

The Parabolic Transform and Some Singular Integral Evolution Equations

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Abstract Some singular integral evolution equations with wide class of closed operators are studied in Banach space. The considered integral equations are investigated without the existence of the resolvent of the closed operators. Also, some non-linear singular evolution equations are studied. An abstract parabolic transform is constructed to study the solutions of the considered ill-posed problems. Applications to fractional evolution equations and Hilfer fractional evolution equations are given. All the results can be applied to general singular integro-differential equations. The Fourier Transform plays an important role in constructing solutions of the Cauchy problems for parabolic and hyperbolic partial differential equations. This means that the Fourier transform is suitable but under conditions on the characteristic forms of the partial differential operators. Also, the Laplace transform plays an important role in studying the Cauchy problem for abstract differential equations in Banach space. But in this case, we need the existence of the resolvent of the considered abstract operators. This note is devoted to exploring the Cauchy problem for general singular integro-partial differential equations without conditions on the characteristic forms and also to study general singular integral evolution equations. Our approach is based on applying the new parabolic transform. This transform generalizes the methods developed within the regularization theory of ill-posed problems.

Keywords Abstract Parabolic Transform, Singular Integral Evolution Equations, Hilfer Fractional Differential Equations

AMS subject classifications: 34-K30, 26A33, 60H15, 47D60, 74D62, 35A05, 34G20.

1. Introduction

Let us consider the following singular evolution equations:

$$u(t) = f(t) + \int_0^t \sum_{i=1}^r K_i(t, \theta) A_i(\theta) u(\theta) d\theta, \quad (1.1)$$

$$u(t) = f(t) + \int_0^t F(V(t, \theta)) d\theta, \quad (1.2)$$

Where $\{A_i(t)\}: t \in J, i = 1, 2, \dots, r\}$ is a family of linear closed operators defined on densely domains in a Banach space $E, J = [0, T]$ is an interval, $T > 0, \{K_i(t, \theta), t, \theta \in J, \theta < t\}$ is a family of linear bounded operated defined on E to E , such that

$$\|K_i(t, \theta)h\| \leq \frac{M}{(t-\theta)^{1-\alpha}} \|h\|, \quad (1.3)$$

For all $h \in E, 0 < \alpha \leq 1, M$ is a positive constant independent on t and $\theta, \|\cdot\|$ is the norm in E ,

$V(t, \theta) = (K_1(t, \theta) A_1(\theta)u(\theta), \dots, K_r(t, \theta) A_r(\theta) u(\theta)), F$ is an abstract function defined on E^r to E , for every $t, \theta \in J, t > \theta, f(t) \in E$ is a given continuous function in $t \in J$.

Let $D(A_i), i = 1, \dots, r$ be the domain of definitions of A_i . It is supposed that the domains $D(A_1), \dots, D(A_r)$ are independent of t .

We assume that all the functions $A_1(t)h, \dots, A_r(t)h$ are continuous on J for every $h \in \cap_{i=1}^r D(A_i)$ and all the functions $K_1(t, \theta)h, \dots, K_r(t, \theta)h$ are continuous on $J \times J, t > \theta$, for every $h \in E$.

It is assumed also that F satisfies the following Lipschitz condition;

$$\|F(\eta) - F(\eta^*)\| \leq M \sum_{i=1}^r \|\eta_i - \eta_i^*\|, \quad (1.4)$$

For all $\eta = (\eta_1, \dots, \eta_r), \eta^* = (\eta_1^*, \dots, \eta_r^*)$ in E^r, M is a positive constant.

Equations (1.1) and (1.2) are studied for a wide class of the closed operators $A_1(t), \dots, A_r(t)$. In general, these operators have no resolvents. This means that we shall study ill-posed problems. See [1-13].

In section 2, we shall define the abstract parabolic transform. Using this transform, we can find a dense set S in E such that if $f(t) \in S$, then equation (1.1) can be solved.

In section 3, we shall solve fractional integral evolution equations and Hilfer fractional integral evolution equations.

