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Global Existence and Nonexistence of Solutions to a Cross Diffusion System with Nonlocal Boundary Conditions

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Abstract Mathematical models of nonlinear cross diffusion are described by a system of nonlinear partial parabolic equations associated with nonlinear boundary conditions. Explicit analytical solutions of such nonlinearly coupled systems of partial differential equations are rarely existed and thus, several numerical methods have been applied to obtain approximate solutions. In this paper, based on a self-similar analysis and the method of standard equations, the qualitative properties of a nonlinear cross-diffusion system with nonlocal boundary conditions are studied. We are constructed various self-similar solutions to the cross diffusion problem for the case of slow diffusion. It is proved that for certain values of the numerical parameters of the nonlinear cross-diffusion system of parabolic equations coupled via nonlinear boundary conditions, they may not have global solutions in time. Based on a self-similar analysis and the comparison principle, the critical exponent of the Fujita type and the critical exponent of global solvability are established. Using the comparison theorem, upper bounds for global solutions and lower bounds for blow-up solutions are obtained.

Keywords Cross-diffusion, Critical Exponents, Global Solvability, Blow-up, Self-similar Analysis

1. Introduction

In this paper, we studied the qualitative properties of solutions of a nonlinear cross-diffusion system associated with nonlocal boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\upsilon^{m_1 - 1} \left| \dfrac{\partial u}{\partial x} \right|^{p-2} \dfrac{\partial u}{\partial x} \right), \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{m_2 - 1} \left| \dfrac{\partial \upsilon}{\partial x} \right|^{p-2} \dfrac{\partial \upsilon}{\partial x} \right), \ x \in R_+, \ t > 0, \end{cases} \quad (1)$$

$$\begin{cases} -\upsilon^{m_1 - 1} \left| \dfrac{\partial u}{\partial x} \right|^{p-2} \dfrac{\partial u}{\partial x}(0,t) = u^{q_1}(0,t), \\ -u^{m_2 - 1} \left| \dfrac{\partial \upsilon}{\partial x} \right|^{p-2} \dfrac{\partial \upsilon}{\partial x}(0,t) = \upsilon^{q_2}(0,t), \ t > 0, \end{cases} \quad (2)$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \quad (3)$$

where $p > \max\{m_1, m_2\} + 1$, $m_i > 1$, $q_i > 0 (i = 1,2)$, u_0 and $\upsilon_0(x)$ – are non-negative continuous functions with compact support in R_+.

Recently, there has been a surge in the analysis and simulation of mathematical models of the reaction-diffusion type in the presence of the so-called cross-diffusion. Cross-diffusion is a process in which a concentration or density gradient of one chemical or biological type induces a flow (linear or non-linear) of another type. The concept of cross-diffusion also includes well-known cases of modeling chemo- and hypo taxis. The application of reaction-cross-diffusion of a system is easily found in literature and includes the pattern formation development in biology [16], electrochemistry, cancer motility [5, 8, 11, 29] and biofilms [10]. By introducing cross-diffusion into standard reaction-diffusion models, it hs shown that there is no cross-diffusion when preventing explosion phenomena associated with such systems [4]. Explicit analytical solutions to these complex and often nonlinearly coupled systems of partial differential equations rarely exist, and thus several numerical methods have been applied to obtain approximate solutions.

Cross-diffusion models are found in various fields of natural science. For example, in physical systems (plasma physics) [1, 12, 23], in chemical systems (dynamics of electrolytic solutions), in biological systems (cross-diffusion transport, dynamics of population systems), in ecology (dynamics of forest age structure), in seismology - Burridge-Knopoff model describing the tectonic plates interaction [13-15, 21, 29]. In recent years, in the study of biological population and the tectonic plates movement, mathematical models with cross - diffusion have been widely used [14, 15, 29-34].

The local existence (in time) of solutions to systems of parabolic equations was established by authors [5, 6, 24-27] in a series of important papers. For existence of solution, not much is known, however. Under the different condition on the cross-diffusion flow, or the initial values u_0 and υ_0 was obtained the global existence [25-27]. In recent years, by the many authors the condition for the global existence of solutions and the condition for the occurrence of a blow-up regime for various boundary value problems have been intensively studied (see [5-26]).

A general treatment of global blow-up, as well as lower bounds for regional and single point blow-up for arbitrary nonlinear equations and systems with the parabolic monotonicity property (including Neumann boundary conditions), was performed by the method of stationary states in [15] and [28], where some estimates of L^∞ norms of the components and their supports are obtained.

