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Hankel Determinant $H_2(3)$ for Certain Subclasses of Univalent Functions

Andy Liew Pik Hern[1], **Aini Janteng**[1,*], **Rashidah Omar**[2]

[1]Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia

[2]Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara Cawangan Sabah, 88997 Kota Kinabalu, Sabah, Malaysia

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Abstract Let S to be the class of functions which are analytic, normalized and univalent in the unit disk $U = \{z : |z| < 1\}$. The main subclasses of S are starlike functions, convex functions, close-to-convex functions, quasi-convex functions, starlike functions with respect to (w.r.t.) symmetric points and convex functions w.r.t. symmetric points which are denoted by S^*, K, C, C^*, S_S^*, and K_S respectively. In recent past, a lot of mathematicians studied about Hankel determinant for numerous classes of functions contained in S. The qth Hankel determinant for $q \geq 1$ and $n \geq 0$ is defined by $H_q(n)$. $H_2(1) = a_3 - {a_2}^2$ is greatly familiar so called Fekete-Szegö functional. It has been discussed since 1930's. Mathematicians still have lots of interest to this, especially in an altered version of $a_3 - \mu {a_2}^2$. Indeed, there are many papers explore the determinants $H_2(2)$ and $H_3(1)$. From the explicit form of the functional $H_3(1)$, it holds $H_2(k)$ provided k from 1-3. Exceptionally, one of the determinant that is $H_2(3) = a_3a_5 - {a_4}^2$ has not been discussed in many times yet. In this article, we deal with this Hankel determinant $H_2(3) = a_3a_5 - {a_4}^2$. From this determinant, it consists of coefficients of function f which belongs to the classes S_S^* and K_S so we may find the bounds of $|H_2(3)|$ for these classes. Likewise, we got the sharp results for S_S^* and K_S for which $a_2 = 0$ are obtained.

Keywords Univalent Functions, Starlike Functions w.r.t. Symmetric Points, Convex Functions w.r.t. Symmetric Points, Hankel Determinant

1 Introduction

Let S denotes the class of normalized analytic univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where $z \in U = \{z : |z| < 1\}$.

In a few years, many mathematicians still looking the result on Hankel determinants for various subclasses of S. The qth Hankel determinant for the conditions of $q \geq 1$ and $n \geq 0$ (see [17]) as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{2}$$

This determinant has been studied by several authors. The classical Fekete-Szegö functional is $H_2(1)$. This functional has been studied since 1930s and until now (see [1–3, 7, 8, 10, 14, 16, 19, 20, 23]). Fekete and Szegö then farther generalized the estimation $\left|a_3 - \mu {a_2}^2\right|$ where μ is real and $f \in S$. For example, in [1] and [3], the researchers generalised the Fekete-Szegö problems by using different operator such as q-Ruscheweyh operator and Komatu integral operator. In fact, the determinants $H_2(2)$ and $H_3(1)$ have been discussed by many mathematicians (see [4, 5, 11–13, 18, 21, 22, 24]). Especially, from [5] and [18], mathematicians extend their ideas to bi-univalent functions instead of univalent functions.

Lately, Zaprawa [25] premeditated the determinant $H_2(3)$ for S. This included well known subclasses of S which are starlike functions, convex functions and functions whose derivative has a positive real part, denoted by S^*, K and R respectively. The determinant $H_2(3)$ is from the explicit form of $H_3(1)$ where $H_3(1) = H_2(3) + a_2(a_3a_4 - a_2a_5) + a_3H_2(2)$. Inspired by this, particularly, in section 3, we obtained the bounds of $|H_2(3)|$ for the class of starlike functions w.r.t. symmetric points, S^*_S and the class of convex functions w.r.t. symmetric points, K_S. Furthermore, in section 4, we obtained the bounds of $|H_2(3)|$ for the class of starlike functions w.r.t. symmetric conjugate points, S^*_{SC} and the class of convex functions w.r.t symmetric conjugate points, K_{SC}.

2 Preliminary Results

First, let P denotes the class of functions consisting of p, such that

$$p(z) = 1 + p_1z + p_2z^2 + ... = 1 + \sum_{n=1}^{\infty} p_nz^n. \tag{3}$$

which are regular in the open unit disc U and satisfy $Re\, p(z) > 0$ for any $z \in U$. Here $p(z)$ is called the Carathe̊dory function [6].

Lemma 2.1. *[9] If $p \in P$ then $|p_n| \leq 2$ for each n.*

Lemma 2.2. *[9] If $p \in P$ then the sharp estimate $|p_n - \mu p_kp_{n-k}| \leq 2$ holds for $n, k = 1, 2, ...,\ n > k$, $\mu \in [0, 1]$.*

From this Lemma 2.2, Livingston [15] proved $|p_n - p_kp_{n-k}| \leq 2$.

