

Logistic Map on the Ring of Multisets and Its Application in Economic Models

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Abstract In this paper, we extend complex polynomial dynamics to a set of multisets endowed with some ring operations (the metric ring of multisets associated with supersymmetric polynomials of infinitely many variables). Some new properties of the ring of multisets are established and a homomorphism to a function ring is constructed. Using complex homomorphisms on the ring of multisets, we proposed a method of investigations of polynomial dynamics over this ring by reducing them to a finite number of scalar-valued polynomial dynamics. An estimation of the number of such scalar-valued polynomial dynamics is established. As an important example, we considered an analogue of the logistic map, defined on a subring of multisets consisting of positive numbers in the interval $[0, 1]$. Some possible application to study the natural market development process in a competitive environment is proposed. In particular, it is shown that using the multiset approach, we can have a model that takes into account credit debt and reinvestments. Some numerical examples of logistic maps for different growth rate multiset $[r]$ are considered. Note that the growth rate $[r]$ may contain both “positive” and “negative” components and the examples demonstrate the influences of these components on the dynamics.

Keywords Logistic Map, Multisets, Supersymmetric Polynomials, Chaos, Market Development Model

1 Introduction

The difference equation $t_{n+1} = rt_n(1 - t_n)$ is known as the logistic map, where $n \in \mathbb{Z}_+$ is the discrete time, $r \in [0, 4]$ is the growth rate and the sequence $(t_n)_{n=0}^\infty$, $0 < t_n < 1$ describes the dynamic of the “population” value t_n at any time n . The logistic map was considered first by R. May[1] in 1976 for

a discrete-demographic model of biological populations. But this equation is well applicable in Economics, Sociology, Cryptography, Physics and etc. There are a lot of publications in different areas of sciences with using dynamic models based on chaotic properties of the logistic map. For example, economic models which involve cycles and chaos were considered in [2]. Applications of the logistic map for investigations of exogenous shocks in economic models were studied in [3]. In [4] was introduced a logistic map with memory and studied its applications in economic models. Some other generalizations of the logistic map can be found in [5]. Complex behavior of two combined logistic models was investigated in [6]. Dynamics of the logistic difference equation on fuzzy numbers were studied in [7, 8].

It is well known (see e. g. [9, 6]) that if $r \in [0, 1)$, the population will eventually die, if $r \in [1, 3]$, the population will approach $\frac{r-1}{r}$. If $3 < r \leq 1 + \sqrt{6}$, from almost all initial conditions, the population will have permanent oscillations between two values (the bifurcation points). Practically, the logistic map becomes chaotic when $r \geq 3.5699$. The number of bifurcation points rapidly grows, when r approaches 3.85. If $3.85 < r < 4$, the population has a chaotic behaviour with periodic windows in the interval $[0, 1]$. If $r = 4$, t_n has a chaotic behavior on the whole interval $[0, 1]$; the dynamics has lost near-all its determinism and the population evolves as a random number generator.

In this paper, we consider the logistic map defined on the ring of multisets, which was introduced in [10]. The ring of multisets can be considered as a natural domain of supersymmetric polynomials of infinitely many variables. Supersymmetric polynomials of infinitely many variables give us some kind of generalization of symmetric polynomials. Symmetric polynomials and analytic functions on infinite dimensional Banach spaces were studied in [11, 12, 13, 14, 15, 16]. In particular, there were considered some semiring algebraic operations on the spectra of algebras of symmetric polynomials which can

be applied to a set of multisets. Supersymmetric polynomials allow to extend this operations to a ring operations on a larger set [10]. Some further investigations in this direction can be found in [17]. Algebras of analytic functions, generated by a sequence of polynomials on a Banach space, in the general case were investigated in [18]. For the general information about algebraic theory of symmetric functions we refer the reader to [19].

In Section 2 we introduce the ring of multisets and consider its basic properties. In Section 3 we investigate general properties of polynomial dynamics on the ring of multisets. In Section 4 we consider the logistic map dynamic on the ring of multisets and its possible application to economical models.

