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Fuzzy Sumudu Decomposition Method for Fuzzy Delay Differential Equations with Strongly Generalized Differentiability

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Abstract Fuzzy delay differential equation has always been a tremendous way to model real-life problems. It has been developed throughout the last decade. Many types of fuzzy derivatives have been considered, including the recently introduced concept of strongly generalized differentiability. However, considering this interpretation, very few methods have been introduced, obstructing the potential of fuzzy delay differential equations to be developed further. This paper aims to provide solution for fuzzy nonlinear delay differential equations and the derivatives considered in this paper is interpreted using the concept of strongly generalized differentiability. Under this method, the calculations will lead to two cases i.e. two solutions, and one of the solutions is decreasing in the diameter. To fulfil this, a method resulting from the elegant combination of fuzzy Sumudu transform and Adomian decomposition method is used, it is termed as fuzzy Sumudu decomposition method. A detailed procedure for solving fuzzy nonlinear delay differential equations with the mentioned type of derivatives is constructed in detail. A numerical example is provided afterwards to demonstrate the applicability of the method. It is shown that the solution is not unique, and this is in accord with the concept of strongly generalized differentiability. The two solutions can later be chosen by researcher with regards to the characteristic of the problems. Finally, conclusion is drawn.

Keywords Fuzzy Differential Equations, Fuzzy Sumudu Transform, Sumudu Decomposition Method, Nonlinear Delay Differential Equations

1 Introduction

Delay differential equations (DDEs) have long being used to model real-world problems comprising many fields such as mathematics, physics and biology. There are useful whenever the model possesses functional terms. Some of recent works using DDEs are done in [1] where the authors studied DDEs under dissipative type conditions. Meanwhile, An et al. investigated the impulsive hybrid interval-valued delay integro-differential equations [2]. As useful as DDEs are, there are still limitations encountered. Ordinary or classical DDEs cannot handle real-worlds problems in the most ideal way due to the presence of uncertainties and fuzziness. This is a common occurence when dealing with our surrounding since we cannot be certain on the measurement taken due to some constraint in our knowledge and senses. At instance, when we consider a model of a population, uncertainties might arise due to birth, death and migrations.

To handle the shortcoming, scientist come to many concept that have been proposed to handle uncertainty, some of which are probabilistic and stochastic theory. One of the newest tool for handling uncertainty is using fuzzy set theory pioneered by Zadeh [3]. At instance, fuzzy differential equations have been established over the years and have been opened to many types of interpretations of derivative. For example, using Zadeh extension principle, differential inclusion and Seikkala derivative. The latest addition to the list is the concept of strongly generalized differentiability. Soon after, fuzzy DDEs is coined out as a result from the combination of DDEs and fuzzy differential equations.

One of the earliest result on fuzzy DDEs was discussed by

Lupulescu and results on existence and uniqueness are presented [4]. The works are driven by Liu process. Since then, many works have been done considering fuzzy DDEs in many areas such as in [5], focusing fuzzy derivatives under the concept of strongly generalized differentiability. On the other hand, in [6], the existence of local and global solutions of fuzzy delay differential inclusions has been studied. Recent works on the topic are in [7], discussing on prey-predator model with delay terms and in [8], investigating fuzzy DDEs under granular differentiability. In [9], Runge-Kutta method have been used to solve DDEs with uncertain parameters and spatial pattern formation has been analysed. If we compare between fuzzy DDEs using Zadeh extension principle and fuzzy DDEs interpreted under the concept of strongly generalized concept, it is obvious that there are very few methods that can be used to the latter. This served as a motivation for the study in this paper to pinpoint all possible extensions and modifications that need to be done in order to handle fuzzy DDEs under the strongly generalized differentiability.

