

Construction of Triangles with the Algebraic Geometry Method

Viliam Ďuriš^{1,*}, Timotej Šumný²

¹Department of Mathematics, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, 94974 Nitra, Slovakia

²Benkova Primary School, Benkova 34, 949 11 Nitra, Slovakia

Received April 25, 2020; Revised May 19, 2020; Accepted June 23, 2020

Copyright©2020 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract The accuracy of geometric construction is one of the important characteristics of mathematics and mathematical skills. However, in geometrical constructions, there is often a problem of accuracy. On the other hand, so-called 'Optical accuracy' appears, which means that the construction is accurate with respect to the drawing pad used. These "optically accurate" constructions are called approximative constructions because they do not achieve exact accuracy, but the best possible approximation occurs. Geometric problems correspond to algebraic equations in two ways. The first method is based on the construction of algebraic expressions, which are transformed into an equation. The second method is based on analytical geometry methods, where geometric objects and points are expressed directly using equations that describe their properties in a coordinate system. In any case, we obtain an equation whose solution in the algebraic sense corresponds to the geometric solution. The paper provides the methodology for solving some specific tasks in geometry by means of algebraic geometry, which is related to cubic and biquadratic equations. It is thus focusing on the approximate geometrical structures, which has a significant historical impact on the development of mathematics precisely because these tasks are not solvable using a compass and ruler. This type of geometric problems has a strong position and practical justification in the area of technology. The contribution of our work is so in approaching solutions of geometrical problems leading to higher degrees of algebraic equations, whose importance is undeniable for the development of mathematics. Since approximate constructions and methods of solution resulting from approximate constructions are not common, the content of the paper is significant.

Keywords Algebraic geometry, Approximation, Biquadratic Equation, Cubic Equations, Euclidean Construction, GeoGebra

1. Introduction

The accuracy of geometric constructions is one of the most important characteristics in mathematics, but geometric tasks often result in two types of accuracy problems. The first level is perfect accuracy, which means that the object can be infinitely expanded or shrunk, but the positional features of its points do not change. This is the result of a logical sequence of steps used in its construction. The basic foundation here is a finite set of true statements, which are based on the Greek-Hellenistic foundations of geometry [1]. On the other hand, there is the so-called optical precision, which means that the design is accurate with respect to the drawing board. To consider such a construction as accurate with respect to the drawing board, the deviation must not be greater than the design tool, and it must be implemented through a finite number of steps. These optically precise structures are termed approximate constructions because they do not achieve perfect accuracy, but the best possible approximation. In general, approximate structures also include any structures that are accurate, but non-Euclidean, i.e. the design uses some special curves (e.g. a fixed conic section) or another instrument (e.g. a thread, etc.). However, due to the current onset of computer geometry, approximative constructions are being phased out.

Many solutions of historical tasks, such as the trisection of an angle, or the quadrature of the circle, are solved by curves. These curves can be approximated through a point cloud, which means that the curves are drawn by means of their individual points and we are trying to determine their intersection. Since the search for the intersection in the point cloud effectively means a search for the points that merge into a single point, it is more preferable to approximate the curve only in a certain limited space around the projected intersection. In addition to curve approximation in the limited space, a mesh grid method can be used to determine the points of a curve more easily. This

curve approximation is possible thanks to the use of straight lines and circular curves. The approximation method also differs according to the required level of accuracy. Apart from the construction itself, this method also offers the possibility of exploring various curves and their properties.

In addition to curve approximation in the limited space, a mesh grid method can be used to determine the points of a curve more easily. In technical practice, graph paper is used for complex curves. Graph paper has a square grid. This grid helps us achieve a better orientation on paper because the grid points are firmly locked. Thanks to this feature of grid points, we can better orientate on the plane, which makes approximate constructions more accurate and easier. Among other things, the search for the intersections of the curves under consideration with the grid points provides an insight into diophantine geometry, which is used in the number theory, study of elliptic curves, but also in quantum systems.