Also, some general singular integro-partial differential equations are studied. The properties of the solutions in all cases are given. In section 4, we shall study equation (1.2).

2. Abstract Parabolic Transforms

The parabolic transform is defined in [14]. Let us, now, try to modify the definition to be suitable for abstract integral equations. Let $Q(t)$ be a strongly continuous semi-group defined on E, with infinitesimal generator B defined on dense set D (B) in E, such that $D(B) \subset \cap_{i=1}^r D(A_i)$.

We assume that there exist constants, M and γ , $M > 0$, $0 < \gamma < \alpha$,

$$\text{Such that: } \|A_i(t_2)Q(t_1)g\| \leq \frac{M}{t_1^\gamma} \|g\|, \tag{2.1}$$

for all $i = 1, \dots, r$, and all $g \in E$, $t_1, t_2 \in J$, $t_1 > 0$.

A parabolic transform of an abstract function $\varphi(t_1, \dots, t_r)$ is defined by

$$\tilde{\varphi}(t_1, \dots, t_r, c_2 + c_1 t) = Q(c_2 + c_1 t) \varphi(t_1, \dots, t_r), \tag{2.2}$$

Where c_1, c_2 are constants, $c_2 + c_1 t \geq 0$, $t, t_i \in J$, $i = 1, \dots, r$, and φ is an abstract function defined on J^r with values in E.

Let us study the following of singular integral evolution equation;

$$v(t) = \tilde{f}(t, ct) + \int_0^t \sum_{i=1}^r K_i(t, \theta) A_i(\theta) \tilde{v}(\theta, ct - c\theta) d\theta, \tag{2.3}$$

c is a positive number.

Theorem (2.1). Under conditions (1.3) and (2.1), there exists a unique solution $v(t) \in E$. This solution is continuous in $t \in J$.

Proof: Let us use the method of successive approximations.

Let $\{v_n\}$ be a sequence defined by:

$$v_{n+1}(t) = \tilde{f}(t, ct) + \int_0^t \sum_{i=1}^r K_i(t, \theta) A_i(\theta) \tilde{v}_n(\theta, ct - c\theta) d\theta$$

The zero approximation $v_0(t)$ is chosen to be identically zero. Using conditions (1.3) and (2.1) and the properties of the operators A, K, Q, it can be proved that the functions $v_1(t), \dots, v_n(t), \dots$ are continuous in $t \in J$ and with values in E.

Again, according to (1.3) and (2.1), we get

$$\|v_{n+1}(t) - v_n(t)\| \leq \frac{M}{c^\gamma} \int_0^t \frac{\|v_n(\theta) - v_{n-1}(\theta)\|}{(t-\theta)^{1-\delta}} d\theta,$$

where; $\delta = \alpha - \gamma$

Thus:

$$\|v_{n+1}(t) - v_n(t)\| \leq \frac{M^{n+1}}{c^{n\gamma}} \frac{t^{n\delta} (\Gamma(\delta))^n}{\Gamma(n\delta + 1)},$$

Where $\Gamma(\cdot)$ is the gamma function. The last inequality leads to the fact that the sequence $\{v_n(t)\}$ uniformly converges in E to a continuous function $v(t)$ on J, which represents the solution of equation (2.3). To prove the uniqueness, let us suppose that there are two solutions $v(t)$ and $v^*(t)$ of (2.3).

Thus;

$$\|v(t) - v^*(t)\| \leq M/c^\gamma \int_0^t \frac{\|v(\theta) - v^*(\theta)\|}{(t-\theta)^{1-\delta}} d\theta$$

By induction, one gets

$$\|v(t) - v^*(t)\| \leq \frac{M^{n+1}}{c^{n\gamma}} \frac{t^{n\delta} (\Gamma(\delta))^n}{\Gamma(n\delta + 1)}$$

As $n \rightarrow \infty$, we get $v(t) = v^*(t)$

The following theorem proves that the solution $v(t)$ of equation (2.3) depends continuously on $f(t)$