One of the methods that have been used to solve fuzzy differential equations in recent literature is fuzzy Sumudu transform (FST) and this can be explored in [10], [11], and [12]. This method possess unity property or often referred as scale preserving property. Using this property, researcher might have an insight or idea on the behaviour of the solutions as the variable approaches certain values or numbers. In other words, the transformed function can be treated as the replica of the original function rather than as dummies which happen when considering different type of fuzzy integral transform such as fuzzy Laplace transform [13]. Furthermore, FST have been applied to several types of fuzzy differential equations, for example, fuzzy differential equations with fractional order derivatives in [14] and [15], and fuzzy partial derivatives [13] as well as fuzzy integral equations [16]. The method however suffers a drawback because an integral transform can only be used to solve linear problems and a lot of a real world problems are actually modelled using nonlinear terms.

In this paper, we focus on the combination of FST and decomposition method, termed as fuzzy Sumudu decomposition method (FSDM). This method has been successfully applied on linear fuzzy differential equations [17]. In this work, we will try implementing the method on nonlinear fuzzy differential equations with delay terms to exhibit the practicality of the method on a wider aspect of fuzzy differential equations. Other than extending the applicability to wider range of DDEs, FSDM also helps in simplifying the workings since the differential equations are reduced using the FST part in the method. This is illustrated throughout the procedures and example provided in this paper.

This paper is divided into several sections. Next section provides some some basic concepts necessary to understand the paper. This is followed by Section 3, providing step-by-step procedures for solving nonlinear fuzzy differential equations. Section 4 demonstrates the applications of the proposed method on a numerical example. Later, conclusion is drawn.

2 Preliminaries

In this section, some important prerequisites are revisited. These involve the concept of fuzzy numbers, fuzzy functions and some previous results of FST.

Definition 1. *[3] A fuzzy number is a mapping $\widetilde{u}:\mathbb{R}\to[0,1]$ that satisfies the following conditions.*

1. *$\forall\,\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is upper semi continuous,*
2. *$\forall\,\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is fuzzy convex, i.e., $\widetilde{u}(\gamma s+(1-\gamma)t)\geq$ min$\{\widetilde{u}(s),\widetilde{u}(t)\}$ for all $s,t\in\mathbb{R}$, and $\gamma\in[0,1]$,*
3. *$\forall\,\widetilde{u}\in\mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is normal,*
4. *supp $\widetilde{u}=\{t\in\mathbb{R}|\widetilde{u}(t)>0\}$ is the support of $\widetilde{u}$, and it has a compact closure cl(supp $\widetilde{u}$).*

Definition 2. *[18] Let $\widetilde{u}\in\mathcal{F}(\mathbb{R})$ and $\alpha\in]0,1]$. The α-level set of $\widetilde{u}$ is the crisp set $\widetilde{u}^\alpha$ that contains all the elements with membership degree greater than or equal to α, i.e.*

$$\widetilde{u}^\alpha=\{x\in\mathbb{R}|\widetilde{u}(t)\geq\alpha\},$$

where $\widetilde{u}^\alpha$ denotes α-level set of fuzzy number $\widetilde{u}$.

Definition 3. *[19] A parametric form of an arbitrary fuzzy number $\widetilde{u}$ is an ordered pair $[\underline{u}^\alpha,\overline{u}^\alpha]$ of functions $\underline{u}^\alpha$ and $\overline{u}^\alpha$, for any $\alpha\in[0,1]$, that fulfil the following conditions.*

i. $\underline{u}^\alpha$ is a bounded left continuous monotonic increasing function in $[0,1]$,

ii. $\overline{u}^\alpha$ is a bounded left continuous monotonic decreasing function in $[0,1]$,

iii. $\underline{u}^\alpha\leq\overline{u}^\alpha$.

Definition 4. *[20] Let $\widetilde{f}:]a,b[\to\mathcal{F}(\mathbb{R})$ be a fuzzy function and $t_0\in]a,b[$. We say that $\widetilde{f}$ is strongly generalized differentiable on t_0, if there exists an element $\widetilde{f}'(t_0)\in\mathcal{F}(\mathbb{R})$, such that*

i. for all $h>0$ sufficiently small, $\exists\widetilde{f}(t_0+h)-^H\widetilde{f}(t_0),\widetilde{f}(t_0)-^H\widetilde{f}(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to0}\frac{\widetilde{f}(t_0+h)-^H\widetilde{f}(t_0)}{h}=\lim_{h\to0}\frac{\widetilde{f}(t_0)-^H\widetilde{f}(t_0-h)}{h}=\widetilde{f}'(t_0),$$