It is known that systems of degenerate equations may not have a classical solution in the region, where $u, \upsilon \equiv 0$. For system (1)-(3), the local existence and the comparison principle of weak solutions are defined in the usual integral way (see [18, 28]). In this paper, we assume that the solution belongs to class

$$0 \le u(x,t), 0 \le \upsilon(x,t) \in C\left(R_+ \times (0,+\infty)\right),$$

$$\upsilon^{m_1-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x},\ u^{m_2-1}\left|\frac{\partial \upsilon}{\partial x}\right|^{p-2}\frac{\partial \upsilon}{\partial x} C\left(R_+ \times (0,+\infty)\right).$$

In [16], the conditions of global solvability and insolubility in time of a solution were studied, and the solution was estimated near the explosion time of a nonlocal diffusion problem

$$u_t = u_{xx},\quad \upsilon_t = \upsilon_{xx},\quad x>0,\ 0<T<0, \tag{4}$$

$$-u_x(0,t) = u^\alpha \upsilon^p,\ \ -\upsilon_x(0,t) = u^q \upsilon^\beta,\ \ 0<t<T, \tag{5}$$

$$u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x),\ x>0. \tag{6}$$

It is proved that if $pq \le (1-\alpha)(1-\beta)$, then every solution to the problem (4)-(6) is global.

In [11], the systems of cross-diffusion equations on a stationary surface of the following form are studied

$$\begin{cases} \dfrac{\partial u_m}{\partial t} - \displaystyle\sum_{k=1}^{r} d_{mk}\Delta_\Gamma u_k = f_m(u_1,...,u_r),\ in\,\Gamma\times(0,T), \\ u_m(x,0) = u_{0,m}(x),\ \forall_x \in \Gamma,\ m=1,...,r \end{cases}$$

where $r \ge 1$. They provide a fully-discrete scheme by applying the Implicit–Explicit Euler method. In addition, they provide sufficient conditions for the existence of polytopal invariant regions for the numerical solution after spatial and full discretization. Furthermore, they prove optimal error bounds for the semi- and fully-discrete methods, that is, the convergence rates are quadratic in the mesh size and linear in the time step.

In [24] the following problem is investigated

$$u_t = \left(u^n\right)_{xx},\quad \upsilon_t = \left(\upsilon^k\right)_{xx},\quad x\in R_+,\ t>0, \tag{7}$$

$$\begin{aligned} -\left(u^n\right)_x(0,t) &= \upsilon^p(0,t), \\ -\left(\upsilon^k\right)_x(0,t) &= u^q(0,t),\quad t>0, \end{aligned} \tag{8}$$

$$u(x,0) = u_0(x),\quad \upsilon(x,0) = \upsilon_0(x),\quad x\in R_+. \tag{9}$$

It is shown that the solution to problem (7)-(8) is global if $pq \le (n+1)(k+1)/4$. Conditions were obtained on the numerical parameters of systems (7) - (9) under which the solution to the problem explodes in a finite time.

The studies given in [10] should also be noted; there, system (7) was studied with the following boundary conditions

$$\begin{aligned} -\left(u^n\right)_x(0,t) &= u^\alpha \upsilon^p(0,t), \\ -\left(\upsilon^k\right)_x(0,t) &= u^q \upsilon^\beta(0,t),\quad t>0. \end{aligned}$$

It is shown that $\min\{y_1 - r_1, y_2 - r_2\} = 0$, where

$$r_1 = \frac{2p+k+1-2\beta}{4pq-(k+1-2\alpha)(n+1-2\beta)},$$

$$r_2 = \frac{2p+n+1-2\beta}{4pq-(k+1-2\alpha)(n+1-2\beta)},$$

$$y_1 = \frac{1-r_1(n-1)}{2},\quad y_2 = \frac{1-r_2(k-1)}{2},$$

are the critical Fujita exponents.

In [23], Yongsheng Mi, Chunlai Mu, and Botao Chen considered the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\right), & x>0,\ 0<T<0, \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\right), & x>0,\ 0<T<0, \end{cases}$$

$$\begin{cases} -\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\Bigg|_{x=0} = \upsilon^{q_1}(0,t), \quad 0<T<0, \\ -\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\Bigg|_{x=0} = u^{q_2}(0,t), \quad 0<T<0, \end{cases}$$

$$u(x,0)=u_0(x),\quad \upsilon(x,0)=\upsilon_0(x),\quad x>0.$$

They showed that the critical global existence exponent and critical Fujita exponent are