Lemma 2.3. *[25] If $p \in P$ then for $\mu \in \mathbb{R}$ the following sharp estimate holds*

$$\left|p_2 - \mu p_1{}^2\right| \leq \begin{cases} 2 - \mu|p_1|^2; & \mu \leq \frac{1}{2} \\ 2 - (1-\mu)|p_1|^2; & \mu \geq \frac{1}{2}. \end{cases} \tag{4}$$

Lemma 2.4. *[25] If $p \in P$ then*

$$|p_3 - p_1p_2| \leq \frac{1}{4}\left(8 - 2|p_1|^2 + |p_1|^3\right).$$

Remark 2.1 Considering $p(z^n)$, we can obtain related versions of these lemmas writing p_{kn} instead of p_k, $k = 1, 2, \cdots$. For example

$$\left|p_4 - \mu p_2{}^2\right| \leq \begin{cases} 2 - \mu|p_2|^2; & \mu \leq \frac{1}{2} \\ 2 - (1-\mu)|p_2|^2; & \mu \geq \frac{1}{2}. \end{cases} \tag{5}$$

and

$$|p_6 - p_2p_4| \leq \frac{1}{4}\left(8 - 2|p_2|^2 + |p_2|^3\right) \tag{6}$$

Lemma 2.5. *[25] If $p \in P$ then $\left|p_2p_4 - p_3{}^2\right| \leq 4$. The equality holds only for functions*

$$p(z) = \frac{1+z^3}{1-z^3}, \tag{7}$$

$$p(z) = \frac{1+z^2}{1-z^2} \tag{8}$$

and their rotations.

Lemma 2.6. *[25] If $p \in P$ then*

$$\left|p_2p_4 - p_3{}^2\right| \leq 4 - \frac{1}{2}|p_2|^2 + \frac{1}{4}|p_2|^3.$$

3 Results on $H_2(3)$ for S^*_S and K_S

In this section, we obtained the bounds of $H_2(3)$ for S^*_S and K_S. Let f be given by (1). Then

$$f \in S^*_S \Leftrightarrow \frac{2zf'(z)}{f(z) - f(-z)} \in P \tag{9}$$

and

$$f \in K_S \Leftrightarrow \frac{2\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'} \in P. \tag{10}$$

As $f \in S^*_S$, we have (9) and $\exists\, p \in P$, which yields

$$2zf'(z) = \left(f(z) - f(-z)\right)p(z) \tag{11}$$

for some $z \in U$. By comparing and equating the coefficients in (11) yields

$$a_2 = \frac{1}{2}p_1, \tag{12}$$

$$a_3 = \frac{1}{2}p_2, \tag{13}$$

$$a_4 = \frac{1}{4}\left(\frac{1}{2}p_1p_2 + p_3\right) \tag{14}$$

$$a_5 = \frac{1}{4}\left(\frac{1}{2}p_2{}^2 + p_4\right) \tag{15}$$

Therefore, if $f \in S^*_S$ then

$$H_2(3) = \frac{1}{64}\left(4p_2{}^3 + 8p_2p_4 - p_1{}^2p_2{}^2 - 4p_1p_2p_3 - 4p_3{}^2\right) \tag{16}$$

Similarly, since $f \in K_S$, we have (10) that $\exists\, p \in P$ such that

$$2\left(zf'(z)\right)' = \left(f(z) - f(-z)\right)'p(z) \tag{17}$$

for some $z \in U$.

By comparing and equating coefficients in (17) yields

$$a_2 = \frac{1}{4}p_1 \tag{18}$$

$$a_3 = \frac{1}{6}p_2 \tag{19}$$

$$a_4 = \frac{1}{16}\left(p_3 + \frac{1}{2}p_1p_2\right) \tag{20}$$

$$a_5 = \frac{1}{20}\left(p_4 + \frac{1}{2}p_2{}^2\right) \tag{21}$$

By substituting (18)-(21) into the definition of $H_2(3)$, we obtain

$$H_2(3) = \frac{1}{15360}\left(128p_2p_4 + 64{p_2}^3 - 60{p_3}^2\right) - 60p_1p_2p_3 - 15{p_1}^2{p_2}^2 \quad (22)$$

From the expression of $H_2(3)$, it is not easy to get the bounds of equations (16) and (22). Thus, we begin with a peculiar case.

Theorem 3.1. *Let f be given by (1) with assumption that $a_2 = 0$.*
(a). If $f \in S_S^*$ then $|H_2(3)| \leq 1$.
(b). If $f \in K_S$ then $|H_2(3)| \leq \frac{1}{15}$.