Procedures similar to those used in arithmetics, i.e. the basic operations such as a sum and difference, multiplication and division, or exponentiation and square roots, can also be considered in geometric designs. Just like the arithmetic expressions gradually give way to algebraic equations, geometric tasks may lead to the theory of algebraic equations. In the work *Geometry* [2] by René Descartes, we see how Descartes uses the known geometrical methods to construct the lines and how he algebraizes them. Subsequently, he concludes that the algebraic equations that can be decomposed to the simplest elements, are associated with geometric constructions. If these elements only include a quadratic form, which can be solved by using the roots, these tasks are solvable by classical geometry using a compass and ruler. These considerations have so far only been verbalized as presumptions and they only pointed to the tasks that can be solved with certainty. However, they have not contributed to the solution of unsolved tasks. On the other hand, it is already known, which tasks are solvable by a compass and ruler, and what the conditions of solvability are. For this, we can use the elements of analytical geometry, which are applied on the basic principles of Euclidean constructions.

Let us consider the lines that can be expressed using a linear equation in the form

$$ax + by + c = 0,$$

and a circle expressed by a quadratic equation

$$(x - m)^2 + (y - n)^2 = r^2.$$

And gradually apply these equations on the principles of Euclidean constructions:

1. We can construct a line determined by two points: if the points $[x_1, y_1]$ and $[x_2, y_2]$ are defined in the coordinate system, then the straight line passing through these points has the form of

$$x(y_2 - y_1) + y(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0.$$

2. Construct an intersection of two divergent lines: if we have two divergent straight lines defined by the equations

$$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0,$$

then the point defined as the intersection of these lines has the coordinates

$$\left[\frac{b_1c_2 - b_2c_1}{a_1b_1 - a_2b_2}, -\frac{a_1c_2 - a_2c_1}{a_1b_1 - a_2b_2} \right].$$

3. Construct a circle with a given center and radius

$$(x - m)^2 + (y - n)^2 = r^2.$$

4. Construct the intersection of a straight line and circle: we use the above equations as the equations representing a circle and straight line:

$$x = -\frac{a(am+bn+c) \pm b\sqrt{(a^2+b^2)r^2 - (am+bn+c)^2}}{a^2+b^2} + m,$$

$$y = -\frac{b(am+bn+c) \pm a\sqrt{(a^2+b^2)r^2 - (am+bn+c)^2}}{a^2+b^2} + n.$$

5. Construct the intersection of two set circles: the circles are defined by a system of equations

$$(x - m_1)^2 + (y - n_1)^2 = r_1^2,$$

$$(x - m_2)^2 + (y - n_2)^2 = r_2^2.$$

The solution of this system and the relevant conditions for the solution can be determined by rearranging the first equation and using the second equation to express the relation for y :

$$(x - m_1)^2 = r_1^2 - (y - n_1)^2,$$

$$y = m_2 \pm \sqrt{r_2^2 - (x - m_2)^2}.$$

For example, let us choose the positive solutions (we could proceed analogically in the case of negative solutions):

$$(x - m_1)^2 = r_1^2 - \left(m_2 + \sqrt{r_2^2 - (x - m_2)^2} - n_1 \right)^2.$$

By reverse squaring, we get an equation of the fourth degree

$$\frac{(2x^2 - 2x(m_1 + m_2) + m_1^2 + m_2^2 - (m_2 - n_1) - r_2^2 - r_1^2)^2}{4(m_2 - n_1)^2} = r_2^2 - (x - m_2)^2.$$

It should be pointed out that when solving this type of equation of the fourth degree, we get a quadratic equation when suitable substitution is used. As is the case with the straight line and circle, this points to the possibility of only construct the root, or possibly compound root, values even with two circles method.