or

ii. for all $h>0$ sufficiently small, $\exists \widetilde{f}(t_0) -^H \widetilde{f}(t_0+h)$, $\widetilde{f}(t_0-h) -^H \widetilde{f}(t_0)$ and the limits (in the metric D)

$$\lim_{h\to 0}\frac{\widetilde{f}(t_0) -^H \widetilde{f}(t_0+h)}{-h} = \lim_{h\to 0}\frac{\widetilde{f}(t_0-h) -^H \widetilde{f}(t_0)}{-h} = \widetilde{f}'(t_0).$$

Theorem 1. *[21] Let $\widetilde{f}:\mathbb{R}\to\mathcal{F}(\mathbb{R})$ be a continuous fuzzy function and $\widetilde{f}(t)=[\underline{f}^\alpha(t),\overline{f}^\alpha(t)]$, for every $\alpha\in[0,1]$. Then*

1. *if the fuzzy function $\widetilde{f}$ is (i)-differentiable, then $\underline{f}^\alpha(t)$ and $\overline{f}^\alpha(t)$ are both differentiable and*

$$f'(t)=[(\underline{f}')^\alpha(t),(\overline{f}')^\alpha(t)],$$

2. *if the fuzzy function $\widetilde{f}$ is (ii)-differentiable, then $\underline{f}^\alpha(t)$ and $\overline{f}^\alpha(t)$ are both differentiable and*

$$f'(t)=[(\overline{f}')^\alpha(t),(\underline{f}')^\alpha(t)].$$

Definition 5. *[10] Let $\widetilde{f}:\mathbb{R}\to\mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Suppose that $\widetilde{f}(ut)\odot e^{-t}$ is improper fuzzy Riemann-integrable on $[0,\infty[$, then $\int_0^\infty \widetilde{f}(ut)\odot e^{-t}dt$ is called fuzzy Sumudu transform and is denoted by*

$$G(u)=\mathcal{S}[\widetilde{f}(t)](u)=\int_0^\infty \widetilde{f}(ut)\odot e^{-t}dt,\quad u\in[-\tau_1,\tau_2],$$

where the variable u is used to factor the variable t in the argument of the fuzzy function and $\tau_1,\tau_2>0$.

Theorem 2. *[13] Let $\widetilde{f}:\mathbb{R}\to\mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function, and $\widetilde{f}$ is the primitive of $\widetilde{f}'$ on $[0,\infty)$. Then*

$$\mathbb{S}[\widetilde{f}'(t)](u)=\frac{\mathbb{S}[\widetilde{f}(t)] -^H \widetilde{f}(0)}{u},$$

where $\widetilde{f}$ is (i)-differentiable, or

$$\mathbb{S}[\widetilde{f}'(t)](u)=\frac{(-\widetilde{f}(0)) -^H (-\mathbb{S}[\widetilde{f}(t)])}{u},$$

where $\widetilde{f}$ is (ii)-differentiable.

3 Fuzzy Sumudu decomposition method for fuzzy nonlinear delay differential equations

In this section, we revisit the required procedures for solving fuzzy DDE. The method is a sophisticated combination of fuzzy Sumudu transform and the Adomian decomposition method by George Adomian [22]. The procedure for solving linear fuzzy differential equations with the said method has been provided in [17]. In this paper, we modified the procedures to handle nonlinear terms in fuzzy DDE. Consider

$$\frac{d\widetilde{y}}{dt}+\widetilde{R}(y)+\widetilde{N}(t-\tau)=\widetilde{f}(t),\tag{1}$$

with the initial condition

$$y(0)=\widetilde{y}_0=[\underline{y}_0,\overline{y}_0],\tag{2}$$

where $\widetilde{y}=\widetilde{y}(t)$, $\widetilde{R}$ is a linear bounded fuzzy operator, $\widetilde{f}(t)$ a continuous function, $\widetilde{N}$ a nonlinear bounded fuzzy operator and $\frac{d\widetilde{y}}{dt}$ the first order fuzzy derivative. Using FST on both sides of Eq. (1),

$$\begin{aligned}S\left[\frac{d\widetilde{y}}{dt}\right](u)+S\left[\widetilde{R}(y)\right](u)+S\left[\widetilde{N}(t-\tau)\right](u)\\ =S\left[\widetilde{f}(t)\right](u).\end{aligned}\tag{3}$$

