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Construction of Bivariate Copulas on a Multivariate Exponentially Weighted Moving Average Control Chart

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Abstract The control chart is an important tool in multivariate statistical process control (MSPC), which for monitoring, control, and improvement of the process control. In this paper, we propose six types of copula combinations for use on a Multivariate Exponentially Weighted Moving Average (MEWMA) control chart. Observations from an exponential distribution with dependence measured with Kendall's tau for moderate and strong positive and negative dependence (where $\tau = \pm 0.5, \pm 0.8$) among the observations were generated by using Monte Carlo simulations to measure the Average Run Length (ARL) as the performance metric and should be sufficiently large when the process is in-control on a MEWMA control chart. In this study, we develop an approach performance on the MEWMA control chart based on copula combinations by using the Monte Carlo simulations.The results show that the out-of-control (ARL_1) values for $\lambda = 0.05$ were less than for $\lambda = 0.10$ in almost all cases. The performances of the Farlie-Gumbel-Morgenstern x Ali-Mikhail-Haq copula combination was superior to the others for all shifts with strong positive dependence among the observations and $\lambda = 0.05$. Moreover, when the magnitudes of the shift were very large, the performance metric values for observations with moderate and strong positive and negative dependence followed the same pattern.

Keywords Marginal, Joint Distribution, Multivariate Control Chart, Monte Carlo Simulation

1. Introduction

Multivariate Statistical Process Control (MSPC) is an important method for process monitoring, control and improvement in many areas such as engineering, economics, environmental statistics, finance and etc. For example, in automotive production quality control depends on correlated variables such as the lifetimes of the components in the engine, etc. A control chart is a common tool for MSPC for detecting changes in the vector means of the process. Multivariate control charts are generalizations of their univariate counterparts [1]. Hotelling's T^2 was the first multivariate control chart [2], followed by the Multivariate Exponentially Weighted Moving Average (MEWMA) control chart as a better alternative for detecting small shifts in the process vector mean [3,4]. Most multivariate detection procedures are based on the assumption that the observations are independent and identically distributed (i.i.d.) and follow a multivariate normal distribution. However, many processes are non-normal and correlated, so multivariate control charts need to be able to cope with related joint distributions. Hence, Kuvattana et al. [5] and Sukparungsee et al. [6] introduced the copula to address this requirement.

Copulas are functions that join multivariate distributions to their one-dimensional marginal distribution functions in which the one-dimensional margins are uniform on the interval (0,1) [7]. They are used to explain the dependence between random variables and are based on a representation of Sklar's theorem [8]. A new way of constructing asymmetric copulas was introduced by Mukherjee et al. [9], and later on copulas have been applied to MSPC [10]. Several other studies have proposed and compared the performance of bivariate copulas on the multivariate control charts [11-14]. Herein, we present the efficiency of the combinations of bivariate copulas constructed for shifts in the process vector mean on a MEWMA control chart when observations follow an exponential distribution.

2. Research Methodology

This paper is organized into the following sections: in section 2.1 the multivariate exponentially weighted moving average (MEWMA) control chart. Section 2.2, we review copulas function and constructing bivariate copulas. Section 2.3 describes the dependence measure of data and finally section 2.4 provides the ARL and the simulation study.

2.1. The Multivariate Exponentially Weighted Moving Average (MEWMA) Control Chart

The MEWMA control chart was first developed by Lowry et al. [4]. The given observations $W_1, W_2, \ldots$ from a d-variate Gaussian distribution $N(\mu, \Sigma)$, for i = 1,2,. . . , can be defined as

$$\mathbf{Z}_i = \Lambda \mathbf{W}_i + (1-\Lambda)\mathbf{Z}_{i-1} \quad (1)$$

where $\mathbf{Z}_i$ is a vector of variable values from the data and Λ is a diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_d$, for $0 \le \lambda \le 1$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_d = \lambda$.

The quantity plotted on the control chart is

$$T_i^2 = \mathbf{Z}_i' \Sigma_i^{-1} \mathbf{Z}_i, \quad (2)$$

where $\Sigma_i = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]\Sigma.$

When $\lambda_1 = \lambda_2 = \ldots = \lambda_d = \lambda$ on the interval (0,1) (as assumed in this study), the control chart signals a shift in the mean vector when $T_i^2 > H$ where H is the control limit chosen for the desired in-control process. Generally, the Average Run Length (ARL) can be used to measure the performance of a MEWMA control chart. It depends on the degree of dependence between the variables measured using the covariance matrix Σ and the scalar-weighted λ associated with the past observations. We consider a bivariate EWMA control chart and the control limit H for the in-control process $ARL_0 = 370$.