2 Ring of Multisets

Let us recall that a *multiset* is an unordered finite collection of nonzero numbers which can have multiple instances for each of its elements. Let us denote by \mathcal{M}_0^+ the set of all finite multisets. That is, if $x \in \mathcal{M}_0^+$, then we can write $x = (x_1, x_2, \dots, x_m)$ for some m , where x_i are, in the general case, nonzero complex numbers. Since we are taking into account only nonzero numbers, we can write $(x_1, \dots, x_m) = (x_1, \dots, x_m, 0) = (x_1, \dots, x_m, 0, 0, \dots)$ and $(0) = (0, 0, \dots)$ will denote the empty multiset. There are natural operations of addition and multiplication on \mathcal{M}_0^+ . For $x = (x_1, \dots, x_m)$, $x' = (x'_1, \dots, x'_k)$ in \mathcal{M}_0^+ we set

$$x \bullet x' = (x_1, \dots, x_m) \cup (x'_1, \dots, x'_k) = (x_1, \dots, x_m, x'_1, \dots, x'_k)$$

and

$$x \diamond x' = (x_i x'_j)_{i=1, j=1}^{m, k}.$$

It is easy to check that these operations are associative, commutative and we have the distributive law. So $\mathcal{M}_0^+(\bullet, \diamond)$ is a semiring (see [20]). To get a ring, we need to have the invertibility of operation “ \bullet ”. Let us consider the Cartesian product $\mathcal{M}_0^+ \times \mathcal{M}_0^+$ and represent each element $u \in \mathcal{M}_0^+ \times \mathcal{M}_0^+$ by $u = (y|x)$, where $x, y \in \mathcal{M}_0^+$. We can extend the introduced operations to $\mathcal{M}_0^+ \times \mathcal{M}_0^+$ by the following way: let $u = (y|x)$ and $u' = (y'|x')$. Then

$$u \bullet u' = (y \bullet y'|x \bullet x')$$

and

$$u \diamond u' = ((x \diamond y') \bullet (x' \diamond y)) | ((x \diamond x') \bullet (y \diamond y')).$$

Let us define the following relation of equivalence on $\mathcal{M}_0^+ \times \mathcal{M}_0^+$:

$$u \sim u' \text{ if and only if there exists } a \in \mathcal{M}_0^+ \text{ such that } u = u' \bullet (a|a) \text{ or } u' = u \bullet (a|a). \text{ In particular, } (a|a) \sim 0 = (0|0) \text{ for every } a \in \mathcal{M}_0^+.$$

We denote by \mathcal{M}_0 the quotient set $\mathcal{M}_0^+ \times \mathcal{M}_0^+ / \sim$ and by $[u]$ the class of equivalence containing u . From [10] we know that the operations

$$[u] + [v] := [u \bullet v]$$

and

$$[u][v] = [u \diamond v]$$

are well defined on \mathcal{M}_0 and $(\mathcal{M}_0, +, \cdot)$ is a commutative ring with the unit $\mathbb{1} = (0|1)$. Let u^- denotes $(x|y)$ if $u = (y|x) \in \mathcal{M}_0^+ \times \mathcal{M}_0^+$. Then

$$[u] + [u^-] = [u \bullet u^-] = 0.$$

That is, $[u^-] = -[u]$.

For every $k \in \mathbb{N}$ we denote

$$T_k([u]) = \sum_n x_n^k - \sum_n y_n^k.$$

From [10] we have that $[u] = [v]$ if and only if $T_k([u]) = T_k([v])$ for all $k \in \mathbb{N}$. Moreover, operators T_k are ring homomorphisms from \mathcal{M}_0 to \mathbb{C} , that is,

$$T_k([u] + [v]) = T_k([u]) + T_k([v])$$

and

$$T_k([u][v]) = T_k([u])T_k([v]),$$

$[u], [v] \in \mathcal{M}_0, k \in \mathbb{N}$. Note that polynomials T_k form the algebra of supersymmetric polynomials [10] and are difference of *symmetric* polynomials

$$F_k(x) = \sum x_n^k$$

and

$$F_k(y) = \sum y_n^k.$$

In [10] is proved that \mathcal{M}_0 is a metric space with respect to the metric