From Theorems 1 and 2, Eq. (3) is separated into two cases to indicate the type of differentiability possessed. First, let consider (i)-differentiable $\widetilde{y}(t)$, then,

$$\begin{aligned}\frac{\mathcal{S}[\widetilde{t}(u)-\widetilde{y}_0]}{u}+\mathcal{S}\left[\widetilde{R}(y)\right](u)+S\left[\widetilde{N}(t-\tau)\right](u)\\ =\mathcal{S}\left[\widetilde{f}(t)\right](u),\end{aligned}\tag{4}$$

represented parametrically as

$$\begin{aligned}\frac{s[\underline{y}(t)](u)-\underline{y}_0]}{u}+s\left[\underline{R}(y)\right](u)+s\left[\underline{N}(t-\tau)\right](u)\\ =s\left[\underline{f}(t)\right](u),\\ \frac{s[\overline{y}(t)](u)-\overline{y}_0]}{u}+s\left[\overline{R}(y)\right](u)+s\left[\overline{N}(t-\tau)\right](u)\\ =s\left[\overline{f}(t)\right](u),\end{aligned}\tag{5}$$

and we obtain,

$$\begin{aligned}s[\underline{y}(t)](u)=&\underline{y}_0-us\left[\underline{R}(y)\right](u)-s\left[\underline{N}(t-\tau)\right](u)\\ &+us\left[\underline{f}(t)\right](u),\\ s[\overline{y}(t)](u)=&\overline{y}_0-us\left[\overline{R}(y)\right](u)-s\left[\overline{N}(t-\tau)\right](u)\\ &+us\left[\overline{f}(t)\right](u).\end{aligned}\tag{6}$$

Then, we define

$$\begin{aligned}\underline{y}(t)&=\sum_{n=0}^{\infty}\underline{y}_n(t),\\ \overline{y}(t)&=\sum_{n=0}^{\infty}\overline{y}_n(t).\end{aligned}\tag{7}$$

The fuzzy nonlinear operator can be further decomposed as the following.

$$\begin{aligned}\underline{N}(t-\tau)&=\sum_{n=0}^{\infty}\underline{A}_n(t),\\ \overline{N}(t-\tau)&=\sum_{n=0}^{\infty}\overline{A}_n(t).\end{aligned}\tag{8}$$

where $[\underline{A}_n, \overline{A}_n]$ is the fuzzy Adomian polynomial of $\widetilde{y}_0, \widetilde{y}_1, \ldots, \widetilde{y}_n$:

$$\begin{aligned} \underline{A}_n &= \frac{1}{n!}\frac{d}{d\lambda}\left[\underline{N}\sum_{n=0}^{\infty}\lambda^n \underline{y}_n\right], \\ \overline{A}_n &= \frac{1}{n!}\frac{d}{d\lambda}\left[\overline{N}\sum_{n=0}^{\infty}\lambda^n \overline{y}_n\right]. \end{aligned} \tag{9}$$

At instance,

$$\begin{aligned} \underline{A}_0 &= f(\underline{y}_0) \\ \overline{A}_0 &= f(\overline{y}_0) \\ \underline{A}_1 &= \underline{y}_1 f^1(\underline{y}_0) \\ \overline{A}_1 &= \overline{y}_1 f^1(\overline{y}_0) \end{aligned} \tag{10}$$

Substituting (7)-(10) into (6),

$$\begin{aligned} s\left[\sum_{n=0}^{\infty}\underline{y}_n(t)\right](u) =& \underline{y}_0 - us\left[\underline{R}\sum_{n=0}^{\infty}\underline{y}_n(t)\right](u) \\ &- u\left[\sum_{n=0}^{\infty}\underline{A}_n(t)\right] + us\left[\underline{f}(t)\right](u), \\ s\left[\sum_{n=0}^{\infty}\overline{y}_n(t)\right](u) =& \overline{y}_0 - us\left[\overline{R}\sum_{n=0}^{\infty}\overline{y}_n(t)\right](u) \\ &- u\left[\sum_{n=0}^{\infty}\overline{A}_n(t)\right] + us\left[\overline{f}(t)\right](u). \end{aligned} \tag{11}$$