2.2. Copulas Function and Constructing Bivariate Copulas

Theoretically, for the copula function according to Sklar's theorem [8] for a bivariate case, let X and Y be continuous variables with joint distribution function G and marginal cumulative distributions $F(x)$ and $F(y)$, respectively. Consequently, $G(x,y) = C(F(x),F(y);\theta)$ with copula $C:[0,1]^2 \to [0,1]$ where θ is a parameter of the copula. Theoretically, let A and B be bivariate copulas. It follows that $C_{\alpha,\beta}(u,v) = A(u^{\alpha}, v^{\beta})B(u^{1-\alpha}, v^{1-\beta})$, where $C_{\alpha,\beta}$ is a copula with parameters $\alpha, \beta \in \mathbf{I}$ and $\alpha \neq \beta$ [15]. If $\alpha = \beta = 1$, then $C_{1,1} = A$, and if $\alpha = \beta = 0$ then $C_{0,0} = B$. Similarly, if $C_{(u,v)} \neq C_{(v,u)}$ we have an asymmetric copula.

In accordance with Khoudraji's device [16], let C be symmetric copula $C \neq \Pi_2$, where Π_2 is independence copula. A family of asymmetric copulas $C_{\alpha,\beta}$ with parameters $0 < \alpha, \beta < 1$, $\alpha \neq \beta$ that includes C as a limiting case is given by

$$C_{\alpha,\beta}(u,v) = u^{\alpha} v^{\beta} \cdot C(u^{1-\alpha}, v^{1-\beta}). \quad (3)$$

2.3. Dependence Measure of the Data

Generally, a copula can be used in the study of the dependence of association between random variables by Kendall's tau, which we implemented in this study (Table-1). Let X and Y be continuous random variables with copula C, then Kendall's tau is given by

$$\tau = 4\iint_{\mathbf{I}^2} C(u,v)\,dC(u,v) - 1.$$

Table 1. Kendall's tau of copula function

Copula	Type	Kendall's tau	Parameter space of θ
Clayton	Asymmetric	$\theta/(\theta/2)$	$[-1,\infty)\setminus\{0\}$
Frank	Asymmetric	$1+\frac{4(\frac{1}{\theta}\int_0^{\theta}\frac{t}{e^t-1}dt-1)}{\theta}$	$(-\infty,\infty)\setminus\{0\}$
FGM	Symmetric	$2\theta/(3-\theta)$	[-1,1]
AMH	Symmetric	$\arcsin(\theta)/\left(\frac{\pi}{2}\right)$	[-1,1]

2.4. The ARL and the Simulation Study

Theoretically, the ARL is an average number of points that must be plotted before the out-of-control condition occurs. ARL is classified into ARL_0 and ARL_1. ARL_0 is the average number of observations before the first out-of-control point, while ARL_1 is the average number of observations when the process is out-of-control. The expectations of ARL_0 and ARL_1 can be respectively expressed as

$$ARL_0 \approx E_\omega(\kappa) \geq K; \quad \text{for } \omega = \infty \tag{4}$$

$$ARL_1 \approx E_\omega(\kappa \mid \kappa \geq 1); \quad \text{for } \omega = 1 \tag{5}$$

where ω is the change point time, κ is the stopping time, and $E_\omega(.)$ is the expectation under the assumption that the change point occurs at ω.

We ran a Monte Carlo simulation using R statistical software [17-20] with the 50,000 rounds and a sample size of 6,000. The observations were generated from a copula based on an exponential distribution with mean = 1 (for the in-control process) and shifts at level 0.01, 0.05, 0.1, 0.5, 1, and 5 (for the out-of-control process). The performance of the MEWMA control chart was assessed for λ = 0.05 and 0.10. For all combinations of copulas, setting ω corresponds to Kendall's tau for moderate and strong positive and negative dependence ($\tau = \pm 0.5$, $\pm$-0.8).

