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3-Vertex Friendly Index Set of Graphs

Girija K. P., Devadas Nayak C*, Sabitha D'Souza, Pradeep G. Bhat

Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India

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Abstract Graph labeling is an assignment of integers to the vertices or the edges, or both, subject to certain conditions. In literature we find several labelings such as graceful, harmonious, binary, friendly, cordial, ternary and many more. A friendly labeling is a binary mapping such that $|v_f(1) - v_f(0)|\leq 1$ where $v_f(1)$ and $v_f(0)$ represents number of vertices labeled by 1 and 0 respectively. For each edge uv assign the label $|f(u) - f(v)|$, then the function f is cordial labeling of G if $|v_f(1) - v_f(0)|\leq 1$ and $|e_{f^*}(1) - e_{f^*}(0)|\leq 1$, where $e_{f^*}(1)$ and $e_{f^*}(0)$ are the number of edges labeled 1 and 0 respectively. A friendly index set of a graph is $\{|e_{f^*}(1) - e_{f^*}(0)|: f^* \ runs\ over\ all\ friendly\ labeling\ f\ of\ G\}$ and it is denoted by $FI(G)$. A mapping $f : V(G) \to \{0, 1, 2\}$ is called ternary vertex labeling and $f(v)$ represents the vertex label for v. In this article, we extend the concept of ternary vertex labeling to 3-vertex friendly labeling and define 3-vertex friendly index set of graphs. The set $FI_{3v}(G) = \{|e_{f^*}(i) - e_{f^*}(j)|: f^* \ runs\ over\ all\ 3-vertex\ friendly\ labeling\ f\ for\ all\ i, j \in \{0, 1, 2\}\}$ is referred as 3-vertex friendly index set. In order to achieve $FI_{3v}(G)$, number of vertices are partitioned into $\{V_0, V_1, V_2\}$ such that $||V_i|-|V_j||\leq 1$ for all $i, j = 0, 1, 2$ with $i \neq j$ and label the edge uv by $|f(u)-f(v)|$ where $f(u), f(v) \in \{0, 1, 2\}$. In this paper, we study the 3-vertex friendly index sets of some standard graphs such as complete graph K_n, path P_n, wheel graph W_n, complete bipartite graph $K_{m,n}$ and cycle with parallel chords PC_n.

Keywords Friendly Labeling, Ternary Vertex Labeling, 3-Vertex Friendly Labeling, 3-Vertex Friendly Index Set

1 Introduction

In this paper, all graphs are assumed to be simple, finite, connected and undirected. All undefined notations and concepts are found in [7]. For a graph $G = (V, E)$, graph labeling is an assignment on vertices or edges or both to integers. For more information on graph labeling, refer to [6].

A mapping $f : V(G) \to \{0, 1\}$ of a graph G is a binary vertex labeling. A labeling $f : V(G) \to \{0, 1\}$ of a graph G is called friendly labeling if $|v_f(1) - v_f(0)|\leq 1$, where $v_f(1)$ and $v_f(0)$ represents number of vertices labeled by 1 and 0 respectively.

For each edge uv assign the label $|f(u) - f(v)|$, then the function f is called cordial labeling of G if $|v_f(1)-v_f(0)|\leq 1$ and $|e_{f^*}(1) - e_{f^*}(0)|\leq 1$, where $e_{f^*}(1)$ and $e_{f^*}(0)$ are the number of edges labeled 1 and 0 respectively.

A friendly index set denoted by $FI(G)$ of a graph G is defined as the set $\{|e_{f^*}(1) - e_{f^*}(0)|: f^* \ runs\ over\ all\ friendly\ labeling\ f\ of\ G\}$.

For more information on friendly index set of a graphs, refer to [3],[4],[5],[6],[8],[9],[10] and [11].

Labeled graphs have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graphs play vital role in the study of x-ray crystallography, communication network and to determine optimal circuit layouts. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb [1].

The present work is aimed to discuss one such labeling known as ternary vertex labeling. In the following section, 3-vertex friendly index set is introduced and studied for some graphs such as complete graph, path, wheel graph, complete bipartite graph and cycle with parallel chords.

A complete graph is a simple graph whose vertices are pairwise adjacent and the complete graph with n vertices is denoted by K_n. A path is a walk in which all vertices are distinct and it is denoted by P_n. The wheel graph is invented by W. T. Tutte, it is defined to be the graph $K_1 + C_{n-1}$ for all $n \geq 4$. A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every edge joining V_1 and V_2, then G is a complete bipartite graph. A cycle with parallel chords PC_n is a graph obtained from the cycle $C_n : v_0v_1v_2, \cdots, v_{n-1}v_0$ for all $n \geq 6$ by adding the chords $v_1v_{n-1}, v_2v_{n-2}, \cdots, v_{\frac{n-2}{2}}v_{\frac{n+2}{2}}$ if n is even, or adding the chords $v_2v_{n-1}, v_3v_{n-2}, \cdots, v_{\frac{n-1}{2}}v_{\frac{n+3}{2}}$ if n is odd.

Definition 1. *[2] Let G be a simple connected graph with n vertices. A mapping $f : V(G) \to \{0, 1, 2\}$ is called ternary vertex labeling and $f(v)$ represents the label for vertex v.*

Definition 2. *[2] A ternary vertex labeling of a graph G is called 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for all $i, j \in \{0, 1, 2\}$.*

For a ternary vertex labeling f, the induced edge labeling $f^* : E(G) \to \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$, where $u, v \in \{0, 1, 2\}$.

With these notations, we now introduce the notion of 3-vertex friendly labeling and 3-vertex friendly index set.

Definition 3. *The ternary vertex labeling is called a 3-vertex friendly labeling if $|v_f(i) - v_f(j)| \leq 1$, for all $i, j \in \{0, 1, 2\}$.*

Let f be a 3-vertex friendly labeling of a graph G. In this paper, we consider $V_0 = \{v : f(v) = 0\}$, $V_1 = \{v : f(v) = 1\}$ and $V_2 = \{v : f(v) = 2\}$. Let $v_f(0)$, $v_f(1)$ and $v_f(2)$ represent the number of vertices with label 0, 1 and 2 respectively, and $e_{f^*}(0)$, $e_{f^*}(1)$ and $e_{f^*}(2)$ represent edges labeled with 0, 1 and 2 respectively.

