \sigma^L(A) \cap B \neq \varnothing$$

since L is an odd integer. Therefore, T_σ is k -type transitive for all $k \in \{1,2,...,2^d\}$.

We list all the results in the Table 1.

Table 1. The Dynamics of $\mathbb{Z}^d$ -action on 1-Step Shifts of Finite Type over Two Symbols Induced by Shift Map

Space	(X_F, σ)	(X_F, T_σ)
$X_2,\ F_2 = \{00\}$	Topologically transitive	k -type transitive for all $k \in \{1,2,...,2^d\}$
$X_3,\ F_3 = \{01\}$	Not topologically transitive	Not k -type transitive
$X_8,\ F_8 = \{00,11\}$	Topologically transitive	k -type transitive for all $k \in \{1,2,...,2^d\}$ with sufficient condition
$X_9,\ F_9 = \{01,10\}$	Not topologically transitive	Not k -type transitive

4. Conclusions

In the study, we had considered four different kinds of 1-step shifts of finite type over two symbols and we want to observe the influence of shift map, σ to the $\mathbb{Z}^d$-action induced by the shift map, T_σ. Table 1 shows that among the four spaces, two of them are not topologically transitive and also not k-type transitive for all $k \in \{1, 2, ..., 2^d\}$. The shift map σ on both X_2 and X_8 is topologically transitive. Then, the $\mathbb{Z}^d$-action on X_2 induced by shift map σ is k-type transitive while the action on X_8 is k-type transitive only whenever a sufficient condition is satisfied. Therefore, the main conclusion we have here is a $\mathbb{Z}^d$-action on 1-step shift of finite type over two symbols induced by shift map is k-type transitive for all $k \in \{1, 2, ..., 2^d\}$ whenever the shift map is topologically transitive and a sufficient condition on homomorphism $r(n)$ is satisfied.

Acknowledgments

The authors would like to thank Universiti Kebangsaan Malaysia and the Center for Research and Instrumentation (CRIM) for the financial funding through GUP-2019-054.

REFERENCES

[1] A. Barzanouni, M.S. Divandar, E. Shah. On properties of expansive group actions, Acta Mathematica Vietnamica, Vol. 44, No. 4, 923-934, 2019.

[2] Z. Wang, G. Zhang. Chaotic behavior of group actions, Contemporary Mathematics, Vol. 669, 299-316, 2016.

[3] G. Cairns, A. Kolganova, A. Nielsen. Topological transitivity and mixing notions for group actions, Rocky Mountain Journal of Mathematics, Vol. 37, No. 2, 371-397, 2007.

[4] H. Wang, X. Long, H. Fu. Sensitivity and chaos of semigroup actions, Semigroup Forum, Vol. 84, 81-90, 2012.

[5] F. Schneider, S. Kerkhoff, M. Behrisch, S. Siegmund. Chaotic actions of topological semigroup, Semigroup Forum, Vol. 87, 590-598, 2013.

[6] S. Shah, R. Das. A note on chaos for $\mathbb{Z}^d$-action, Dynamics of Continuous, Discrete and Impulsive System Series A: Mathematical Analysis, Vol. 22, No. 2, 95-103, 2015.

[7] S. Shah, R. Das. On different types of chaos for $\mathbb{Z}^d$-actions, Journal of Mathematics Research, Vol. 7, No. 3, 191-197, 2015.

[8] S. Shah, R. Das. On collective sensitivity for $\mathbb{Z}^d$-actions, Dynamical systems, Vol. 31, No. 2, 1-7, 2016.

[9] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey. On Devaney's definition of chaos, The American Mathematical Monthly, Vol. 99, No. 4, 332-334, 1992.

[10] D. Kim, S. Lee. Spectral decomposition of k-type nonwandering sets for $\mathbb{Z}^2$-actions, Bulletin of the Korean Mathematical Society, Vol. 51, No. 2, 387-400, 2014.

[11] Y. Lima. $\mathbb{Z}^d$-actions with prescribed topological and ergodic properties, Ergodic theory and dynamical systems, Vol. 32, No. 1, 191-209, 2012.

[12] K. Schmidt. Multi-dimensional symbolic dynamical systems, Codes, Systems and Graphical Models, Vol. 123, 67-82, 2001.

[13] M. Hochman, T. Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type, Annals of Mathematics, Vol. 171, 2011-2038, 2010.

[14] M. Boyle, M. Schraudner. $\mathbb{Z}^d$ shifts of finite type without equal entropy full shift factors, Journal of Difference Equations and Applications, Vol. 15, No. 1, 47-52, 2009.

[15] R. Pavlov. A class of nonsofic $\mathbb{Z}^d$ shift spaces, Proceedings of the American Mathematical Society, Vol. 141, No. 3, 987-996, 2013.

[16] D. Lind, B. Marcus. Introduction to Symbolic Dynamics and Coding, Cambridge University Press, Cambridge, 1995.