3. Results

The simulation results are reported in Tables 2 to 9, in which the results are only empirical. The aim of the study was to optimize the parameters for constructing bivariate copulas (α, β), as shown in Equation (3), for which we used the Maximum pseudo-likelihood estimator method [21]. For the in-control process on the MEWMA control chart, the desired ARL_0 = 370 was set for each copula combination. The results in Tables 2 and 3 indicate moderate positive dependence among the observations ($\tau = 0.5$), Tables 4 and 5 strong positive dependence ($\tau = 0.8$), Tables 6 and 7 moderate negative dependence ($\tau = -0.5$), and Tables 8 to 9 show strong negative dependence ($\tau = -0.8$).

Table 2. ARL_1 of the MEWMA control chart with moderate positive dependence (τ = 0.5, λ= 0.05)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	**329.14**	330.20	329.25	332.24	332.61	329.24
0.05	236.15	240.22	**233.78**	242.38	234.52	241.08
0.10	194.51	197.77	**194.46**	200.24	197.11	199.02
0.50	12.74	14.41	13.39	**10.27**	12.87	10.48
1.00	1.74	1.92	1.81	2.10	**1.70**	2.19
5.00	**1.02**	**1.02**	1.09	1.14	1.07	1.03
UCL	10.69	12.24	11.21	14.03	10.66	15.30
α	0.566	0.858	0.953	0.855	0.161	0.045
β	0.617	0.466	0.906	0.841	0.128	0.032

Note that: Copula combinations i.e.
[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 3. ARL_1 of the MEWMA control chart with moderate positive dependence (τ = 0.5, λ= 0.10)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	**330.39**	332.88	332.15	334.11	333.3	332.28
0.05	243.41	246.95	245.29	248.99	**242.75**	251.73
0.10	204.81	209.86	**137.03**	211.93	208.05	211.82
0.50	15.30	16.52	20.87	17.30	15.69	**11.86**
1.00	2.06	2.27	2.16	2.43	**2.03**	2.52
5.00	1.02	1.03	**1.01**	1.04	1.09	1.20
UCL	13.69	15.56	14.26	17.77	13.73	19.35
α	0.566	0.858	0.953	0.855	0.161	0.045
β	0.617	0.466	0.906	0.841	0.128	0.032

Note that: Copula combinations i.e.
[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 4. ARL$_1$ of the MEWMA control chart with strong positive dependence (τ = 0.8, λ= 0.05)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	329.43	331.64	328.91	334.11	**326.23**	331.25
0.05	237.85	243.71	238.09	244.72	**210.57**	243.13
0.10	194.65	203.84	196.25	205.23	**126.87**	202.94
0.50	15.03	16.48	14.07	17.48	**7.84**	11.83
1.00	2.01	2.36	1.93	2.47	**1.67**	2.47
5.00	1.03	1.07	1.10	1.10	**1.02**	1.04
UCL	13.11	17.84	11.99	20.52	10.24	20.70
α	0.457	0.567	0.635	0.95	0.405	0.069
β	0.457	0.779	0.652	0.949	0.676	0.007

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 5. ARL$_1$ of the MEWMA control chart with strong positive dependence (τ = 0.8, λ= 0.10)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	331.30	**330.08**	333.33	335.59	330.10	336.65
0.05	246.74	249.45	246.12	256.45	240.26	**232.98**
0.10	207.94	211.95	**139.75**	217.58	203.87	152.70
0.50	16.52	18.31	**10.68**	19.68	15.17	13.33
1.00	2.37	2.67	2.24	2.87	**2.01**	2.88
5.00	1.04	1.08	1.13	1.12	**1.00**	1.32
UCL	16.57	22.50	15.15	26.31	13.22	26.65
α	0.457	0.567	0.635	0.95	0.405	0.069
β	0.457	0.779	0.652	0.949	0.676	0.007

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 6. ARL$_1$ of the MEWMA control chart with moderate negative dependence (τ = -0.5, λ= 0.05)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	**324.71**	328.54	326.98	325.05	330.92	328.38
0.05	**232.22**	235.22	235.14	233.45	235.47	235.28
0.10	192.74	192.98	191.74	**191.16**	193.04	193.90
0.50	14.47	16.13	14.30	15.92	**13.97**	16.06
1.00	1.85	2.45	1.82	2.36	**1.80**	2.29
5.00	**1.02**	**1.02**	**1.02**	**1.02**	**1.02**	**1.02**
UCL	11.32	14.53	11.1	13.95	10.97	13.40
α	0.982	0.999	0.99	0.919	1.000	0.149
β	0.999	0.999	0.998	1.000	0.915	0.029