$$q_1q_2 = \frac{(2p_1-1+m_1)(2p_2-1+m_2)}{p_1p_2}$$

and

$$\min\{l_1-k_1, l_2-k_2\}=0,$$

where

$$k_1 = \frac{(p_2-1)p_1q_1+(p_1-1)(2q_2+m_2+1)}{q_1q_2p_1p_2-(2q_1+m_1+1)(2q_2+m_2+1)},$$

$$k_2 = \frac{(p_1-1)p_2q_2+(p_2-1)(2q_1+m_1+1)}{q_1q_2p_1p_2-(2q_1+m_1+1)(2q_2+m_2+1)},$$

$$l_1 = \frac{k_2q_1-k_1(p_1+m_1-2)}{p_1-1},$$

$$l_2 = \frac{k_1q_2-k_2(p_2+m_2-2)}{p_2-1}.$$

In [33] considered cross-diffusion induced pattern formation in a prey–predator model with Rosenzweig–MacArthur type reaction kinetics in a one- dimensional spatial domain. Authors this work investigated the bifurcation of travelling wave solution into Turing patterns and transition of one pattern into another in the presence of cross-diffusion.

Esther S. Daus, Ansgar Jüngel and Bao Quoc Tang [34] studied the large-time asymptotics of weak solutions to Maxwell–Stefan diffusion systems for chemically reacting fluids with different molar masses and reversible reactions. They obtained the exponential decay to the unique equilibrium with a rate that is constructive up to a finite-dimensional inequality.

The purpose of this study is to find the conditions of existence and nonexistence of solutions to problem (1) -(3) over time based on self-similar analysis. Various self-similar solutions to the problem (1)-(3) are constructed, estimates of the solutions are obtained, the critical Fujita exponents and critical exponents for the global existence of the solution are established.

Introduce the notation

$$\beta = \frac{(q_1-1)(q_2-1)-(p-2)(q_2-1)-(m_1-1)(q_1-1)}{p(q_1-1)(q_2-1)-(p-2)(q_2-1)-(m_1-1)(q_1-1)} = $$

$$= \frac{(q_1-1)(q_2-1)-(p-2)(q_1-1)-(m_2-1)(q_2-1)}{p(q_1-1)(q_2-1)-(p-2)(q_1-1)-(m_2-1)(q_2-1)},$$

$$\alpha_1 = \frac{(p-1)(q_2-1)}{l_1},\quad \alpha_2 = \frac{(p-1)(q_1-1)}{l_2},$$

$$l_1 = p(q_1-1)(q_2-1)-(p-2)(q_1-1)-(m_2-1)(q_2-1),$$

$$l_2 = p(q_1-1)(q_2-1)-(p-2)(q_2-1)-(m_1-1)(q_1-1).$$

2. Main Results

Theorem 2.1. Let $\min\{l_1,l_2\}>0$, then, any solution to problem (1)-(3) is unbounded for sufficiently large initial data.

Proof. Introduce new functions $\hat{u}(x,t)$ and $\vartheta(x,t)$ of the types:

$$\begin{cases} \hat{u}(x,t) = (T-t)^{-\alpha_1}\varphi(\xi), \\ \vartheta(x,t) = (T-t)^{-\alpha_2}\phi(\xi),\ \xi = x(T-t)^{-\beta},\ x\ge 0,\ t\ge 0, \end{cases} \tag{10}$$

which are self-similar ones. Theorem 1 can be proved at $T>0$. Substituting these functions in (1)-(3), we obtain a self-similar problem consisting of the following systems of equations relative to $\varphi(\xi)$ and $\phi(\xi)$:

$$\begin{cases} \dfrac{d}{d\xi}\left(\phi^{m_1-1}\left|\dfrac{d\varphi}{d\xi}\right|^{p-2}\dfrac{d\varphi}{d\xi}\right) - \beta\xi\dfrac{d\varphi}{d\xi} - \alpha_1\varphi = 0, \\ \dfrac{d}{d\xi}\left(\varphi^{m_2-1}\left|\dfrac{d\phi}{d\xi}\right|^{p-2}\dfrac{d\phi}{d\xi}\right) - \beta\xi\dfrac{d\phi}{d\xi} - \alpha_2\phi = 0, \end{cases} \tag{11}$$

$$\begin{cases} -\phi^{m_1-1}\left|\dfrac{d\varphi}{d\xi}\right|^{p-2}\dfrac{d\varphi}{d\xi}(0) = \varphi^{q_1}(0), \\ -\varphi^{m_2-1}\left|\dfrac{d\phi}{d\xi}\right|^{p-2}\dfrac{d\phi}{d\xi}(0) = \phi^{q_2}(0). \end{cases} \tag{12}$$