$$\rho([u], [v]) = \|[u] - [v]\|,$$

where

$$\|[u]\| = \inf \left\{ \sum_i |x_i| + \sum_j |y_j| : [(y|x)] = [u] \right\}.$$

The completion of \mathcal{M}_0 with respect to ρ is denoted by \mathcal{M} . The operation of \mathcal{M}_0 and complex homomorphisms $T_k, k = 1, 2, \dots$ are continuous with respect to the metric ρ and can be continuously extended to the completion \mathcal{M} (see [10]). On the other hand, there are discontinuous complex homomorphisms of \mathcal{M}_0 which can not be extended to \mathcal{M} . For example, let

$$T_0([u]) = \sum_n x_n^0 - \sum_n y_n^0,$$

where we assume that $0^0 = 0$. Then

$$T_0(y_1, \dots, y_j | x_1, \dots, x_m) = m - j$$

if all numbers $x_i, 1 \leq i \leq m$ and $y_l, 1 \leq l \leq j$ are nonzero. So T_0 is well defined on \mathcal{M}_0 and simple calculations show that T_0 is additive and multiplicative. But the completion \mathcal{M} of \mathcal{M}_0 contains elements with infinitely many “coordinates” like $u = (0|1/2, 1/4, \dots, 1/2^n, \dots)$, where T_0 is not defined.

We need, also, describe some homomorphisms and subrings of \mathcal{M}_0 . For every $k \in \mathbb{N}$ the map

$$\Phi_k([u]) = [(y_1^k, \dots, y_j^k | x_1^k, \dots, x_m^k)], \tag{1}$$

where $u = (y_1, \dots, y_j | x_1, \dots, x_m)$, is a ring homomorphism from \mathcal{M}_0 to \mathcal{M}_0 . We will use also notation $[u]^{(k)} := \Phi_k([u])$. If U is a subset of \mathbb{C} which is closed with respect to the multiplication in \mathbb{C} , then

$$\mathcal{M}_0^U = \{[(y|x)] : x_i \in U, y_i \in U, i \in \mathbb{N}\}$$

is a subring in \mathcal{M}_0 . The case when $U = \Delta := [0; 1]$ is important for us. Note that if $0 < \varepsilon < 1$, then $\mathcal{M}_0^{\Delta_\varepsilon}$ is an ideal in \mathcal{M}_0^Δ , where $\Delta_\varepsilon = [0; \varepsilon]$. Thus setting

$$\gamma_\varepsilon = \begin{cases} a & \text{if } a > \varepsilon, \\ 0 & \text{if } a \leq \varepsilon, \end{cases} \quad a \in \mathbb{R}_+$$

we have that the following map

$$\Gamma_\varepsilon(u) = [(\gamma_\varepsilon(y_1), \dots, \gamma_\varepsilon(y_j), \dots | \gamma_\varepsilon(x_1), \dots, \gamma_\varepsilon(x_m), \dots)]$$

is a homomorphism from \mathcal{M}_0^Δ onto the quotient ring $\mathcal{M}_0^\Delta / \mathcal{M}_0^{\Delta_\varepsilon}$.

Let us observe another representation of \mathcal{M}_0 . For every multiset $u = (y|x) \in \mathcal{M}_0$

$$u = (\underbrace{y_1, \dots, y_1}_{m_1}, \dots, \underbrace{y_j, \dots, y_j}_{m_j} | \underbrace{x_1, \dots, x_1}_{n_1}, \dots, \underbrace{x_i, \dots, x_i}_{n_i}), \tag{2}$$

where all y_k are mutually different and all x_l are mutually different, we assign a map \mathfrak{A}_u from \mathbb{C} to \mathbb{Z} defined by the following way: $\mathfrak{A}_u(0) = 0$ and

$$\mathfrak{A}_u(\lambda) = \begin{cases} n_l - m_k & \text{if } \lambda = x_l \text{ and } \lambda = y_k \\ n_l & \text{if } \lambda = x_l \text{ and } \lambda \notin y \\ -m_k & \text{if } \lambda \notin x \text{ and } \lambda = y_k \\ 0 & \text{otherwise,} \end{cases}$$

$l = 1, \dots, i; \quad k = 1, \dots, j$ if $\lambda \neq 0$. In other words, $\mathfrak{A}_u(\lambda)$ counts the number of entries of λ in x minus the number of entries of λ in y . From the definition it follows that $\mathfrak{A}_u = \mathfrak{A}_v$ if and only if $[u] = [v]$. Also, it is easy to see that $\mathfrak{A}_{u \bullet v}(\lambda) = \mathfrak{A}_u(\lambda) + \mathfrak{A}_v(\lambda)$, $u, v \in \mathcal{M}_0$. From this point of view we can consider elements of \mathcal{M}_0^Δ as some discrete analogues of fuzzy numbers with the support in Δ .