Comparing coefficients of $\widetilde{y}$, we obtain

$$\begin{aligned} &s\left[\underline{y}_0\right](u) = \underline{y}_0 + us\left[\underline{f}(t)\right](u), \\ &s\left[\overline{y}_0\right](u) = \overline{y}_0 + us\left[\overline{f}(t)\right](u), \\ &s\left[\underline{y}_n\right](u) = -us\left[\underline{R}\underline{y}_{n-1}(t)\right](u) - us\left[\underline{A}_{n-1}(t)\right], \\ &s\left[\overline{y}_n\right](u) = -us\left[\overline{R}\overline{y}_{n-1}(t)\right](u) - us\left[\overline{A}_{n-1}(t)\right]. \end{aligned} \tag{12}$$

Applying inverse FST, we conclude

$$\begin{aligned} \underline{y}_0 &= s^{-1}[\underline{y}_0] + s^{-1}\left[us\left[\underline{f}(t)\right](u)\right], \\ \overline{y}_0 &= s^{-1}[\overline{y}_0] + s^{-1}\left[us\left[\overline{f}(t)\right](u)\right], \\ \underline{y}_n &= -s^{-1}\left[us\left[\underline{R}\underline{y}_{n-1}(t) + us\left[\underline{A}_{n-1}(t)\right](u)\right](u)\right], \\ \overline{y}_n &= -s^{-1}\left[us\left[\overline{R}\overline{y}_{n-1}(t) + us\left[\overline{A}_{n-1}(t)\right](u)\right](u)\right]. \end{aligned} \tag{13}$$

Next, we consider $\widetilde{y}(t)$ to be (ii)-differentiable. From Theorems 1 and 2, then

$$\begin{aligned} \frac{-\widetilde{y}_0 -^H (-\mathcal{S}[\widetilde{y}(t)](u))}{u} + \mathcal{S}\left[\widetilde{R}(y)\right](u) + S\left[\widetilde{N}(t-\tau)\right](u) \\ = \mathcal{S}\left[\widetilde{f}(t)\right](u). \end{aligned} \tag{14}$$

This can be represented in parametric form as follows.

$$\begin{aligned} \frac{-\overline{y}_0 - (-s[\overline{y}(t)](u))}{u} + s\left[\underline{R}(y)\right](u) + s\left[\underline{N}(t-\tau)\right](u) \\ = s\left[\underline{f}(t)\right](u), \\ \frac{-\underline{y}_0 - (-s[\underline{y}(t)](u))}{u} + s\left[\overline{R}(y)\right](u) + s\left[\overline{N}(t-\tau)\right](u) \\ = s\left[\overline{f}(t)\right](u). \end{aligned} \tag{15}$$

Analogous to the previous case, we have

$$\begin{aligned} \underline{y}_0 &= s^{-1}[\underline{y}_0] + s^{-1}\left[us\left[\overline{f}(t)\right](u)\right], \\ \overline{y}_0 &= s^{-1}[\overline{y}_0] + s^{-1}\left[us\left[\underline{f}(t)\right](u)\right], \\ \underline{y}_n &= -s^{-1}\left[us\left[\overline{R}\overline{y}_{n-1}(t) + us\left[\overline{A}_{n-1}(t)\right](u)\right](u)\right], \\ \overline{y}_n &= -s^{-1}\left[us\left[\underline{R}\underline{y}_{n-1}(t) + us\left[\underline{A}_{n-1}(t)\right](u)\right](u)\right]. \end{aligned} \tag{16}$$

4 Numerical example

Consider the following delay differential equation under fuzzy settings adapted from [23].

$$\widetilde{y}' = 1 - 2\widetilde{y}^2(t/2) \tag{17}$$

where

$$\widetilde{y}(0) = [\alpha - 1, 1 - \alpha] \tag{18}$$

We apply FST on both sides of (17) to get

$$\begin{aligned} \mathcal{S}(\widetilde{y}') &= \mathcal{S}(1 - 2\widetilde{y}^2(t/2)) \\ &= \mathcal{S}(1) - \mathcal{S}(2\widetilde{y}^2(t/2)) \\ &= 1 - 2\mathcal{S}(\widetilde{y}^2(t/2)) \end{aligned} \tag{19}$$

Then, Equation 19 can be divided into two cases as stated previously.