This knowledge of geometric structures and basic knowledge of algebraic structures can help us define the theory of construction of geometric objects using a compass and ruler, which is independent of the skills and knowledge of a geometer, but its limits stem directly from the properties of the algebraic body. Each geometric task

begins with the baseline state, in which the fixed points determine the initial structure – the so-called minimum body T_0 – consisting of real numbers. Then all other Euclidean points belong to the same minimum body T_0 or the body T_1 , which was created as an adjunct of the clearly defined square of a positive element from T_0 .

2. Solution of Algebraic Equations

Geometric tasks correspond to algebraic equations in two ways. The first method is based on the structure of algebraic expressions, which change into an equation. The second method is based on the methods of analytic geometry where the geometric objects and points are directly expressed by the equations that describe their properties in a coordinate system. In both cases, we get an equation whose solution in the algebraic sense corresponds to the solution in the geometric sense.

In the equation of the first degree in the form $ax + b = 0$ where $a, b \in \mathbb{R}$, the solution can be found simply by rearranging the equation into the form $x = -\frac{b}{a}$. In the equation of the second degree in the form $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$ we make a rearrangement into the form $x^2 + px + q = 0$ where $p = \frac{b}{a}$, $q = \frac{c}{a}$. Then, we use a rearrangement using the identity $(m + n)^2 = m^2 + 2mn + n^2$ and get the equation in the form $(x + \frac{p}{2})^2 + q - \frac{p^2}{4} = 0$. Thence $x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$, and the expression $q - \frac{p^2}{4} = D$ is the so-called discriminant of a quadratic equation. If the discriminant assumes negative values, the solution is in the set of complex numbers. The second way to determine the roots of a quadratic equation is the use of Viet relations [3] $x^2 + px + q = (x - x_1)(x - x_2) = 0$. Then $x^2 - (x_2 + x_1)x + x_1x_2 = 0$. By way of comparison, we get the relations $x_2 + x_1 = -p$, $x_1x_2 = q$. Based on these findings, we have several ways to graphically represent the quadratic equation:

1. The quadratic equation can be interpreted as a parabolic equation whose intersections with the axis x (line $y = 0$) are the roots of the considered equation. For this solution, we need to know the coordinates of the vertex of the parabola, which can be determined from the quadratic equation once rearranged into the form $(x + \frac{p}{2})^2 + q - \frac{p^2}{4} = 0$. Then, the vertex of the parabola in the Cartesian coordinates is $V[-\frac{p}{2}, q - \frac{p^2}{4}]$.
2. If the quadratic equation is replaced by a Viet relation $x_2 + x_1 = -p$, $x_1x_2 = q$, we can interpret these relations as a task to determine the intersections of the lines and the hyperbole defined by the equations $x + y = -p$, $xy = q$. The intersections with the coordinates $[x, y]$ determine the solutions $x = x_1$, $y = x_2$ and $x = x_2$, $y = x_1$.

These graphical methods are disadvantageous when there is no solution in the set of real numbers and the intersection of the considered geometric shapes does not exist. Let us now consider an equation of the third degree in the form $az^3 + bz^2 + cz + d = 0$ where $a, b, c, d, z \in \mathbb{R}$. By substituting $x = z - \frac{b}{3a}$, we rearrange the equation to the form $x^3 + px + q = 0$ and introduce substitution with two new unknowns $x = u + v$. Then $(u + v)^3 + p(u + v) + q = 0$, and after rearrangement, we get $u^3 + v^3 + (u + v)(3uv + p) + q = 0$. Equality is also maintained with the conditions $u^3 + v^3 = -q$, $uv = -\frac{p}{3}$ that resemble the Viet relations for quadratic equations. To use these, we must rearrange them into the form $u^3 + v^3 = -q$, $u^3v^3 = -\frac{p^3}{27}$. We can now use the substitution $u^3 = y_1$, $v^3 = y_2$ to obtain the Viet relations for the quadratic equation in the form $y_1 + y_2 = -q$, $y_1y_2 = -\frac{p^3}{27}$. From the above, we can make a quadratic equation (a quadratic resolvent of the cubic equation) $y^2 + qy - \frac{p^3}{27} = 0$. Then, $D = q^2 + \frac{4p^3}{27}$ is the discriminant of the quadratic resolvent. Subsequently, we get the roots $y_{1,2} = \frac{-q \pm \sqrt{D}}{2}$ and the solution of the quadratic equation can be arrived at by using the substitution $x_k = u_k + v_k$, $k = 1, 2, 3$. Thus, we get the equation $u^3 - y_1 = 0$, $v^3 - y_2 = 0$. These tasks can then be seen as an extension of the equation $\varepsilon^3 - 1 = 0$. One root $\varepsilon_1 = 1$ can be determined trivially, and by decomposing it into the product form, we get the relation $(\varepsilon - 1)(\varepsilon^2 + \varepsilon + 1) = 0$. By solving this quadratic equation, we get the remaining roots $\varepsilon_{2,3} = \frac{-1 \pm i\sqrt{3}}{2}$. Let us then return to the solution in the form