Definition 4. *The set $FI_{3v}(G) = \{|e_{f^*}(i) - e_{f^*}(j)|$: f^* runs over all $3-vertex\ friendly\ labeling\ f\ for\ all$ $i, j \in \{0, 1, 2\}\}$ is called 3-vertex friendly index set.*

The proofs in the following section make use of partitioning the number of vertices of a graph into three sets V_0, V_1 and V_2.

2 3-Vertex Friendly Index Set of Some classes of Graphs

Theorem 5. *The 3-vertex friendly index set of the complete graph K_n is*

$$(1) \left\{\frac{n(n+9)}{18}, \frac{n^2}{9}, \frac{|n(n-9)|}{18}\right\} \ if\ n \equiv 0(mod\ 3)$$

$$(2) \left\{\frac{n^2+7n-8}{18}, \frac{|n^2-11n+10|}{18}, \frac{n^2+13n-14}{18}, \frac{n^2+4n-5}{9}, \frac{(n-1)^2}{9}, \frac{|n^2-5n+4|}{18}\right\} \ if\ n \equiv 1(mod\ 3)$$

$$(3) \left\{\frac{|n^2-7n+10|}{18}, \frac{(n+1)^2}{9}, \frac{|n^2-4n-5|}{9}, \frac{n^2+11n-8}{18}, \frac{n^2+5n-14}{18}, \frac{|n^2-13n+4|}{18}\right\}$$
$if\ n \equiv 2(mod\ 3)$.

Proof. Let K_n be a complete graph on n vertices. The proof involves the following three cases.

Case 1. $n \equiv 0(mod\, 3)$.
To satisfy 3-vertex friendly labeling, n is partitioned into $\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right)$. Each of these $\frac{n}{3}$ vertices of V_i are adjacent to $\frac{n}{3} - 1$ vertices of the same class, where $i = 0, 1, 2$. Therefore $e_{f^*}(0) = \frac{3}{2}\left[\frac{n}{3}\left(\frac{n}{3} - 1\right)\right] = \frac{n(n-3)}{6}$. Now each of the $\frac{n}{3}$ vertices of V_1 are adjacent to the $\frac{n}{3}$ vertices of V_0 and $\frac{n}{3}$ vertices of V_2. Therefore $e_{f^*}(1) = \frac{2n^2}{9}$. Also $\frac{n}{3}$ vertices of V_0 are adjacent to $\frac{n}{3}$ vertices of V_2, so $e_{f^*}(2) = \frac{n^2}{9}$. Thus $|e_{f^*}(0) - e_{f^*}(1)| = \frac{n(n+9)}{18}$, $|e_{f^*}(1) - e_{f^*}(2)| = \frac{n^2}{9}$ and $|e_{f^*}(2) - e_{f^*}(0)| = \frac{|n(n-9)|}{18}$. Hence, $FI_{3v}(K_n) = \left\{\frac{n(n+9)}{18}, \frac{n^2}{9}, \frac{|n(n-9)|}{18}\right\}$ for $n \equiv 0(mod\, 3)$.

Case 2. $n \equiv 1(mod\, 3)$.
To satisfy 3-vertex friendly labeling n is partitioned in three different ways as follows, $X_1 = \left(\frac{n-1}{3}, \frac{n+2}{3}, \frac{n-1}{3}\right)$, $X_2 = \left(\frac{n+2}{3}, \frac{n-1}{3}, \frac{n-1}{3}\right)$ and $X_3 = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3}\right)$. We find the total number of edges labeled with 0, 1 and 2 as explained in Case 1. Now from X_1 we have $e_{f^*}(0) = \frac{n^2-3n+2}{6}$, $e_{f^*}(1) = \frac{2(n^2+n-2)}{9}$ and $e_{f^*}(2) = \frac{(n-1)^2}{9}$. So $|e_{f^*}(0) - e_{f^*}(1)| = \frac{n^2+13n-14}{18}$, $|e_{f^*}(1) - e_{f^*}(2)| = \frac{n^2+4n-5}{9}$ and $|e_{f^*}(2) - e_{f^*}(0)| = \frac{|n^2-5n+4|}{18}$. For partitions X_2 and X_3, $e_{f^*}(0) = \frac{n^2-3n+2}{6}$, $e_{f^*}(1) = \frac{2n^2-n-1}{9}$ and $e_{f^*}(2) = \frac{n^2+n-2}{9}$. Now $|e_{f^*}(0) - e_{f^*}(1)| = \frac{n^2+7n-8}{18}$, $|e_{f^*}(1) - e_{f^*}(2)| = \frac{(n-1)^2}{9}$ and $|e_{f^*}(2) - e_{f^*}(0)| = \frac{|n^2-11n+10|}{18}$. By considering all possible partitions of vertex set satisfying 3-vertex friendly labeling we obtain,

$$FI_{3v}(K_n) = \left\{\frac{n^2+7n-8}{18}, \frac{(n-1)^2}{9}, \frac{|n^2-11n+10|}{18}, \frac{n^2+13n-14}{18}, \frac{n^2+4n-3}{9}, \frac{|n^2-5n+4|}{18}\right\},$$

for all $n \equiv 1(mod\, 3)$.