Proposition 2.1 For every $[u] \in \mathcal{M}_0$ and $k = 0, 1, 2, \dots$ the following equality holds

$$T_k([u]) = \sum_{\lambda \in \mathbb{C}} \lambda^k \mathfrak{A}_u(\lambda).$$

Proof. Note first that the sum on the right side consists only of a finite numbers of terms. So, it is well defined for every $[u] \in \mathcal{M}_0$. Let u be as in (2), then

$$\sum_{\lambda \in \mathbb{C}} \lambda^k \mathfrak{A}_u(\lambda) = \sum_{s=1}^i n_s x_s^k - \sum_{l=1}^j m_l x_l^k = T_k([u]).$$

□

3 Dynamics of Complex Polynomials on \mathcal{M}_0

Let $q: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of a complex variable t . Then the dynamic of q with the initial condition $t_0 \in \mathbb{C}$ is the sequence

$$t_1 = q(t_0), \quad t_2 = q(t_1), \quad \dots, \quad t_{n+1} = q(t_n), \quad \dots,$$

that is, $t_{n+1} = \underbrace{q \circ q \circ \dots \circ q}_{n+1}(t_0)$. Let $[a_0], [a_1], \dots, [a_m]$ be some fixed elements in \mathcal{M}_0 . We consider a formal polynomial $p([u]), p: \mathcal{M}_0 \rightarrow \mathcal{M}_0$ defined by

$$p([u]) = [a_m][u]^m + [a_{m-1}][u]^{m-1} + \dots + [a_0].$$

So the dynamic of p with the initial condition $[u_0] \in \mathcal{M}_0$ is the sequence

$$[u_1] = p([u_0]), [u_2] = p([u_1]), \dots, [u_{n+1}] = p([u_n]), \dots$$

Proposition 3.1 Let φ be a ring homomorphism from \mathcal{M}_0 to \mathbb{C} . Then

$$\varphi([u_{n+1}]) = \underbrace{\tilde{\varphi}(p) \circ \tilde{\varphi}(p) \circ \dots \circ \tilde{\varphi}(p)}_{n+1}(\varphi([u_0])), \tag{3}$$

where $\tilde{\varphi}(p)$ is a polynomial on \mathbb{C} defined by

$$\tilde{\varphi}(p)(t) = \varphi([a_m])t^m + \varphi([a_{m-1}])t^{m-1} + \dots + \varphi([a_0]).$$

Proof. Clearly (3) is true for $n = 0$. So we can use the simple induction. □

Let $[u] \in \mathcal{M}_0$ and $u = (y|x) = (y_1, \dots, y_m | x_1, \dots, x_k)$. We denote by $\text{card}(x) = k$ the cardinality of x , $\text{card}(y) = m$ and $\text{card}(u) = k + m$. From [10] we know that any element $[u] \in \mathcal{M}$ is completely defined by the number sequence $(T_n(u))_{n=1}^\infty$. On the other hand, if $[u] \in \mathcal{M}_0$, that is a representative $u \in [u]$ is finite, then $[u]$ can be defined by a finite sequence $(T_n(u))_{n=1}^N$, where the number N should depend of the cardinality of u .

A representative $v = (y|x)$ of $[u] \in \mathcal{M}_0$ is called *irreducible* if for every $v' \in [u]$, $v' = (y'|x')$ there is $a \in \mathcal{M}_0^+$ such that $x' = x \bullet a$ and $y' = y \bullet a$.

Proposition 3.2 For every $[u] \in \mathcal{M}_0$ there is an irreducible representative $v = (y|x) \in [u]$ and if $v' = (y'|x')$ is another irreducible representative of $[u]$, then $x = x'$ and $y = y'$ as multisets.