$$x_k = u_k + v_k, \quad k = 1, 2, 3. \quad \text{Then } u = \sqrt[3]{-\frac{q}{2} + \sqrt{q^2 + \frac{4p^3}{27}}},$$

$$v = \sqrt[3]{-\frac{q}{2} - \sqrt{q^2 + \frac{4p^3}{27}}} \quad \text{and we get } u_1 = u, \quad u_2 = u\varepsilon_2,$$

$u_3 = u\varepsilon_3$, $v_1 = v$, $v_2 = v\varepsilon_3$, $v_3 = v\varepsilon_2$ as the solutions. The properties of the solutions depend on the discriminant. If $D > 0$, the cubic equation has one real root and the other two roots are complex-compound. If $D = 0$, the equation has all three real roots, and the root is double or triple. If $D < 0$, the equation has three different but real roots.

A cubic equation can also be solved trigonometrically even if the quadratic discriminant resolvent is a negative number (this case is historically termed as *cassus irreducibilis* [4]). Our task is to determine the third root of a complex number. In this case, it is appropriate to use a goniometric form of the complex number $y_1 = -\frac{q}{2} + i\sqrt{D} = r(i \sin \varphi + \cos \varphi)$, and the equality of complex numbers results in $r \sin \varphi = \sqrt{D}$, $r \cos \varphi = -\frac{q}{2}$. By squaring and summing these equalities, we determine $r = \sqrt{(\frac{q}{2})^2 - D}$, and after rearrangement, we get $r =$

$\sqrt{-\frac{p^3}{27}}$. We then continue with the solution $y_1 = \sqrt{-\frac{p^3}{27}}(\cos \varphi + i \sin \varphi)$, $y_2 = \sqrt{-\frac{p^3}{27}}(\cos \varphi - i \sin \varphi)$. Then $y_1 = u_k^3, y_2 = v_k^3$. When using the Moivre's formula [5], we get $k = 1, 2, 3$:

$$u_k = \sqrt{-\frac{p}{3}} \left(\cos \left(\frac{\varphi + 2(k-1)\pi}{3} \right) + i \sin \left(\frac{\varphi + 2(k-1)\pi}{3} \right) \right),$$

$$v_k = \sqrt{-\frac{p}{3}} \left(\cos \left(\frac{\varphi + 2(k-1)\pi}{3} \right) - i \sin \left(\frac{\varphi + 2(k-1)\pi}{3} \right) \right).$$

Then $x_k = \sqrt{-\frac{p}{3}} \cos \left(\frac{\varphi + 2(k-1)\pi}{3} \right)$. In addition to the algebraic solution, one can also point to the geometric method using the conic sections, which is used by the Persian mathematician Omar Chajjan [1]. Take the equation $ax^3 + bx^2 + cx + d = 0$, which we divide by the expression ax and we get the relation $x^2 + \frac{b}{a}x + \frac{c}{a} + \frac{d}{x} = 0$. We then rearrange the equation so that the expressions on the right and left are positive, and we get the equation $x^2 + px + q = \frac{r}{x}$. We then look for the common point of the curves $y_1 = x^2 + px + q, y_1 = \frac{r}{x}$.