Proof. Let f be given by (1). With assumption that $a_2 = 0$, (12) gives $p_1 = 0$. If $f \in S_S^*$, then (16) yields

$$|H_2(3)| = \frac{1}{16}\left|{p_2}^3 + p_2p_4 + \left(p_2p_4 - {p_3}^2\right)\right| \quad (23)$$

Similarly to $f \in K_S$, (22) gives

$$|H_2(3)| = \frac{1}{15360}\left|64{p_2}^3 + 68p_2p_4 + 60\left(p_2p_4 - {p_3}^2\right)\right| \quad (24)$$

By applying Lemma 2.1 and 2.5, we obtain the bounds 1 and 1/15 for S_S^* and K_S respectively.

The expression (23) equals to 1 if and only if $|p_2| = 2$, $|p_4| = 2$ and $|p_2p_4 - {p_3}^2| = 4$. It is particularly for the rotations of (8) so we have the extremal functions $f(z) = \frac{z}{1-z^2}$ and its rotations.

This may apply to the estimation of $|H_2(3)|$ for K_S and this drives that for the particularly rotations of $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ there is $|H_2(3)| = 1/15$.

□

Next, two general theorems are proven.

Theorem 3.2. *Let f be given by (1).*
(a). If $f \in S_S^*$ then $|H_2(3)| \leq \frac{13}{16}$.
(b). If $f \in K_S$ then $|H_2(3)| \leq \frac{17}{240}$.

Proof. Let $f \in S_S^*$. From (16), it follows that

$$H_2(3) = \frac{1}{64}\left[4\left(p_2p_4 - {p_3}^2\right) + 4p_2(p_4 - p_1p_3) + 3{p_2}^3\right] + {p_2}^2\left(p_2 - {p_1}^2\right) \quad (25)$$

Similarly, if $f \in K_S$, then from (22) gives

$$H_2(3) = \frac{1}{15360}\left[60\left(p_2p_4 - {p_3}^2\right) + 60p_2(p_4 - p_1p_3) + 15{p_2}^2\left(p_2 - {p_1}^2\right) + 8p_2\left(p_4 - {p_2}^2\right) + 57{p_2}^3\right] \quad (26)$$

In both situations, we applied the triangle inequality and Lemmas 2.1, 2.2, 2.3 and 2.6. Let $q = |p_2|$, then from (25), we have

$$|H_2(3)| \leq \frac{1}{64}\left[4\left(4 - \frac{1}{2}|p_2|^2 + \frac{1}{4}|p_2|^3\right) + 4|p_2|(2) + 3|p_2|^3 + |p_2|^2(2)\right] \quad (27)$$

$$= \frac{1}{64}\left(4q^3 - 2q^2 + 10q + 16\right) \quad (28)$$

It achieves the maximum value in the range of $[0, 2]$ for $q = 2$. Thence, we may get the result as shown in the theorem. Similarly, from (26), we have

$$|H_2(3)| \leq \frac{1}{15360}\left[60\left(4 - \frac{1}{2}|p_2|^2 + \frac{1}{4}|p_2|^3\right) + 60|p_2|(2) + 15|p_2|^2(2) + 8|p_2|(2) + 57|p_2|^3\right] \quad (29)$$

$$= \frac{1}{15360}\left(72q^3 + 136q + 240\right) \quad (30)$$

which reaches the maximum value in the range of $[0, 2]$ for $q = 2$. The result follows. The proof of Theorem 3.2 has been shown.

□

4 Results on $H_2(3)$ for S_{SC}^* and K_{SC}

In this section, we acquired the bounds of $H_2(3)$ for S_{SC}^* and K_{SC}. Let f be given by (1). Then

$$f \in S_{SC}^* \Leftrightarrow \frac{2zf'(z)}{f(z) - \overline{f}(-\overline{z})} \in P \quad (31)$$

and

$$f \in K_{SC} \Leftrightarrow \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f}(-\overline{z})\right)'} \in P. \quad (32)$$

Similar lines of proof in Theorem 3.1 and Theorem 3.2. The bounds of $|H_2(3)|$ for S_{SC}^* and K_{SC} are given as follows:

Theorem 4.1. *Let f be given by (1) with assumption that $a_2 = 0$ and a_n are real numbers.*
(a). If $f \in S_{SC}^*$ then $|H_2(3)| \leq 1$.
(b). If $f \in K_{SC}$ then $|H_2(3)| \leq \frac{1}{15}$.

Next, we get two general theorems for S_{SC}^* and K_{SC}.

Theorem 4.2. *Let f be given by (1) with assumption that a_n are real numbers.*
(a). If $f \in S_{SC}^*$ then $|H_2(3)| \leq \frac{13}{16}$.
(b). If $f \in K_{SC}$ then $|H_2(3)| \leq \frac{17}{240}$.

5 Conclusions

In conclusion, this article has obtained the bounds of Hankel determinant $H_2(3) = a_3a_5 - {a_4}^2$ for functions f belongs to the class of S_S^*, K_S, S_{SC}^* and K_{SC}.

Conflicts of Interest

We as authors affirm that, there are no conflicts of interest as regards to the publication of this article.

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