Theorem 2.2. Let $\varepsilon > 0$. If $\|f(t)\| \leq \varepsilon$, for all $t \in J$, then

$$\|v(t)\| \leq \varepsilon E_\sigma \left[\frac{M\Gamma(\sigma)t^\sigma}{c^\sigma} \right], \text{ Where; } E_\sigma \text{ is the Mittag-Leffler function defined by } E_\alpha(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\alpha + 1)},$$

Proof: If $\|f(t)\| \leq \varepsilon$, we get

$$\|v(t)\| \leq \varepsilon + \frac{M}{c^\gamma} \int_0^t \frac{\|v(\theta)\|}{(t-\theta)^{1-\sigma}} d\theta,$$

The last inequality leads to the required result.

Let us now try to discuss the ill-posed problem (1.1).

For this purpose, we need some of additional conditions. Suppose that for every $i = 1, \dots, r$ and every $t_1, t_2, t_3 \in J$, the operator $Q(t_1)$ commutes with $K_i(t_2, t_3)$ and commutes with $A_i(t_3)$:

$$Q(t_1) K_i(t_2, t_3)f = K_i(t_2, t_3) Q(t_1) f, \tag{2.4}$$

For all $f \in E$, $t_1, t_2, t_3 \in J$, $t_2 > t_3$, and

$$Q(t_1) A_i(t_3) f = A_i(t_3) Q(t_1) f, \tag{2.5}$$

For all $f \in \cap_{i=1}^r A_i$, $t_1, t_3 \in J$.

Theorem 2.3. Under all the previous conditions, there exists a dense set S in E, such that if $f(t) \in S$, for all $t \in J$, then equation (1.1) can be solved. Moreover, this solution is unique and continuously depending on f.

Proof: Let S be the set of all elements, of the form $f_n(t) = Q\left(\frac{1}{n}\right) f(t)$, $n = 1, 2, \dots$, $f(t) \in E$, $t \in J$. It is clear that S is dense in E. According to theorem 2.1, there exists for every n a unique continuous solution $v_n(t)$ of the equation

$$= \tilde{f}\left(t, \frac{t}{n\Gamma}\right) + \int_0^t \sum_{i=1}^r K_i(t, \theta) A_i(\theta) \tilde{v}\left(\theta, \frac{t}{n\Gamma} - \frac{\theta}{n\Gamma}\right) d\theta$$

Set $u_n(t) = Q\left(\frac{1}{n} - \frac{t}{n\Gamma}\right) v_n(t)$ and using conditions (2.4)

and (2.5), we find that for every n , $u_n(t)$ satisfies the equation

$$u_n(t) = Q\left(\frac{1}{n}\right) f + \int_0^t \sum_{i=1}^r K_i(t, \theta) A_i(\theta) u_n(\theta) d\theta$$

Notice that according to theorem (2.2), $u_n(t)$ depends continuously on $f(t)$.

As a direct application, we consider the following abstract Hilfer fractional integral equation:

$$u(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha-1} A_1(\theta) u(\theta) d\theta, \quad (2.6)$$

where;

$$f(t) = \frac{t^{(\gamma-1)(\alpha-1)}}{\Gamma(\gamma(1-\alpha)+\alpha)} g, \quad g \in E, \quad (2.7)$$

$$0 < \alpha \leq 1, 0 < \gamma \leq 1.$$

Equation (2.6) can be solved if we replace g by $Q\left(\frac{1}{n}\right) g$.

3. Singular Integro-differential Equations

Consider the following equation

$$u(x, t) = f(x, t) + \int_0^t \sum_{i=1}^r \sum_{|q| \leq m} K_i(t, \theta) a_{iq}(\theta) D^q u(x, \theta) d\theta \quad (3.1)$$

where; $x = (x_1, \dots, x_k) \in \mathbb{R}^k$, \mathbb{R}^k is the k -dimensional Euclidean space,

$q = (q_1, \dots, q_k)$ is a multi-index, $|q| = q_1 + \dots + q_k$,

$$D^q = D_1^{q_1} \dots D_k^{q_k}, D_j = \frac{\partial}{\partial x_j}, \quad j = 1, \dots, k.$$

The coefficients a_{1q}, \dots, a_{rq} are real continuous functions on J , for all $|q| \leq m$.