Case 3. $n \equiv 2(mod\, 3)$.
To satisfy 3-vertex friendly labeling n is partitioned in three different ways as $Y_1 = \left(\frac{n+1}{3}, \frac{n+1}{3}, \frac{n-2}{3}\right)$, $Y_2 = \left(\frac{n-2}{3}, \frac{n+1}{3}, \frac{n+1}{3}\right)$ and $Y_3 = \left(\frac{n+1}{3}, \frac{n-2}{3}, \frac{n+1}{3}\right)$. The total number of edges labeled with 0, 1 and 2 are computed as mentioned in Case 1. Now for the partitions Y_1 and Y_2, $e_{f^*}(0) = \frac{n^2-3n+2}{6}$, $e_{f^*}(1) = \frac{2n^2+n-1}{9}$ and $e_{f^*}(2) = \frac{n^2-n-2}{9}$. Hence $|e_{f^*}(0) - e_{f^*}(1)| = \frac{n^2+11n-8}{18}$, $|e_{f^*}(1) - e_{f^*}(2)| = \frac{(n+1)^2}{9}$ and $|e_{f^*}(2) - e_{f^*}(0)| = \frac{|n^2-7n+10|}{18}$. From Y_3, $e_{f^*}(0) = \frac{n^2-3n+2}{6}$,

$e_{f^*}(1) = \dfrac{2(n^2-n-2)}{9}$ and $e_{f^*}(2) = \dfrac{(n+1)^2}{9}$. So $|e_{f^*}(0)-e_{f^*}(1)| = \dfrac{n^2+5n-14}{18}$, $|e_{f^*}(1)-e_{f^*}(2)| = \dfrac{|n^2-4n-5|}{9}$ and $|e_{f^*}(2)-e_{f^*}(0)| = \dfrac{|n^2-13n+4|}{18}$. By considering all possible partitions of the vertex set satisfying 3-vertex friendly labeling we obtain,

$$FI_{3v}(K_n) = \left\{ \frac{|n^2-7n+10|}{18}, \frac{(n+1)^2}{9}, \frac{|n^2-4n-5|}{9}, \frac{n^2+11n-8}{18}, \frac{n^2+5n-14}{18}, \frac{|n^2-13n+4|}{18} \right\}$$

for $n \equiv 2(mod\,3)$. □

Corollary 6. *If G is a graph with n vertices, then $FI_{3v}(G) \subseteq \left\{0, 1, 2, \cdots, \left\lfloor \dfrac{2(n-1)(n+2)}{9} \right\rfloor \right\}$.*

Proof. Let G be any graph on n vertices and f be a mapping from set of vertices of G to $\{0, 1, 2\}$. It is known that the number of edges of any graph cannot exceed the number of edges of complete graph.
The maximum number of edges labeled by 1 from Case 1, Case 2 and Case 3 of Theorem 2.1 are $\left(\dfrac{2n^2}{9}\right)$, $\left\lfloor \dfrac{2(n-1)(n+2)}{9} \right\rfloor$, and $\left\lfloor \dfrac{2n^2+n-1}{9} \right\rfloor$ respectively.
Therefore $max(e_{f^*}(1)) \le \left\lfloor \dfrac{2(n-1)(n+2)}{9} \right\rfloor$ and every element of friendly index set of any graph cannot exceed the value $e_{f^*}(1)$. So $FI_{3v}(G) \subseteq \left\{0, 1, 2, \ldots \left\lfloor \dfrac{2(n-1)(n+2)}{9} \right\rfloor \right\}$. □

Theorem 7. *The 3-vertex friendly index set of path is $FI_{3v}(P_n) \subseteq \{0, 1, 2, \ldots, n-3\}$ for all $n > 10$.*