Proof. According to [10, Corollary 3] any element $u = (y|x) = (y_1, \dots, y_m | x_1, \dots, x_k)$ can be represented as a rational function

$$w(u)(t) = \frac{(1 + x_1 t)(1 + x_2 t) \dots (1 + x_k t)}{(1 + y_1 t)(1 + y_2 t) \dots (1 + y_m t)}. \tag{4}$$

Moreover, if $u' = (y'|x') = (y'_1, \dots, y'_s | x'_1, \dots, x'_r)$, then

$$w(u \bullet v)(t) = w(u)(t)w(v)(t) \tag{5}$$

and

$$w(u \diamond v)(t) = \frac{\prod_{i=1}^k \prod_{j=1}^r (1 + x_i x'_j t) \prod_{i=1}^m \prod_{j=1}^s (1 + y_i y'_j t)}{\prod_{i=1}^k \prod_{j=1}^s (1 + x_i y'_j t) \prod_{i=1}^m \prod_{j=1}^r (1 + y_i x'_j t)}. \quad (6)$$

From (4) and (5) we can see that the representative u of $[u]$ is irreducible if and only if the fraction in (4) is irreducible. Clearly that the irreducible fraction is unique. \square

Note that from the representation $[u] \mapsto w(u)(t)$ it follows that a polynomial dynamic on \mathcal{M}_0 is equivalent to a dynamics defined on rational functions (4) with “addition” (5) and “multiplication” (6).

Theorem 3.3 *Let $[u] \in \mathcal{M}_0$ and $u = (y|x) = (y_1, \dots, y_m|x_1, \dots, x_k)$ be the irreducible representative of $[u]$. Then the multisets x and y are completely determined by the following finite sequence of numbers $(T_1([u]), T_2([u]), \dots, T_{k+m}([u]))$. That is, if $u' = (y'|x') = (y'_1, \dots, y'_m|x'_1, \dots, x'_k)$ is the irreducible representative of $[u']$, then $[u] = [u']$ if and only if $T_j([u]) = T_j([u'])$, $j = 1, 2, \dots, k + m = \text{card}(u)$.*

Proof. Let us suppose that $T_j([u]) = T_j([u'])$, $j = 1, 2, \dots, m + k$. We set $[v] = [u] - [u'] = [(y \bullet x'|x \bullet y')]$. So $T_j([v]) = 0$ and thus $F_j(y \bullet x') = F_j(x \bullet y')$ for all $j = 1, 2, \dots, m + k$. It is well known that symmetric polynomials F_1, \dots, F_{m+k} form an algebraic basis in the algebra of symmetric polynomials on \mathbb{C}^{m+k} . Since F_j are symmetric, each F_j , $j > m + k$ can be represented as an algebraic combination of F_1, \dots, F_{m+k} . So $F_j(y \bullet x') = F_j(x \bullet y')$ for all $j \in \mathbb{N}$. But it implies that $T_j([u]) = T_j([u'])$ for all $j \in \mathbb{N}$. Hence $[u] = [u']$. The converse implication is trivial. \square

We can use, also, the alternative power map $[u] \mapsto [u]^{(k)} = \Phi_k([u])$, where $\Phi_k([u])$ is defined in (1), to introduce a polynomial dynamics on the ring of multisets \mathcal{M}_0 .

Proposition 3.4 *The power map $[u]^{(k)} = \Phi_k([u])$ has the following properties:*

1. $([u] + [v])^{(k)} = [(u)]^{(k)} + [(v)]^{(k)}$;
2. $([u][v])^{(k)} = [(u)]^{(k)}[(v)]^{(k)}$;
3. $(-[u])^{(k)} = -[u]^{(k)}$;
4. $T_n([(u)]^{(k)}) = T_{kn}([(u)])$.

Proof. The proof obviously follows from definitions. \square

4 Logistic Map and Market Development Model

Now we consider the dynamic of logistic map

$$t_{n+1} = r t_n (1 - t_n) \quad (7)$$

(the scalar case) and the corresponding dynamic on $\mathcal{M}_0^\Delta \subset \mathcal{M}_0$:

$$[u_{n+1}] = [r][u_n]([(0|1]) - [u_n]). \quad (8)$$

We call (8) *the logistic map on the ring of multisets \mathcal{M}_0^Δ* .