Case 1: First, suppose $\widetilde{y}(t)$ is (i)-differentiable, then

$$\begin{aligned} s\left[\underline{y}'(t)\right](u) &= 1 - 2s(\underline{y}^2(t/2))(u), \\ s\left[\overline{y}'(t)\right](u) &= 1 - 2s(\overline{y}^2(t/2))(u). \end{aligned} \tag{20}$$

From part (i) of Theorem 2, we have

$$\begin{aligned} \frac{s[\underline{y}(t)](u) - \underline{y}(0)}{u} &= 1 - 2s(\underline{y}^2(t/2)(u), \\ \frac{s[\overline{y}(t)](u) - \overline{y}(0)}{u} &= 1 - 2s(\overline{y}^2(t/2))(u). \end{aligned} \tag{21}$$

By rearrangement to solve for $s[y(t)](u)$, hence

$$\begin{aligned} s[\underline{y}(t)](u) &= (1-\alpha) + u - 2us(\underline{y}^2(t/2))(u), \\ s[\overline{y}(t)](u) &= (\alpha - 1) + u - 2us(\overline{y}^2(t/2))(u). \end{aligned} \tag{22}$$

Then inverse FST is applied to both sides of (22),

$$\begin{aligned}\underline{y}(t) &= (1-\alpha) + t - 2s^{-1}(us(\underline{y}^2(t/2))(u)),\\ \overline{y}(t) &= (\alpha-1) + t - 2s^{-1}(us(\overline{y}^2(t/2))(u)).\end{aligned} \tag{23}$$

and

$$\begin{aligned}\underline{y}_0(t) &= (\alpha-1) + t,\\ \overline{y}_0(t) &= (1-\alpha) + t,\\ \underline{y}_{n+1}(t) &= -2s^{-1}\left[us\left[\underline{A}_n\right](u)\right],\\ \overline{y}_{n+1}(t) &= -2s^{-1}\left[us\left[\overline{A}_n\right](u)\right].\end{aligned} \tag{24}$$

From FSDM proposed,

$$\underline{A}_0 = \underline{y}_0^2(\frac{t}{2}) = (\alpha-1)^2 + \frac{t^2}{4} + 2(\alpha-1)(\frac{t}{2}), \tag{25}$$

$$\overline{A}_0 = \overline{y}_0^2(\frac{t}{2}) = (1-\alpha)^2 + \frac{t^2}{4} + 2(1-\alpha)(\frac{t}{2}), \tag{26}$$

$$\underline{A}_1 = 2\underline{y}_0(t/2)\underline{y}_1(t/2), \tag{27}$$

$$\overline{A}_1 = 2\overline{y}_0(t/2)\overline{y}_1(t/2). \tag{28}$$

For $n = 0$,

$$\underline{y}_1(t) = -2\left[(\alpha-1)^2 t + \frac{t^3}{12} + (\alpha-1)\frac{t^2}{2}\right] \tag{29}$$

and

$$\overline{y}_1(t) = -2\left[(1-\alpha)^2 t + \frac{t^3}{12} + (1-\alpha)\frac{t^2}{2}\right] \tag{30}$$

From (29) and (30),

$$\begin{aligned}\underline{y}_1\left(\frac{t}{2}\right) &= -2\left[\frac{(\alpha-1)^2}{2}t + \frac{t^3}{96} + (\alpha-1)\frac{t^2}{8}\right]\\ \overline{y}_1\left(\frac{t}{2}\right) &= -2\left[\frac{(1-\alpha)^2}{2}t + \frac{t^3}{96} + (1-\alpha)\frac{t^2}{8}\right]\end{aligned} \tag{31}$$