For $t > \theta$, the kernels $K_1(t, \theta), \dots, K_r(t, \theta)$ are real continuous functions such that;

$$|K_i(t, \theta)| \leq \frac{M}{(t - \theta)^{1-\alpha}}, \quad (3.2)$$

M is a positive constant, $0 < \alpha \leq 1, t > \theta, t, \theta \in J$.

Let $C^m(\mathbb{R}^k)$ be the set of all real continuous functions on \mathbb{R}^k , which have continuous derivatives of order less than or equal to m .

Let $W^m(\mathbb{R}^k)$ be the completion of $C^m(\mathbb{R}^k)$ with respect to the norm:

$$\|g\|_m^2 = \sum_{|q| \leq m} \int_{\mathbb{R}^k} |D^q g(x)|^2 dx.$$

Let $E = L_2(\mathbb{R}^k)$ be the space of all square integral functions on \mathbb{R}^k .

It is assumed that f is a real function defined on \mathbb{R}^k such

that for every $t \in J, f \in L_2(\mathbb{R}^k)$.

In this case we can choose the operator B by

$$B = \left[\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_k^2} \right]^{2N+1}$$

The strongly continuous semi group $Q(t)$ with the infinitesimal generator B is given by

$$(Q(t)\phi)(x) = \int_{\mathbb{R}^k} G(x - y, t) \phi(y) dy,$$

Where; G is the fundamental solution of the following equation

$$\frac{\partial u(x, t)}{\partial t} = \left[\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_k^2} \right]^{2N+1} u(x, t)$$

For sufficiently large N , we can find constants $M > 0, \gamma > 0, 0 < \gamma < \alpha$, such that

$$\|D^q Q(t)\phi\| \leq \frac{M}{t^\gamma} \|\phi\|, \quad (3.3)$$

$$|q| \leq m,$$

Where $\|\cdot\|$ is the norm in $L_2(\mathbb{R}^k)$,

$$\|\phi\|^2 = \int_{\mathbb{R}^k} \phi^2(x) dx.$$

The domain of definitions of the operators $A_i(t) = \sum_{|q| \leq m} a_{q_i}(t) D^q$ and B are $W^m(\mathbb{R}^k)$ and $W^{N+2}(\mathbb{R}^k)$ respectively.

These domains of definitions are dense in $L_2(\mathbb{R}^k)$ and the operators $A_i(t)$ and B are closed in $L_2(\mathbb{R}^k)$.

It is clear that the operators $A_1(\theta), \dots, A_r(\theta)$ are commute with $Q(t)$, for all $t, \theta \in J$.

The given function f as a function of t with values in $L_2(\mathbb{R}^k)$ is supposed to be continuous in $t \in J$ with respect to the norm in $L_2(\mathbb{R}^k)$.

Now all the conditions of section 2 are satisfied.

As in theorem (2.1), we solve in $L_2(\mathbb{R}^k)$ the following equation:

$$v_n(x, t) = \tilde{f}(x, t, ct) + \int_0^t \sum_{i=1}^r \sum_{|q| \leq m} K_i(t, \theta) D^q \tilde{v}_n(\theta, ct - c\theta) d\theta,$$

Where; $c = \frac{1}{n\Gamma}$, we find that for every $n, u_n = Q\left(\frac{1}{n} - \frac{1}{n\Gamma}\right) v_n$, solves the equation

$$u_n(x, t) = f_n(x, t) + \int_0^t \sum_{i=1}^r \sum_{|q| \leq m} K_i(t, \theta) a_{iq}(\theta) D^q u_n(x, \theta) d\theta,$$