Proof. We consider three cases in sequence to prove this theorem. To obtain the maximum element of the index set we follow the labeling pattern as mentioned below in all 3 cases.
Case 1. $n \equiv 0(mod\,3)$.
In this case to satisfy 3-vertex friendly labeling n is partitioned into $\left(\dfrac{n}{3}, \dfrac{n}{3}, \dfrac{n}{3}\right)$. For $n \ge 11$, label $v_1, v_2, v_3, \cdots, v_{\frac{n}{3}}$ by 0, $v_{\frac{n+3}{3}}, v_{\frac{n+6}{3}}, v_{\frac{n+9}{3}}, \cdots, v_{\frac{2n}{3}}$ by 1 and $v_{\frac{2n+3}{3}}, v_{\frac{2n+6}{3}}, v_{\frac{2n+9}{3}}, \cdots, v_n$ by 2. Therefore $e_{f^*}(0) = n-3$, $e_{f^*}(1) = 2$ and $e_{f^*}(2) = 0$. For all $n \ge 11$, $e_{f^*}(0) > e_{f^*}(1)$. Hence, $FI_{3v}(P_n) \subseteq \{0, 1, 2, \ldots, n-3\}$ for all $n \equiv 0(mod\,3)$.
Case 2. If $n \equiv 1(mod\,3)$.
In this case we consider 3 partitions of $V(G)$ as $X_1 = \left(\dfrac{n+2}{3}, \dfrac{n-1}{3}, \dfrac{n-1}{3}\right)$, $X_2 = \left(\dfrac{n-1}{3}, \dfrac{n+2}{3}, \dfrac{n-1}{3}\right)$ and $X_3 = \left(\dfrac{n-1}{3}, \dfrac{n-1}{3}, \dfrac{n+2}{3}\right)$.
If $n \ge 11$, for X_1 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+2}{3}}$ by 0, $v_{\frac{n+5}{3}}, v_{\frac{n+8}{3}}, v_{\frac{n+11}{3}}, \cdots, v_{\frac{2n+1}{3}}$ by 1 and $v_{\frac{2n+4}{3}}, v_{\frac{2n+7}{3}}, v_{\frac{2n+10}{3}}, \cdots, v_n$ by 2, for the partition X_2 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{3}}$ by 0, $v_{\frac{n+2}{3}}, v_{\frac{n+5}{3}}, v_{\frac{n+8}{3}}, \cdots, v_{\frac{2n+1}{3}}$ by 1 and $v_{\frac{2n+4}{3}}, v_{\frac{2n+7}{3}}, v_{\frac{2n+10}{3}}, \cdots, v_n$ by 2, and for X_3 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{3}}$ by 0, $v_{\frac{n+2}{3}}, v_{\frac{n+5}{3}}, v_{\frac{n+8}{3}}, \cdots, v_{\frac{2n-2}{3}}$ by 1 and $v_{\frac{2n+1}{3}}, v_{\frac{2n+4}{3}}, v_{\frac{2n+7}{3}}, \cdots, v_n$ by 2. Thus $e_{f^*}(0) = n-3$, $e_{f^*}(1) = 2$ and $e_{f^*}(2) = 0$. So, for all $n \ge 11$, $e_{f^*}(0) > e_{f^*}(1)$. Hence, $FI_{3v}(P_n) \subseteq \{0, 1, 2, \ldots, n-3\}$ for all $n \equiv 1(mod\,3)$.
Case 3. $n \equiv 2(mod\,3)$.
In this case n is partitioned in 3 different ways as $Y_1 = \left(\dfrac{n+1}{3}, \dfrac{n+1}{3}, \dfrac{n-2}{3}\right)$, $Y_2 = \left(\dfrac{n-2}{3}, \dfrac{n+1}{3}, \dfrac{n+1}{3}\right)$ and $Y_3 = \left(\dfrac{n+1}{3}, \dfrac{n-2}{3}, \dfrac{n+1}{3}\right)$.
If $n \ge 11$, for the partition Y_1 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{3}}$ by 0, $v_{\frac{n+4}{3}}, v_{\frac{n+7}{3}}, v_{\frac{n+10}{3}}, \cdots, v_{\frac{2n+2}{3}}$ by 1 and $v_{\frac{2n+5}{3}}, v_{\frac{2n+8}{3}}, v_{\frac{2n+11}{3}}, \cdots, v_n$ by 2, for Y_2 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-2}{3}}$ by 0, $v_{\frac{n+1}{3}}, v_{\frac{n+4}{3}}, v_{\frac{n+7}{3}}, \cdots, v_{\frac{2n-1}{3}}$ by 1 and $v_{\frac{2n+2}{3}}, v_{\frac{2n+5}{3}}, v_{\frac{2n+8}{3}}, \cdots, v_n$ by 2 and for Y_3 label the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{3}}$ by 0, $v_{\frac{n+4}{3}}, v_{\frac{n+7}{3}}, v_{\frac{n+10}{3}}, \cdots, v_{\frac{2n-1}{3}}$ by 1 and $v_{\frac{2n+2}{3}}, v_{\frac{2n+5}{3}}, v_{\frac{2n+8}{3}}, \cdots, v_n$ by 2. Thus $e_{f^*}(0) = n-3$, $e_{f^*}(1) = 2$ and $e_{f^*}(2) = 0$. So, for all $n \ge 11$, $e_{f^*}(0) > e_{f^*}(1)$. Hence, $FI_{3v}(P_n) \subseteq \{0, 1, 2, \ldots, n-3\}$ for all $n \equiv 2(mod\,3)$. □

Remark. *The 3-vertex friendly index set of path for $n \le 10$ is subset of*

$$\begin{cases} \left\{0, 1, 2, \cdots, \dfrac{2n}{3}\right\} & \text{if } n \equiv 0(mod\,3)\,, \\ \left\{0, 1, 2, \cdots, \dfrac{2(n+2)}{3}\right\} & \text{if } n \equiv 1(mod\,3)\,, \\ \left\{0, 1, 2, \cdots, \dfrac{2(n+1)}{3}\right\} & \text{if } n \equiv 2(mod\,3)\,. \end{cases}$$

Theorem 8. *The 3-vertex friendly index set of wheel W_n,*

$$FI_{3v}(W_n) \subseteq \begin{cases} \left\{0, 1, 2, \cdots, \dfrac{(4n-6)}{3}\right\} & \text{if } n \equiv 0(mod\,3), \\ \left\{0, 1, 2, \cdots, \dfrac{(4n-4)}{3}\right\} & \text{if } n \equiv 1(mod\,3), \\ \left\{0, 1, 2, \cdots, \dfrac{(4n-5)}{3}\right\} & \text{if } n \equiv 2(mod\,3). \end{cases}$$

for all $n > 6$.

Proof. Let $v_1, v_2, \cdots, v_{n-1}$ be the rim vertices and v_n be the central vertex of the wheel graph W_n. The proof involves the following three cases.
Case 1. $n \equiv 0(mod\,3)$
In this case consider the partition $\left(\dfrac{n}{3}, \dfrac{n}{3}, \dfrac{n}{3}\right)$. To get the maximum element of the 3-vertex friendly index set, vertices of W_n are labeled as in the Table 1.

Table 1. Vertex labeling $f(v_i)$ of W_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n}{3}$	$1 \le i \le \frac{n}{3}$
1	$i = n, \frac{n+3}{3}$ and $2k+1$, where $k = \frac{n+12}{6}, \frac{n+18}{6}, \frac{n+24}{6}, \cdots, \frac{n-2}{2}$	$i = n, \frac{n+3}{3}$ and $2k$, where $k = \frac{n+15}{6}, \frac{n+21}{6}, \frac{n+27}{6}, \cdots, \frac{n-1}{2}$
2	$\frac{n+6}{3} \le i \le \frac{n+12}{3}$ $and\ 2k$ where $k = \frac{n+18}{6}, \frac{n+24}{6}, \frac{n+30}{6}, \cdots, \frac{n-2}{2}$	$\frac{n+6}{3} \le i \le \frac{n+12}{3}$ $and\ 2k+1$ where $k = \frac{n+15}{6}, \frac{n+21}{6}, \frac{n+27}{6}, \cdots, \frac{n-3}{2}$

So $e_{f^*}(0) = \dfrac{2n}{3}$, $e_{f^*}(1) = \dfrac{4n-6}{3}$ and $e_{f^*}(2) = 0$. Thus $FI_{3v}(W_n) \subseteq \left\{0, 1, 2, \cdots, \dfrac{(4n-6)}{3}\right\}$ for all $n \equiv 0(mod\ 3)$ where $n > 6$.