For $n = 1$,

$$\begin{aligned}\underline{y}_2(t) =& 8\left[(\alpha-1)^3\frac{t^2}{4} + 3(\alpha-1)^2\frac{t^3}{24} + 7(\alpha-1)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right]\end{aligned} \tag{32}$$

and

$$\begin{aligned}\overline{y}_2(t) =& 8\left[(1-\alpha)^3\frac{t^2}{4} + 3(1-\alpha)^2\frac{t^3}{24} + 7(1-\alpha)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right]\end{aligned} \tag{33}$$

The fuzzy series solution is given as the following.

$$\begin{aligned}\underline{y}(t) =& \underline{y}_0(t) + \underline{y}_1(t) + \underline{y}_2(t) + \ldots,\\ =& (\alpha-1) + t - 2\left[(\alpha-1)^2 t + \frac{t^3}{12} + (\alpha-1)\frac{t^2}{2}\right]\\ &+ 8\left[(\alpha-1)^3\frac{t^2}{4} + 3(\alpha-1)^2\frac{t^3}{24} + 7(\alpha-1)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right] + \ldots\end{aligned} \tag{34}$$

and

$$\begin{aligned}\overline{y}(t) =& \overline{y}_0(t) + \overline{y}_1(t) + \overline{y}_2(t) + \ldots,\\ =& (1-\alpha) + t - 2\left[(1-\alpha)^2 t + \frac{t^3}{12} + (1-\alpha)\frac{t^2}{2}\right]\\ &+ 8\left[(1-\alpha)^3\frac{t^2}{4} + 3(1-\alpha)^2\frac{t^3}{24} + 7(1-\alpha)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right] + \ldots\end{aligned} \tag{35}$$

The result obtained is illustrated in Figure 1. For the sake of simplicity, only first, second and third term of $\widetilde{y}(t)$ is taken into account.

Case 2: Now we consider $\widetilde{y}(t)$ to be (ii)-differentiable, then

$$\begin{aligned}s\left[\overline{y}'(t)\right](u) &= 1 - 2s(\underline{y}^2(t/2))(u),\\ s\left[\underline{y}'(t)\right](u) &= 1 - 2s(\overline{y}^2(t/2))(u).\end{aligned} \tag{36}$$

As in part (ii) in Theorem 2,

$$\begin{aligned}\frac{s(\overline{y}(t)) - (1-\alpha)}{u} &= 1 - 2s(\underline{y}^2(t/2))(u),\\ \frac{s(\underline{y}(t)) - (1-\alpha)}{u} &= 1 - 2s(\overline{y}^2(t/2))(u).\end{aligned} \tag{37}$$

Computing Case 2 using analogous step-by-step procedure for Case 1, we have

$$\begin{aligned}\underline{y}(t) =& \underline{y}_0(t) + \underline{y}_1(t) + \underline{y}_2(t) + \ldots,\\ =& (1-\alpha) + t - 2\left[(\alpha-1)^2 t + \frac{t^3}{12} + (\alpha-1)\frac{t^2}{2}\right]\\ &+ 8\left[(\alpha-1)^3\frac{t^2}{4} + 3(\alpha-1)^2\frac{t^3}{24} + 7(\alpha-1)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right] + \ldots\end{aligned} \tag{38}$$

and

$$\begin{aligned}\overline{y}(t) =& \overline{y}_0(t) + \overline{y}_1(t) + \overline{y}_2(t) + \ldots,\\ =& (\alpha-1) + t - 2\left[(1-\alpha)^2 t + \frac{t^3}{12} + (1-\alpha)\frac{t^2}{2}\right]\\ &+ 8\left[(1-\alpha)^3\frac{t^2}{4} + 3(1-\alpha)^2\frac{t^3}{24} + 7(1-\alpha)\frac{t^4}{384}\right.\\ &\left.+\frac{t^5}{960}\right] + \ldots\end{aligned} \tag{39}$$

Considering the first, second and third term of our fuzzy series solution, the result is depicted in Figure 2.