obtained by substituting (10) into (1)-(3) and some simplifications. Define the conditions under which (10) is an unbounded lower solution to problem (1)-(3). As the compared functions, we choose the following ones:

$$\begin{cases} \tilde{\varphi}(\xi) = A_1(a-\xi)^k, \\ \tilde{\phi}(\xi) = A_2(a-\xi)^z, \end{cases} \tag{13}$$

where $A_i>0\ (i=1,2)$,

$$k = \frac{(p-1)(p-m_1-1)}{(p-2)^2-(m_1-1)(m_2-1)},\quad z = \frac{(p-1)(p-m_2-1)}{(p-2)^2-(m_1-1)(m_2-1)}.$$

To apply the comparison theorem, the following inequalities are required:

$$\begin{cases} \dfrac{d}{d\xi}\left(\tilde{\varphi}^{m_1-1}\left|\dfrac{d\tilde{\varphi}}{d\xi}\right|^{p-2}\dfrac{d\tilde{\varphi}}{d\xi}\right)-\beta\xi\dfrac{d\tilde{\varphi}}{d\xi}-\alpha_1\tilde{\varphi}\ge 0, \\ \dfrac{d}{d\xi}\left(\tilde{\varphi}^{m_2-1}\left|\dfrac{d\tilde{\phi}}{d\xi}\right|^{p-2}\dfrac{d\tilde{\phi}}{d\xi}\right)-\beta\xi\dfrac{d\tilde{\phi}}{d\xi}-\alpha_2\tilde{\phi}\ge 0, \end{cases}$$

$$\begin{cases} -\tilde{\phi}^{m_1-1}\left|\dfrac{d\tilde{\varphi}}{d\xi}\right|^{p-2}\dfrac{d\tilde{\varphi}}{d\xi}(0)\le\tilde{\varphi}^{q_1}(0), \\ -\tilde{\varphi}^{m_2-1}\left|\dfrac{d\tilde{\phi}}{d\xi}\right|^{p-2}\dfrac{d\tilde{\phi}}{d\xi}(0)\le\tilde{\phi}^{q_2}(0). \end{cases}$$

Two systems of inequalities with respect to A_1 and A_2 are obtained

$$\begin{cases} A_1^{p-2}A_2^{m_1-1}-\alpha_1 a+(\alpha_1+k\beta)\xi\ge 0, \\ A_1^{m_2-1}A_2^{p-2}-\alpha_2 a+(\alpha_2+z\beta)\xi\ge 0, \end{cases} \tag{I}$$

$$\begin{cases} A_1^{p-1}A_2^{m_1-1}k^{p-1}a^k\le A_1^{q_1}a^{kq_1}, \\ A_2^{p-1}A_1^{m_2-1}z^{p-1}a^z\le A_2^{q_2}a^{zq_2}. \end{cases} \tag{II}$$

$a\le\min\left\{\dfrac{A_1^{p-2}A_2^{m_1-1}k^p}{\alpha_1},\dfrac{A_1^{m_2-1}A_2^{p-2}z^p}{\alpha_2}\right\},\quad \min\{l_1,l_2\}>0.$ Similarly, we can find the condition for systems (II); the upper bound for parameter a is:

$$a\ge\max\left\{\left(\frac{A_2^{m_1-1}k^{p-1}}{A_1^{q_1-p+1}}\right)^{\frac{1}{k(q_1-1)}},\left(\frac{A_1^{m_2-1}z^{p-1}}{A_2^{q_2-p+1}}\right)^{\frac{1}{z(q_2-1)}}\right\}.$$

Thus, choosing the parameters A_1, A_2, a we can obtain the system of inequalities (I, II) under condition $\min\{l_1,l_2\}>0$. By the principle of comparing solutions, we have estimates for the initial data, with respect to the lower self-similar solutions (10), (13):

$$\begin{cases} u_0(x)\ge T^{-\alpha_1}A_1\left(a-xT^{-\beta}\right)_+^k, \\ \upsilon_0(x)\ge T^{-\alpha_2}A_2\left(a-xT^{-\beta}\right)_+^z. \end{cases}$$

It follows that the solution to problem (1) - (3) is unbounded

$$\begin{cases} u(x,t)\ge(T-t)^{-\alpha_1}\tilde{\varphi}(0)\to\infty,\ t\to T, \\ \upsilon(x,t)\ge(T-t)^{-\alpha_2}\tilde{\phi}(0)\to\infty,\ t\to T. \end{cases}$$

at $\min\{l_1,l_2\}>0$. The theorem is proved.