According to Theorem 3.3, in order to describe the dynamic of $[u_n]$ we have describe the dynamics of $T_k([u_n])$, $k = 1, 2, \dots, N$ for a finite but maybe large number N .

Let $[u_n] = [(y_n|x_n)]$ and $[r] = [(q|z)]$, where $q = (q_1, q_2, \dots)$ and $z = (z_1, z_2, \dots)$ are some finite multisets. It is naturally to assume that $0 < T_k([r]) \leq 4$ for every $k \in \mathbb{N}$. Therefore, we have $0 \leq q_i \leq 1$ and $0 \leq z_i \leq 1$ for all $i \in \mathbb{N}$, that is the reason why (8) is defined on \mathcal{M}_0^Δ .

The scalar logistic map (7) good describes the dynamics of biological populations that can use for their growing the really existing available resources. In particular, if $t_0 = 1$, then $t_n = 0$, $n > 0$ and the population will die because it will exhausts the whole supply of the resources at the moment of time $n = 1$. In economic systems of market development, subjects of economic activity can use a “virtual” money which can be obtained as credit resources. Some times the credit resources are comparable with the marked volume. In this situation the multiset logistic map may be more appropriate for modelling of the market dynamic. Let us consider the following example.

Example 4.1 *Let $r = [(0|1, 1)]$, $u_0 = [(0|1/2, 1/2)]$. Then $T_1([u_0]) = 1$, and so $T_1([u_n]) = 0$ for $n > 0$. But $T_k([u_0]) = 1/2^{k-1}$ and so $T_k([u_n])$ has an equilibrium at $1/2$ for all $k > 1$. That is, $[u_n]$ has a nontrivial and stable dynamic.*

Let us consider the following model of the market development in a competitive environment. We suppose that at any time n the maximal possible income on a given market is equal to 1. Any element $[u_n] = [(y_{n1}, \dots, y_{nj}|x_{n1}, \dots, x_{nm})] \in \mathcal{M}_0^\Delta$ represents a state of the market at time n . The numbers x_{n1}, \dots, x_{nm} represent income transactions at time n when manufactured products are sold and numbers y_{n1}, \dots, y_{nj} represent expense transactions (production costs) at time n . The numbers y_{n1}, \dots, y_{nj} can be considered as credit debts. Note that every number-valued function f of the state of the market at time n does not depend of the order of entries x_{n1}, \dots, x_{nm} and does not depend of the order of entries y_{n1}, \dots, y_{nj} as well, and it is invariant with respect to the transform

$$\begin{aligned} (y_{n1}, \dots, y_{nj}|x_{n1}, \dots, x_{nm}) &\mapsto \\ &\mapsto (y_{n1}, \dots, y_{nj}, a|x_{n1}, \dots, x_{nm}, a). \end{aligned}$$

Hence, if f is a polynomial, then it is supersymmetric and so can be represented as an algebraic combination of polynomials T_1, \dots, T_{m+j} . From this point of view, $T_1([u_n])$ is the total profit of the market at time n , which is equal to the total income minus the total credit debt.

The possible reinvestment of costs x_{ni} is proportional to the income and to the part of these costs on the market. So, it is proportional to x_{ni}^2 . Thus, we can assume that $T_2([u_n])$ describes the total profit obtained by the reinvestment at time n . By the way, $T_3([u_n])$ describes the total profit obtained by the reinvestment of the reinvestment and so on.

Example 4.1 shows that even if the market is completely exhausted by several actors, using credit costs we can obtain a stable profit from the reinvestment.

The multiset $[r] \in \mathcal{M}_0^\Delta$ is the growth rate and has a “positive” part (z_1, z_2, \dots) and a “negative” part (q_1, q_2, \dots) . The

“negative” part is responsible to return some part of the profit to costumers. It may be considered as a tax which returns some money to support the capacity of the market. We suppose that $T_k([z]) \geq 0$ for every k . If all components of $[r]$ are less than 1, then $T_k([r])$ tends to zero as $k \rightarrow \infty$.