Case 2. $n \equiv 1(mod\ 3)$.

In this case to satisfy 3-vertex friendly labeling, n is partitioned in 3 different ways as $X_1 = \left(\dfrac{n+2}{3}, \dfrac{n-1}{3}, \dfrac{n-1}{3}\right)$, $X_2 = \left(\dfrac{n-1}{3}, \dfrac{n+2}{3}, \dfrac{n-1}{3}\right)$ and $X_3 = \left(\dfrac{n-1}{3}, \dfrac{n-1}{3}, \dfrac{n+2}{3}\right)$. To obtain the maximum element of 3-vertex friendly index set, we consider the partition X_2 because $|V_1|$ in X_2 is maximum compared to X_1 and X_3, also it is observed that $e_{f^*}(0)$ and $e_{f^*}(2)$ cannot exceed $e_{f^*}(1)$ in W_n. The vertices of W_n are labeled as in the Table 2.

Table 2. Vertex labeling $f(v_i)$ of W_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n-1}{3}$	$1 \le i \le \frac{n-1}{3}$
1	$i = n, \frac{n+2}{3}$ and $2k+1$ where $k = \frac{n+8}{6}, \frac{n+14}{6}, \frac{n+20}{6}, \cdots, \frac{n-2}{2}$	$i = n, \frac{n+2}{3}$ and $2k$ where $k = \frac{n+11}{6}, \frac{n+17}{6}, \frac{n+23}{6}, \cdots, \frac{n-1}{2}$
2	$\frac{n+5}{3}$ $and\ 2k$ where $k = \frac{n+8}{6}, \frac{n+14}{6}, \frac{n+20}{6}, \cdots, \frac{n-2}{2}$	$\frac{n+5}{3}$ $and\ 2k+1$ where $k = \frac{n+5}{6}, \frac{n+11}{6}, \frac{n+17}{6}, \cdots, \frac{n-3}{2}$

So $e_{f^*}(0) = \dfrac{2n-2}{3}$, $e_{f^*}(1) = \dfrac{4n-4}{3}$ and $e_{f^*}(2) = 0$. Hence, $FI_{3v}(W_n) \subseteq \left\{0, 1, 2, \cdots, \dfrac{(4n-4)}{3}\right\}$ for all $n \equiv 1(mod\ 3)$ where $n > 4$.

Case 3. $n \equiv 2(mod\ 3)$

In this case to satisfy 3-vertex friendly labeling, n is partitioned in 3 different ways as $Y_1 = \left(\dfrac{n+1}{3}, \dfrac{n+1}{3}, \dfrac{n-2}{3}\right)$, $Y_2 = \left(\dfrac{n-2}{3}, \dfrac{n+1}{3}, \dfrac{n+1}{3}\right)$ and $Y_3 = \left(\dfrac{n+1}{3}, \dfrac{n-2}{3}, \dfrac{n+1}{3}\right)$. To obtain maximum element of 3-vertex friendly index set, we consider either the partition Y_1 or Y_2 because $|V_1|$ in Y_1 and Y_2 is maximum compared to Y_3 , also it is observed that $e_{f^*}(0)$ and $e_{f^*}(2)$ cannot exceed $e_{f^*}(1)$ in W_n. The vertices of W_n are labeled as in the Table 3. Here we consider the partition Y_2.

Table 3. Vertex labeling $f(v_i)$ of W_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n-2}{3}$	$1 \le i \le \frac{n-2}{3}$
1	$i = n, \frac{n+1}{3}$ and $2k+1$ $\forall\, k = \frac{n+10}{6}, \frac{n+16}{6}, \frac{n+22}{6}, \cdots, \frac{n-2}{2}$	$i = n, \frac{n+1}{3}$ and $2k$ $\forall\, k = \frac{n+13}{6}, \frac{n+19}{6}, \frac{n+25}{6}, \cdots, \frac{n-1}{2}$
2	$\frac{n+4}{3} \le i \le \frac{n+10}{3}$ $and\ 2k$ $\forall\, k = \frac{n+16}{6}, \frac{n+22}{6}, \frac{n+28}{6}, \cdots, \frac{n-2}{2}$	$\frac{n+4}{3} \le i \le \frac{n+10}{3}$ $and\ 2k$ $+1\ \forall\, k = \frac{n+13}{6}, \frac{n+19}{6}, \frac{n+25}{6}, \cdots, \frac{n-3}{2}$

Thus $e_{f^*}(0) = \dfrac{2n-1}{3}$, $e_{f^*}(1) = \dfrac{4n-5}{3}$ and $e_{f^*}(2) = 0$. Hence, $FI_{3v}(W_n) \subseteq \left\{0, 1, 2, \cdots, \dfrac{(4n-5)}{3}\right\}$ for all $n \equiv 2(mod\ 3)$ where $n > 5$.

□

Theorem 9. *The 3-vertex friendly index set of the complete bipartite graph $K_{m,n}$, $FI_{3v}(K_{m,n})$ is subset of the following set*

1. $\{0, 1, 2, \cdots, mn\}$ *if* $m = \dfrac{n}{2}$,
2. $\left\{0, 1, 2, \cdots, \left\lfloor \dfrac{5n^2 + 2m^2 - 2mn - 4m + 11n + 2}{9} \right\rfloor\right\}$ *if* $m > \dfrac{n}{2}$,
3. $\left\{0, 1, 2, \cdots, \left\lfloor \dfrac{n(m+n-1)}{3} \right\rfloor\right\}$ *if* $m < \dfrac{n}{2}$.

Proof. Let $K_{m,n}$ be the complete bipartite graph with $m + n$ vertices and mn edges. Without loss of generality, we consider $m \le n$. The proof involves the following three cases,

Case 1. $m + n \equiv 0(mod\ 3)$

To satisfy 3-vertex friendly labeling $m + n$ is partitioned into $\left(\dfrac{m+n}{3}, \dfrac{m+n}{3}, \dfrac{m+n}{3}\right)$. While writing the friendly index set of complete bipartite graph there arises three sub cases.