5 Discussion

From the results, we may conclude that the solutions are inline with the concept of strongly generalized differentiability in general. For Case 1, the results diverge as the value of t increases, meanwhile, for Case 2, the results contracts as we choose higher values of t. This two behaviours of the solutions is in accord with the concept of strongly generalized differentiability. We can see that unlike other type of concept for fuzzy derivatives, the solution we have is not unique. This is true for this type of differentiability we chose. Even so, this situation allows engineers and researchers to choose the best solution according to the characteristic of the problem. We can also see that the solution obtain when $\alpha = 1$ is a crisp solution and it represents the solution for non-fuzzy DDEs.

We can also see that a switching point happens at $t = 0$. This means that for value of t greater than 0, the solution is (i)-differentiable while when the value of t is greater than 0, the solution is (ii)-differentiable. For further discussion on switching point, please refer [24].

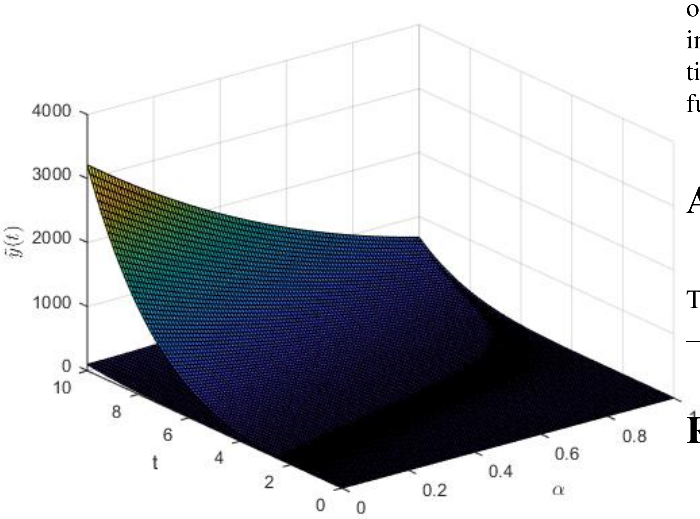

Figure 1. The solutions when $\widetilde{y}(t)$ is (i)-differentiable.

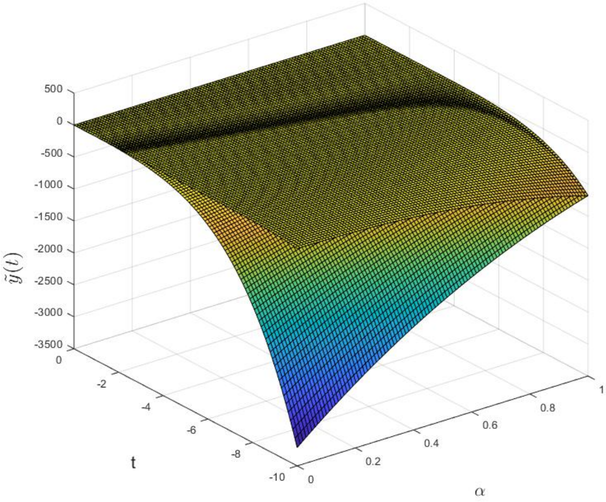

Figure 2. The solutions when $\widetilde{y}(t)$ is (ii)-differentiable.

6 Conclusion and Recommendations

In this paper, FSDM have been successfully used to find the solutions of fuzzy nonlinear DDEs. There might be some previous paper discussing on the same topic, but this paper considered a more recent type of fuzzy derivatives, the strongly generalized differentiability. Procedure for finding the solutions has been constructed and the numerical example has illustrated that this method is applicable in practice. It has also been shown that this research simplified the workings by reducing the fuzzy DDEs under the strongly generalized differentiability, where it can be seen in the literature, previous studies on the topic always involve tedious calculations. For future research, we recommend further discussion on the switching point of the fuzzy solutions obtained as well as the application on more complex fuzzy differential equations such as the one involving higher order derivatives. Other type of initial conditions such as nonlinear fuzzy number can also be used in the future to examine the behaviour of the solutions.

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