Example 4.2 *The following example shows that for some special choice of $[r]$ we can guaranty that only some finite number of functions T_k will be essential for the behavior of $[u_n]$ what is natural for economic systems. Let*

$$[r] = \left[\left(0 \mid \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right].$$

Then $T_1([r]) = 5/2$ and for almost all initial conditions

$$T_1([u_n]) \rightarrow \frac{T_1([r]) - 1}{T_1([r])} = \frac{3}{5} \quad \text{as } n \rightarrow \infty,$$

$T_2([r]) = 5/4$ and for almost all initial conditions

$$T_2([u_n]) \rightarrow \frac{T_2([r]) - 1}{T_2([r])} = \frac{1}{5} \quad \text{as } n \rightarrow \infty,$$

and $T_k([r]) < 1$ for $k > 2$ and so $T_k([u_n]) \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, the situation can be different in the general case. If $[r] = [(0|1, 1)]$ and $[u_0]$ is such that $T_1([u_0]) \neq 0$, then for every k , $T_k([u_n])$ will approach $1/2$ for almost all initial conditions. If $[r] = [(0|1, 1, 1, 0.9)]$, then $T_1([u_n])$ has a chaotic behavior, but for $k > 10$, $T_k([u_n])$ has just one equilibrium point. If $[r] = [(0.9|1, 1, 1, 1)]$, then $T_1([u_n])$ is stable, but for large k , $T_k([u_n])$ has a chaotic behavior.

The mapping $T_0([u_n])$ counts the number of “positive” transactions minus the number of “negative” transactions at the moment n and can be computed in the general case. If

$$[u] = [(y_1, \dots, y_j | x_1, \dots, x_m)]$$

and

$$[r] = [(q_1, \dots, q_s | z_1, \dots, z_l)],$$

then

$$T_0(u) = l(m - j) + s(j - m) + (s - l)(m^2 + j^2) + 2mj(l - s).$$

In particular, it is equal to zero, if $m = j$ and $s = l$.

5 Remarks

(i) Note that equation (8) is not a unique way to extend the logistic map (7) to multisets. If we rewrite (7) by

$$t_{n+1} = rt_n - rt_n^2,$$

then we can consider dynamic

$$[u_{n+1}] = [r][u_n] - [r][u_n]^{(2)}, \tag{9}$$

where the power function $[u] \mapsto [u]^{(2)} = \Phi_2([u])$ is defined by (1) for $k = 2$. It is easy to see that $[u]^{(2)} \neq [u]^2$ in general. So the dynamic (9) is not equivalent to (8) and is interesting in

the mathematical sense. However, the simple example $[u_0] = (0|1/2)$, $[r] = (0|1, 1)$ shows that $T_1([u_3]) < 0$ and so (9) does not describe a reasonable model of development.

(ii) The dynamic of logistic map (8) can be defined also on the quotient ring $\mathcal{M}_0^\Delta / \mathcal{M}_0^{\Delta \varepsilon}$ for some small positive ε . Practically it means that all transactions x_i and y_j are automatic vanishing if they are less than ε .

6 Conclusion

This paper is an invitation to investigations of polynomial dynamics on multisets and their applications in economic models. As an example, we considered the logistic map dynamic on a ring of multisets. We can see that the logistic map on multisets is a generalization of the classical scalar logistic map and the multiset logistic map dynamic contains the scalar dynamic. Thus, the multiset dynamic is applicable to practical problems where one can use classical dynamic systems. However, the proposed approach allows as to use the logistic map dynamic for modeling the natural market development process which takes into account credit debt and reinvestments. We show that in this case, the growth rate $[r]$ may contain both “positive” and “negative” components and demonstrate examples of the influences of these components on the dynamic.