Sub case 1.1 If $m = \dfrac{n}{2}$.

To find the maximum element of 3-vertex friendly index set of a complete bipartite graph, fix $\dfrac{m+n}{3}$ vertices in the first partite with label 1, and these $\frac{m+n}{3}$ vertices are adjacent to $\frac{2(m+n)}{3}$ vertices in the second partite labeled 0 and 2, so the total number of edges labeled with 1 is mn. There exists no edge with label 0 and 2, hence maximum element of the index set is mn. Therefore $FI_{3v}(K_{m,n}) \subseteq \{0, 1, 2, \cdots, mn\}$.

Sub case 1.2 If $m > \frac{n}{2}$.

To find the maximum element of 3-vertex friendly index set of a complete bipartite graph, fix $\frac{m+n}{3}$ vertices by label 1 in the first partite. To obtain minimum number of edges labeled by 0 and 2, label remaining $\frac{2m-n}{3}$ vertices by 0 in the first partite, $\left(\frac{m+n}{3}\right)$ vertices by 2 in the second partite and remaining $\left(\frac{2n-m}{3}\right)$ by 0 in the second partite. So $e_{f^*}(0) = \left(\frac{2m-n}{3}\right)\left(\frac{2n-m}{3}\right)$, $e_{f^*}(1) = \frac{n(m+n)}{3}$ and $e_{f^*}(2) = \left(\frac{m+n}{3}\right)\left(\frac{2m-n}{3}\right)$. Thus, the maximum element of the 3-vertex friendly index set is $\frac{5n^2+2m^2-2mn}{9}$.

Hence $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \ldots, \frac{5n^2+2m^2-2mn}{9}\right\}$.

Sub case 1.3 If $m < \frac{n}{2}$.

Label all the m vertices in the first partite set by 1. These m vertices labeled by 1 are adjacent to $\frac{n-2m}{3}$ vertices labeled by 1, $\frac{m+n}{3}$ vertices labeled by 0 and $\frac{m+n}{3}$ vertices labeled by 2 in the second partite. The total number of edges labeled by 1 is $\frac{2m(m+n)}{3}$ and the total number of edges labeled by 0 is $\frac{m(n-2m)}{3}$. Since in the first partite all m vertices labeled by 1, no edge has label 2. Hence, $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \ldots, \frac{2m(m+n)}{3}\right\}$.

Case 2. If $m + n \equiv 1(mod\, 3)$.

To satisfy 3-vertex friendly labeling $m + n$ is partitioned as $X_1 = \left(\frac{m+n+2}{3}, \frac{m+n-1}{3}, \frac{m+n-1}{3}\right)$, $X_2 = \left(\frac{m+n-1}{3}, \frac{m+n+2}{3}, \frac{m+n-1}{3}\right)$ and $X_3 = \left(\frac{m+n-1}{3}, \frac{m+n-1}{3}, \frac{m+n+2}{3}\right)$.

To prove this case we go through the following two sub cases.

Sub case 2.1 If $m > \frac{n}{2}$.

It is clear that $m \geq \frac{m+n+2}{3}$. To obtain maximum element of the index set consider the partition X_2 because number of vertices labeled by 1 are more in X_2 when compared to the vertices labeled by 1 in the partitions X_1 and X_3.

Suppose $m = \frac{m+n+2}{3}$, fix $\frac{m+n+2}{3}$ vertices in the first partite by label 1. Hence there does not exist edges with label by 0 and 2. Number of edges labeled by 1 is $\frac{n(m+n+2)}{3}$.

Therefore $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{n(m+n+2)}{3}\right\}$.

Suppose $m > \frac{m+n+2}{3}$, label $\frac{m+n+2}{3}$ vertices by 1 and remaining $\frac{2m-n-2}{3}$ vertices by 0 in the first partite. In the second partite $\frac{m+n-1}{3}$ and $\frac{2n-m+1}{3}$ vertices are labeled by 2 and 0 respectively. So edges with label 0, 1 and 2 are $\frac{(2m-n-2)(2n-m+1)}{9}$, $\frac{n(m+n+2)}{3}$ and $\frac{(2m-n-2)(m+n-1)}{9}$ respectively.

Now $|e_{f^*}(0)-e_{f^*}(1)| = \frac{5n^2+2m^2-2mn-4m+11n+2}{9}$, $|e_{f^*}(1) - e_{f^*}(2)| = \frac{4n^2-2m^2+2mn+4m+7n-2}{9}$ and $|e_{f^*}(2) - e_{f^*}(0)| = \frac{n^2+4m^2-4mn-8m+4n+4}{9}$.

Therefore for all $m > \frac{n}{2}$, $FI_{3v}(K_{m,n})$ is subset of $\left\{0, 1, 2, \cdots, \frac{5n^2+2m^2-2mn-4m+11n+2}{9}\right\}$.

Sub case 2.2 If $m < \frac{n}{2}$.

It is clear that $m \leq \frac{m+n-1}{3}$. To obtain the maximum element of the index set, consider the partition X_1 or X_3 because if we consider the partition X_2, there is a possibility of getting least $e_{f^*}(1)$ compared to the partitions X_1 or X_3.

Suppose $m = \frac{m+n-1}{3}$, edges with label 0, 1 and 2 are 0, $\frac{n(m+n-1)}{3}$ and 0 respectively. Therefore $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{n(m+n-1)}{3}\right\}$

Suppose $m < \frac{m+n-1}{3}$, label all the m vertices in the first partite by 1. These m vertices are adjacent to $\frac{n-2m-1}{3}$ vertices labeled by 1 in the second partite. Now $\frac{m+n-1}{3}$ vertices are labeled by 0 and $\frac{m+n+2}{3}$ vertices be labeled by 2 or vice versa in the second partite. Hence $e_{f^*}(0) = \frac{m(n-2m-1)}{3}$ and $e_{f^*}(1) = \frac{m(2m+2n+1)}{3}$ and $e_{f^*}(2) = 0$.