Theorem 2.2. Let $\max\{\alpha_1-\beta,\alpha_2-\beta\}<0$ and the initial data are sufficiently small, then any solution to problem (1)-(3) is global.

Proof. Constructing bounded upper solutions, we can determine the conditions of solvability in time in the following way:

$$\begin{cases} u_+(x,t)=(T+t)^{-\alpha_1}f(\xi), \\ \upsilon_+(x,t)=(T+t)^{-\alpha_2}g(\xi),\ \xi=x(T+t)^{-\beta}, \end{cases} \tag{14}$$

where $T>0$, $f(\xi)$ and $g(\xi)$ are the sought for functions, which, by the solution comparison theorem, must satisfy the system of inequalities:

$$\begin{cases} \dfrac{d}{d\xi}\left(g^{m_1-1}\left|\dfrac{df}{d\xi}\right|^{p-2}\dfrac{df}{d\xi}\right)+\beta\xi\dfrac{df}{d\xi}+\alpha_1 f\le 0, \\ \dfrac{d}{d\xi}\left(f^{m_2-1}\left|\dfrac{dg}{d\xi}\right|^{p-2}\dfrac{dg}{d\xi}\right)+\beta\xi\dfrac{dg}{d\xi}+\alpha_2 g\le 0, \end{cases} \tag{15}$$

$$\begin{cases} -g^{m_1-1}\left|\dfrac{df}{d\xi}\right|^{p-2}\dfrac{df}{d\xi}(0)\ge f^{q_1}(0), \\ -f^{m_2-1}\left|\dfrac{dg}{d\xi}\right|^{p-2}\dfrac{dg}{d\xi}(0)\ge g^{q_2}(0). \end{cases} \tag{16}$$

Along with this, consider the following

$$\begin{cases} \bar{f}(\xi)=A_1\left(a-(\xi+h)^{\frac{p}{p-1}}\right)^k, \\ \bar{g}(\xi)=A_2\left(a-(\xi+h)^{\frac{p}{p-1}}\right)^z, \end{cases} \tag{17}$$

where $h\in\left(0,a^{\frac{p-1}{p}}\right)$, $a>0$, $A_2^{m_1-1}A_1^{p-2}\left(\dfrac{kp}{p-1}\right)^{p-1}=\beta$, $A_1^{m_2-1}A_2^{p-2}\left(\dfrac{zp}{p-1}\right)^{p-1}=\beta$.

The solvability of systems of inequalities (15)-(16) with respect to unknown parameters a, h, and under conditions $q_1>m_2+1$, $q_2>m_1+1$ is shown as follows. Substituting functions (17) into (15) and (16), we obtain

$$\begin{cases} (\alpha_1-\beta)\left(a-(\xi+h_1)^{\frac{p}{p-1}}\right)\le 0, \\ (\alpha_2-\beta)\left(a-(\xi+h_2)^{\frac{p}{p-1}}\right)\le 0. \end{cases}$$

hence the condition for the restrictions on $\max\{\alpha_1-\beta,\alpha_2-\beta\}<0$ and conditions for further calculations of a, h are given in the form:

$$a\le h^{\frac{p}{p-1}}+\min\left\{\left(A_1^{1-q_1}h\beta\right)^{\frac{1}{k(q_1-1)}},\left(A_2^{1-q_2}h\beta\right)^{\frac{1}{z(q_2-1)}}\right\} \tag{18}$$

Given this, we can conclude that if $\max\{\alpha_1-\beta,\alpha_2-\beta\}<0$ and the initial functions $u_0(x)$ and $\upsilon_0(x)$ satisfy the following inequalities:

$$\begin{cases} u_0(x) \le T^{-\alpha_1} A_1 \left(a - \left(xT^{-\beta} + h \right)^{\frac{p}{p-1}} \right)^k, \\ \upsilon_0(x) \le T^{-\alpha_2} A_2 \left(a - \left(xT^{-\beta} + h \right)^{\frac{p}{p-1}} \right)^z, \end{cases}$$

then the solution to problem (1) - (3) is global. The values of parameters a, h are selected from condition (18).

Theorem 2.3. If $q_1 \le 1$, $q_2 \le 1$, then every solution to problem (1)-(3) is global.

Theorem 3 is proved by the method described in [27].

Note 1. Theorem 3 shows that the critical exponents of the global existence of a solution are $q_{10} = 1$, $q_{20} = 1$.

Note 2. Theorem 1 shows that the critical Fujita exponents are $\min\{\alpha_1 - \beta, \alpha_2 - \beta\} = 0$.

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