Where;

$$f_n(x, t) = \int_{\mathbb{R}^k} G(x - y, \frac{1}{n}) f(y, t) dy$$

As a special case, we can solve the following Hilfer fractional integro-partial differential equation

$$u_n(x, t) = f_n(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha-1} \sum_{|q| \leq m} a_q(\theta) D^q u_n(x, \theta) d\theta, \quad (3.4)$$

where;

$$f_n(x, t) = \frac{t^{(\gamma-1)(\alpha-1)}}{\Gamma(\gamma(1-\alpha)+\alpha)} g_n(x),$$

$$0 < \gamma \leq 1, \quad 0 < \alpha \leq 1,$$

$$g_n(x) = \int_{R^k} G(x-y, \frac{1}{n}) g(y) dy, \quad g \in L_2(R^k),$$

Notice that the solution of (3.1) or (3.2) are elements of $W^{4N+2}(R^k)$, for every n and every

$$t < T. \text{ see [4,6,7-18].}$$

Notice also that equations (3.1) and (3.4) are solved without any restrictions on the characteristic forms of the principle parts of the partial operators

Let us study the following equation:

$$u(x, t) = f(x, t) + \int_0^t K(t, \theta) \sum_{|q| \leq 2m} a_q(x) D^q u(x, \theta) d\theta, \tag{3.5}$$

where; K is continuous for $t > \theta$ and satisfies condition (3.2) and f satisfies the conditions as in theorem (3.1).

let B be the operator defined by:

$$B = A^{2N+1},$$

where;

$$A = \sum_{|q| \leq 2m} a_q(x) D^q,$$

Let us suppose some regularity conditions on $a_k; D^q a_q \in C_b(R^k), |q| \leq N^*$, for sufficiently large $N^*, N^* > 2m(2N+1)$,

where;

$C_b(R^k)$ is the set of all continuous bounded functions on R^k .

Let us suppose also that

$$(-1)^{m+1} \sum_{|q|=2m} a_q(x) y^q \geq M |y|^{2m}, \tag{3.6}$$

For all $x \in R^k$, where M is a positive constant, $|y|^2 = y_1^2 + \dots + y_k^2$

The operators A and B are closed operators in $L_2(R^k)$ with domain of definitions $W^{2m}(R^k)$ and $W^{2m(2N+1)}(R^k)$ respectively.

The operator $B = A^{2N+1}$ is the infinitesimal semi-group $Q(t)$ defined by

$$(Q(t) \phi)(x) = \int_{R^k} G(x-y, t) \phi(y) dy,$$

Where G is the fundamental solution of the equation

$$\frac{\partial u(x, t)}{\partial t} = \left[\sum_{|q|=2m} a_q(x) D^q \right]^{2N+1} u(x, t)$$

According to the properties of the fundamental solution G, we deduce that the operator Q satisfies condition (3.3), (see [19-22])

Lemma 3.1 let $f \in W^{2m}(R^k)$, Then $A Q(t) f = Q(t) A f$, for all $t \in J$.

Proof: It is easy to see that $A_\mu B_\lambda f = B_\lambda A_\mu f$,

Where A_μ and B_λ are the linear bounded operators

$$A_\mu = -\mu [I - \mu (I_\mu - A)^{-1}],$$

$$B_\lambda = -\lambda [I - \lambda (I_\lambda - B)^{-1}].$$

Thus;

$$\lim_{\mu \rightarrow \infty} A_\mu B_\lambda f = A B_\lambda f = \lim_{\mu \rightarrow \infty} B_\lambda A_\mu f = B_\lambda A f.$$

Thus;

$$A e^{tB_\lambda} f = e^{tB_\lambda} A f$$

$$\text{Now } \lim_{\lambda \rightarrow \infty} e^{tB_\lambda} f = Q(t) f,$$

$$\text{And } \lim_{\lambda \rightarrow \infty} A e^{tB_\lambda} f = Q(t) A f,$$

But A is closed, thus

$$A Q(t) f = Q(t) A f$$

Notice that equation (3.5) is still ill-posed, but we can now solve the following equation:

$$u_n(x, t) = f_n(x, t) + \int_0^t K(t, \theta) \sum a_q(x) D^q u_n(x, \theta) d\theta,$$

where;

$$f_n(x, t) = \int_{R^k} G(x, y, \frac{1}{n}) f(y, t) dy,$$

$$u_n(x, t) = \int_{R^k} G(x, y, \frac{1}{n} - \frac{t}{nT}) v_n(y, t) dy,$$

$u_n \in W^{2m(2N+1)}(R^k)$, for every $t \in J$ and every $n = 1, 2, \dots$ (v_n will be determined as in theorem (2.2)).