Hence, $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{m(2m+2n+1)}{3}\right\}$.

Case 3. If $m + n \equiv 2(mod\, 3)$.

To satisfy 3-vertex friendly labeling $m + n$ is partitioned into $Y_1 = \left(\frac{m+n+1}{3}, \frac{m+n+1}{3}, \frac{m+n-2}{3}\right)$, $Y_2 = \left(\frac{m+n-2}{3}, \frac{m+n+1}{3}, \frac{m+n+1}{3}\right)$ and $Y_3 = \left(\frac{m+n+1}{3}, \frac{m+n-2}{3}, \frac{m+n+1}{3}\right)$. In this case we will consider the following two sub cases.

Sub case 3.1 If $m > \frac{n}{2}$.

It is clear that $m \geq \frac{m+n+1}{3}$. To obtain maximum element of the index set, consider the partition Y_1 or Y_2 to label the set of vertices of $K_{m,n}$ because number of vertices labeled by 1 are more in Y_1 and Y_2 compared to the vertices labeled by 1 in the partition Y_3.

Suppose $m = \frac{m+n+1}{3}$, fix $\frac{m+n+1}{3}$ vertices in the first partite by label 1. Number of edges labeled by 1 is $\frac{n(m+n+1)}{3}$. It is observed that there does not exist edges labeled by 0 and 2. Therefore $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{n(m+n+1)}{3}\right\}$.

Suppose $m > \frac{m+n+1}{3}$, for the partition Y_1 label $\frac{m+n+1}{3}$ vertices by 1 and remaining $\frac{2m-n-1}{3}$ vertices by 0 in the first partite. In the second partite $\frac{m+n-2}{3}$ vertices are labeled by 2 and $\frac{2n-m+2}{3}$ vertices are labeled by 0. Edges with label 0, 1 and 2 are $\frac{(2m-n-1)(2n-m+2)}{9}$, $\frac{n(m+n+1)}{3}$ and $\frac{(2m-n-1)(m+n-2)}{9}$ respectively. For the partition Y_2, interchange the vertex labeling of 0 and 2 as in Y_1 and vertices with label 1 remains same as in Y_1.

Now $|e_{f^*}(0)-e_{f^*}(1)|= \frac{5n^2+2m^2-2mn-5m+7n+2}{9}$, $|e_{f^*}(1)-e_{f^*}(2)|= \frac{4n^2-2m^2+2mn+5m+2n-2}{9}$ and $|e_{f^*}(2)-e_{f^*}(0)|= \frac{4m^2+n^2-4mn-10m+5n+4}{9}$.

Therefore $FI_{3v}(K_{m,n})$ is the subset of $\left\{0, 1, 2, \cdots, \frac{2m^2+5n^2-2mn-5m+7n+2}{9}\right\}$.

Sub case 3.2 If $m < \frac{n}{2}$.

It is clear that $m \leq \frac{m+n-2}{3}$. To obtain the maximum element of the index set, consider the partition Y_3. Since $m < \frac{n}{2}$ all the vertices in the first partite be labeled by 1.

Suppose $m = \frac{m+n-2}{3}$, edges with label 0, 1 and 2 are 0, $\frac{n(m+n-2)}{3}$ and 0 respectively. Therefore $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{n(m+n-2)}{3}\right\}$

Suppose $m < \frac{m+n-2}{3}$. All the vertices will be labeled by 1 in the first partite. Label $\frac{m+n+1}{3}$ by 0, $\frac{n-2m-2}{3}$ by 1 and $\frac{m+n+1}{3}$ by 2 in the second partite. Hence $e_{f^*}(0) = \frac{m(n-2m-2)}{3}$, $e_{f^*}(1) = \frac{m(2m+2n+2)}{3}$ and $e_{f^*}(2) = 0$. Thus $FI_{3v}(K_{m,n}) \subseteq \left\{0, 1, 2, \cdots, \frac{m(2m+2n+2)}{3}\right\}$. □

Theorem 10. *The 3-vertex friendly index set of cycle with parallel chords is* $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \ldots, n+p-4\}$.

Proof. Let PC_n be the cycle with p parallel chords on n vertices and $v_1v_2v_3\ldots v_nv_1$ be the cycle of PC_n.

Case 1. $n \equiv 0(mod\, 3)$.
In this case to satisfy 3-vertex friendly labeling n is partitioned into $\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right)$. To obtain the maximum element of the 3-vertex friendly index set of PC_n, set of vertices are labeled as in Table 4.

Table 4. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$1 \leq i \leq \frac{n+6}{6}$ and $\frac{5n+12}{6} \leq i \leq n$	$1 \leq i \leq \frac{n+3}{6}$ and $\frac{5n+9}{6} \leq i \leq n$
1	$\frac{n+12}{6} \leq i \leq \frac{2n+6}{6}$ and $\frac{4n+12}{6} \leq i \leq \frac{5n+6}{6}$	$\frac{n+9}{6} \leq i \leq \frac{2n+6}{6}$ and $\frac{4n+12}{6} \leq i \leq \frac{5n+3}{6}$
2	$\frac{2n+12}{6} \leq i \leq \frac{4n+6}{6}$	$\frac{2n+12}{6} \leq i \leq \frac{4n+6}{6}$

Thus $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-5\}$.

Case 2. $n \equiv 1(mod\, 3)$.
In this case to satisfy 3-vertex friendly labeling n is partitioned into three different ways as follows, $X_1 = \left(\frac{n+2}{3}, \frac{n-1}{3}, \frac{n-1}{3}\right)$, $X_2 = \left(\frac{n-1}{3}, \frac{n+2}{3}, \frac{n-1}{3}\right)$ and $X_3 = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3}\right)$.
For the partition X_1, label set of vertices of PC_n as per the pattern depicted in Table 5.