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REFERENCES

- [1] R. M. May. Simple mathematical models with very complicated dynamics. *Nature*. Vol. 261 No. 5560, (1976) 459-467. doi:10.1038/261459a0
- [2] D. Kelsey. *The Economics of Chaos or the Chaos of Economics*, Oxford Economic Papers, Vol. 40, (1988) 1-31.
- [3] M. Akhmet, Z. Akhmetova, M. Fen. Chaos in Economic Models with Exogenous Shocks, *Journal of Economic Behavior & Organization*, Vol. 106(C), (2014) 95–108.
- [4] V.V. Tarasova, V.E. Tarasov. Logistic map with memory from economic model, *Chaos, Solitons and Fractals* Vol. 95, (2017) 84–91. <https://doi.org/10.1016/j.chaos.2016.12.012>
- [5] A. Radwan. On some generalized discrete logistic maps, *Journal of Advanced Research*, Vol. 4 (2013), 163-171.
- [6] R. Lopez-Ruiz, D. Fournier-Prunaret. Complex behaviour in a discrete logistic model for the symbiotic interaction of two

- species, *Math. Biosci. Eng.*, Vol. 1, No. 2, (2004) 307–324. doi:10.1016/j.jmaa.2012.04.087
- [7] A. Khastan. Fuzzy Logistic difference equation, *Iranian J. Fuzzy Syst.* Vol. 15 No. 7, (2018) 55-66.
- [8] Q. Zhang, W. Zhang, J. Liu, Y. Shao. On a fuzzy logistic difference equation, *Wseas transaction on mathematics*, Vol. 13 (2014), 282-285.
- [9] G. Boeing. Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals, Self-Similarity and the Limits of Prediction, *Systems*, Vol. 4, No 4, (2016) 37. doi:10.3390/systems4040037
- [10] F. Jawad, A. Zagorodnyuk. Supersymmetric Polynomials on the Space of Absolutely Convergent Series, *Symmetry* Vol. 11 No 9, (2019) 1111. doi: 10.3390/sym11091111
- [11] R. Alencar, R. Aron, P. Galindo, A. Zagorodnyuk. Algebra of symmetric holomorphic functions on ℓ_p , *Bull. Lond. Math. Soc.* Vol. 35, (2003) 55-64.
- [12] R. Aron, P. Galindo, D. Pinasco, I. Zalduendo. Group-symmetric holomorphic functions on a Banach space, *Bull. Lond. Math. Soc.* Vol 48 No 5, (2016) 779–796.
- [13] I. Chernega, P. Galindo, A. Zagorodnyuk. Some algebras of symmetric analytic functions and their spectra, *Proc. Edinb. Math. Soc.*, Vol. 55, (2012) 125-142. doi:10.1017/S0013091509001655
- [14] I. Chernega, P. Galindo, A. Zagorodnyuk. The convolution operation on the spectra of algebras of symmetric analytic functions. *J. Math. Anal. Appl.*, Vol. 395 No 2, (2012) 569-577.
- [15] I. Chernega, P. Galindo, A. Zagorodnyuk. A multiplicative convolution on the spectra of algebras of symmetric analytic functions, *Rev. Mat. Complut.*, Vol. 27 No. 2, (2014) 575-585. doi:10.1007/s13163-013-0128-0
- [16] I. Chernega, O. Holubchak, Z. Novosad, A. Zagorodnyuk. Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces, *European Journal of Mathematics* Vol 6, (2020) 153–163. https://doi.org/10.1007/s40879-019-00390-z
- [17] F. Jawad, H. Karpenko, A. Zagorodnyuk. Algebras generated by special symmetric polynomials on ℓ_1 , *Carpathian Math. Publ.*, Vol. 11, No. 2, (2019) 335–344. doi:10.15330/cmp.11.2.335-344
- [18] S. I. Halushchak. Spectra of some algebras of entire functions of bounded type, generated by a sequence of polynomials, *Carpathian Math. Publ.*, Vol. 11, No. 2, (2019) 311–320. doi:10.15330/cmp.11.2.311-320
- [19] I. G. Macdonald. *Symmetric Functions and Orthogonal Polynomials*, University Lecture Series, Vol. 12, AMS, Providence, RI, 1997.
- [20] I.V. Chernega. A semiring in the spectrum of the algebra of symmetric analytic functions in the space ℓ_1 . *J. Math. Sci.* Vol. 212, (2016) 38-45. doi:10.1007/s10958-015-2647-3. Translated from *Matematychni Metody ta Fyzyko-Mekhanichni Polya*, Vol 57, (2014) 35-40.