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 7. ARL_1 of the MEWMA control chart with moderate negative dependence (τ = -0.5, λ= 0.10)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	328.63	**323.78**	326.92	330.17	326.36	326.79
0.05	233.24	228.14	233.79	231.26	233.64	**227.13**
0.10	191.54	**185.83**	190.78	187.44	191.02	186.32
0.50	16.39	16.20	**16.10**	16.32	16.11	16.27
1.00	2.26	2.80	2.22	2.73	**2.19**	2.65
5.00	**1.02**	1.03	**1.02**	1.03	**1.02**	**1.02**
UCL	14.25	17.68	14.00	17.14	13.86	16.38
α	0.982	0.999	0.990	0.919	1.000	0.149
β	0.999	0.999	0.998	1.000	0.915	0.029

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 8. ARL_1 of the MEWMA control chart copulas with strong negative dependence (τ = -0.8, λ= 0.05)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	326.37	328.95	327.01	326.35	326.71	**322.98**
0.05	234.28	232.98	**232.22**	235.52	235.31	233.37
0.10	192.17	**191.68**	194.22	192.31	194.65	191.90
0.50	14.44	16.25	14.31	15.92	**14.28**	15.89
1.00	1.84	2.58	**1.82**	2.50	**1.82**	2.42
5.00	1.02	**1.01**	**1.01**	**1.01**	1.02	**1.01**
UCL	11.32	15.53	11.09	14.95	11.12	14.25
α	0.996	1.000	0.995	0.806	0.999	0.261
β	0.987	0.858	0.998	0.999	0.999	0.001

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

Table 9. ARL_1 of the MEWMA control chart with strong negative dependence (τ = -0.8, λ= 0.10)

Shift	Copula combinations					
	[1]	[2]	[3]	[4]	[5]	[6]
0.01	326.44	327.60	325.84	**324.42**	328.11	327.16
0.05	234.44	**226.75**	232.27	228.30	235.30	229.35
0.10	193.59	187.43	190.09	186.49	192.91	**185.96**
0.50	16.35	**15.83**	16.02	15.91	16.34	15.99
1.00	2.26	2.88	**2.22**	2.82	**2.22**	2.75
5.00	1.02	**1.01**	1.02	**1.01**	1.02	**1.01**
UCL	14.25	18.77	13.98	18.12	14.05	17.40
α	0.996	1.000	0.995	0.806	0.999	0.261
β	0.987	0.858	0.998	0.999	0.999	0.001

Note that: Copula combinations i.e.

[1] Clayton × FGM [2] Clayton × Frank [3] Clayton × AMH [4] FGM × Frank [5] FGM × AMH [6] Frank × AMH

The results in Tables 2 to 9 show that the ARL_1 values for $\lambda = 0.05$ were less than those for $\lambda = 0.10$ in almost all cases. The results in Tables 2 and 3 indicate that the Clayton X Ali-Mikhail-Haq (AMH) copula combination was superior to the others in almost all cases. Meanwhile, with strong positive dependence ($\tau = 0.8$) and $\lambda = 0.05$, Farlie-Gumbel-Morgenstern (FGM) X AMH attained the minimum ARL_1 with all shifts (Table 4). Meanwhile, for moderate negative dependence ($\tau = -0.5$), Clayton X FGM attained the minimum ARL_1 with shift values at 0.01 and 0.05 (Table 6). For the results for strong negative dependence ($\tau = -0.8$) and $\lambda = 0.05$ (Table 8), the performance of FGM X AMH was superior to the others with shift values at 0.5 and 1. However, when the magnitude of the shift was large ($\delta \geq 5$), the performances of all of the copula combinations for moderate and strong positive and negative dependence were the same.

4. Conclusions

In this study, we investigated closed-form approximations of the ARL for MEWMA control charts using bivariate copulas constructed via Khoudraji's device, and we used Monte Carlo simulation when the marginal of the variables was exponential with $\mu = 1$. The simulation results suggest that there were no meaningful differences between the performances of the bivariate copulas at a very large shift ($\delta \geq 5$) when the observations had moderate and strong positive and negative dependence. In addition, the performances of the constructed bivariate copulas were superior to a single copula [5] for a moderate shift in a process on a MEWMA control chart. For further research, we could use the real data to compare the simulation results.

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