Table 5. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$1 \leq i \leq \frac{n+2}{6}$ and $\frac{5n+4}{6} \leq i \leq n$	$1 \leq i \leq \frac{n+5}{6}$ and $\frac{5n+7}{6} \leq i \leq n$
1	$\frac{n+8}{6} \leq i \leq \frac{2n+4}{6}$ and $\frac{4n+8}{6} \leq i \leq \frac{5n-2}{6}$	$\frac{n+11}{6} \leq i \leq \frac{2n+4}{6}$ and $\frac{4n+8}{6} \leq i \leq \frac{5n+1}{6}$
2	$\frac{2n+10}{6} \leq i \leq \frac{4n+2}{6}$	$\frac{2n+10}{6} \leq i \leq \frac{4n+2}{6}$

Thus $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-5\}$ for all $n \geq 10$.
For the partition X_2, label set of vertices of PC_n as per the pattern stated in Table 6.

Table 6. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n+2}{6}$ and $\frac{5n+10}{6} \le i \le n$	$1 \le i \le \frac{n+5}{6}$ and $\frac{5n+13}{6} \le i \le n$
1	$\frac{n+8}{6} \le i \le \frac{2n+4}{6}$ and $\frac{4n+8}{6} \le i \le \frac{5n+4}{6}$	$\frac{n+11}{6} \le i \le \frac{2n+4}{6}$ and $\frac{4n+8}{6} \le i \le \frac{5n+7}{6}$
2	$\frac{2n+10}{6} \le i \le \frac{4n+2}{6}$	$\frac{2n+10}{6} \le i \le \frac{4n+2}{6}$

Thus $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-4\}$ for all $n > 10$.
For the partition X_3, label set of vertices of PC_n as per the pattern shown in Table 7.

Table 7. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$\frac{2n+10}{6} \le i \le \frac{4n+2}{6}$	$\frac{2n+10}{6} \le i \le \frac{4n+2}{6}$
1	$\frac{n+8}{6} \le i \le \frac{2n+4}{6}$ and $\frac{4n+8}{6} \le i \le \frac{5n-2}{6}$	$\frac{n+11}{6} \le i \le \frac{2n+4}{6}$ and $\frac{4n+8}{6} \le i \le \frac{5n+1}{6}$
2	$1 \le i \le \frac{n+2}{6}$ and $\frac{5n+4}{6} \le i \le n$	$1 \le i \le \frac{n+5}{6}$ and $\frac{5n+7}{6} \le i \le n$

Thus $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-5\}$ for all $n \ge 10$.

Case 3. $n \equiv 2(mod\, 3)$.
To satisfy 3-vertex friendly labeling n is partitioned into $Y_1 = \left(\frac{n+1}{3}, \frac{n+1}{3}, \frac{n-2}{3}\right)$, $Y_2 = \left(\frac{n-2}{3}, \frac{n+1}{3}, \frac{n+1}{3}\right)$ and $Y_3 = \left(\frac{n+1}{3}, \frac{n-2}{3}, \frac{n+1}{3}\right)$.

For the partition Y_1, label set of vertices of PC_n as per the pattern stated in Table 8.

Table 8. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n+4}{6}$ and $\frac{5n+8}{6} \le i \le n$	$\frac{2n+8}{6} \le i \le \frac{4n+4}{6}$
1	$\frac{n+10}{6} \le i \le \frac{2n+8}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+2}{6}$	$\frac{n+7}{6} \le i \le \frac{2n+2}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+5}{6}$
2	$\frac{2n+14}{6} \le i \le \frac{4n+4}{6}$	$1 \le i \le \frac{n+1}{6}$ and $\frac{5n+11}{6} \le i \le n$

Thus $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-4\}$ for all $n > 11$.
For the partition Y_2, label set of vertices of PC_n as per the pattern depicted in Table 9.

Table 9. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$\frac{2n+14}{6} \le i \le \frac{4n+4}{6}$	$1 \le i \le \frac{n+1}{6}$ and $\frac{5n+11}{6} \le i \le n$
1	$\frac{n+10}{6} \le i \le \frac{2n+8}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+2}{6}$	$\frac{n+7}{6} \le i \le \frac{2n+2}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+5}{6}$
2	$1 \le i \le \frac{n+4}{6}$ and $\frac{5n+8}{6} \le i \le n$	$\frac{2n+8}{6} \le i \le \frac{4n+4}{6}$

Thus $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-4\}$ for all $n > 11$.
For the partition Y_3, label set of vertices of PC_n as per the pattern shown in Table 10.

Table 10. Vertex labeling $f(v_i)$ of PC_n

Label	n is even	n is odd
0	$1 \le i \le \frac{n+4}{6}$ and $\frac{5n+8}{6} \le i \le n$	$1 \le i \le \frac{n+7}{6}$ and $\frac{5n+11}{6} \le i \le n$
1	$\frac{n+10}{6} \le i \le \frac{2n+2}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+2}{6}$	$\frac{n+13}{6} \le i \le \frac{2n+2}{6}$ and $\frac{4n+10}{6} \le i \le \frac{5n+5}{6}$
2	$\frac{2n+8}{6} \le i \le \frac{4n+4}{6}$	$\frac{2n+8}{6} \le i \le \frac{4n+4}{6}$

Thus $e_{f^*}(0) = n+p-5$, $e_{f^*}(1) = 5$ and $e_{f^*}(2) = 0$ when n is odd, and $e_{f^*}(0) = n+p-4$, $e_{f^*}(1) = 4$ and $e_{f^*}(2) = 0$ when n is even. Hence $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \cdots, n+p-4\}$ for all $n \geq 11$.
By considering all cases, $FI_{3v}(PC_n) \subseteq \{0, 1, \ldots, n+p-6, n+p-5, n+p-4\}$.

□

Remark. *For* $n \leq 12$, $FI_{3v}(PC_n) \subseteq \{0, 1, 2, \ldots, n+p-1\}$.

3 Conclusion

In this paper we have obtained 3-vertex friendly index set of complete graph, path, wheel, complete bipartite graph and cycle with parallel chords. To investigate 3-vertex friendly index set for remaining class of graph is an open area of research.

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