4. Nonlinear Equations

The results in sections 2 and 3 are obtained after suitable modification on the given function f, which plays in some special cases the role of initial condition.

For the non-linear equation (1.2), we can obtain solutions with stability properties, without conditions (2.4) and (2.5), but after suitable modifications on f and F. see [23-37].

Let us consider in the Banach space E, the following equation;

$$v(t) = Q(ct) f(t) + Q(ct) \int_0^t F(V^*(t, \theta)) d\theta, \tag{4.1}$$

where

$$V^*(t, \theta) = (K_1(t, \theta) A_1(\theta) \tilde{V}(\theta, c - \frac{c\theta}{T}), \dots, K_r(t, \theta) A_r(\theta) \tilde{V}(\theta, c - \frac{c\theta}{T})),$$

Where $c > 0$,

$$V^*(t, \theta) = (K_1(t, \theta) A_1(\theta) \tilde{V}(\theta, c - \frac{c\theta}{T}) = Q(c - \frac{c\theta}{T}) v(\theta).$$

Theorem 4.1 under the conditions on f, F, Q, K_i, A_i of sections 1 and 2, (but without conditions (2.4) and (2.5)), there exists a unique solution $v(t) \in E$ of equation (4.1). This solution is continuous in $t \in J$ and is continuously depending on f, (with respect to the norm in E).

Proof: Using condition (1.4), we find that the proof is similar to the proof in theorems (2.1) and (2.2).

Let $c = \frac{1}{n}$, $u_n(t) = Q(c - \frac{ct}{T}) v_n(t)$, where; $v_n(t)$ is the

solution of (4.1) with $c = \frac{1}{n}$, $0 < t < T$.

It is clear that for every n , $u_n(t)$ satisfies the following equation, (without conditions (2.4) and (2.5)),

$$u_n(t) = f_n(t) + \int_0^t F_n(U_n(t, \theta)) d\theta,$$

Where;

$$U_n(t, \theta) = (K_1(t, \theta) A_1(\theta) u_n(\theta), \dots, K_r(t, \theta) A_r(\theta) u_n(\theta)),$$

$$f_n = Q\left(\frac{1}{n}\right) f, \quad F_n = Q\left(\frac{1}{n}\right)$$

As an application, we can study the following equation in the space $L^2(\mathbb{R}^k)$;

$$u(x, t) = f(x, t) + \int_0^t F(V(x, t, \theta)) d\theta, \quad (4.2)$$

Where u satisfies the conditions in section 3 and V is the following:

$$V(x, t, \theta) = (K_1(t, \theta) A_1(\theta) u(\theta), \dots, K_r(t, \theta) A_r(\theta) u(\theta)),$$

K_1, \dots, K_r are real functions and satisfies the conditions in section 3.

The operators A_1, \dots, A_r are

$$A_i(\theta) = \sum_{|q| \leq m} a_{qi}(x, \theta) D^q, \quad i = 1, \dots, r$$

$$a_{qi} \in C_b(\mathbb{R}^k \times J), \quad |q| \leq m,$$

In this case, we can use the strongly continuous semi-group $Q(t)$ with the infinitesimal generator

$$B = \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_k^2} \right)^{2N+1}$$

The well posed version of equation (4.2) is given by:

$$u_n(x, t) = f_n(x, t) + \int_0^t F_n(V_n(x, t, \theta)) d\theta, \quad (4.3)$$

Where; $f_n(x, t) = \int Q(x - y, \frac{1}{n}) f(y, t) dy$

5. Conclusions

A general singular integral evaluation equation is studied in Banach space. The abstract parabolic transform is constructed to investigate a wide class of ill-posed problems. The Hilfer fractional integro-partial differential equations are studied without any restrictions on the characteristic forms.

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