Let us now consider an equation of the fourth degree in the form $z^4 + az^3 + bz^2 + cz + d = 0$, which we rearrange by substitution $x = \frac{z-a}{4}$ into the form $x^4 + Ax^2 + Bx + C = 0$. Similarly to the cubic equation, we use substitution $x = u + v + w$ with more variables and we get the equation

$$(u^2 + v^2 + w^2)^2 + 4(u^2v^2 + u^2w^2 + v^2w^2) + A(u + v + w) + C + (uv + uw + vw)[4(u^2 + v^2 + w^2) + 2A] + (u + v + w)(8uvw + B) = 0.$$

Let us determine the conditions $u^2 + v^2 + w^2 = -\frac{A}{2}$, $u^2v^2 + u^2w^2 + v^2w^2 = \frac{A^2 - 4C}{16}$, $uvw = -\frac{B}{8}$ and just like in the cubic equation where the quadratic equation is determined by the Viet relations, we will do the same for the construction of cubic equation $y^3 + \frac{A}{2}y^2 + \frac{A^2 - 4C}{16}y - \frac{B^2}{64} = 0$. This cubic equation is an Euler resolvent [6] of an algebraic equation of the fourth degree. The roots are $y_1 = u^2, y_2 = v^2, y_3 = w^2$ and the relations can be written as $z_1 = \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}, z_2 = \sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}, z_3 = -\sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}, z_4 = -\sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3}$. We see that to solve a fourth degree equation, it is necessary to master the methods to solve cubic equations. The solution properties depend directly on the solution properties of the cubic equation. If the cubic resolvent had positive real numbers in all three roots, the solution of the contemplated equation would be a quadruplet of real numbers. If the cubic equation has real numbers as roots, but two roots take negative values, all four roots will have complex roots. If the cubic resolvent has two complex roots, then all four roots are complex.

Since the algebraic approach is quite demanding, a graphical solution with a circle and a fixed parabola was used in the past. Let us consider the equations $(x - m)^2 + (y - n)^2 = r^2, y = x^2$ and $x^4 + Ax^2 + Bx + C = 0$ while m, n are the coordinates of the circle center and r is its radius. By rearrangement, we get the relations $(x - m)^2 + (x^2 - n)^2 = r^2, x^4 + x^2(1 - 2n) - 2mx + m^2 + n^2 - r^2 = 0$ and by comparing the coefficients, we get the necessary data for the geometric solution $m = -\frac{A}{2}$,

$n = \frac{1-A}{2}, r = \sqrt{C - \frac{1-2A+2A^2}{2}}$. We see that the geometric solution is much faster, but in terms of Euclidean constructions, it is tied to the construction of a parabola, which is either predetermined, or merely approximated in the envisaged solutions.

For a general solution of the equations of a higher degree, algebra does not provide us with predetermined formulas as is the case in the fourth degree equations. However, special cases of equations of higher degrees are solved, such as binomial and trinomial, reciprocal and antireciprocal equations. What is more, we need to be able to determine whether the equation at least has rational roots for the Euclidean structures. Even in the case of cubic equations, we transitioned to a simple binomial equation. Since it is a low degree equation in an appropriate form, we can easily get one solution, and the procedure is aimed at the decomposition into a lower degree polynomial. Furthermore, when solving *cassus irreducibilis*, Moivre's formula can be used, which facilitates the calculation of any square (square root) of a complex number. By combining this knowledge, we can solve the equation $x^n - z = 0, x, z \in \mathbb{C}$. Using the Moivre's formula, is true that $x_k = \sqrt[n]{|z|} \left(\cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right)$ while $z = a + bi, |z| = \sqrt{a^2 + b^2}$. As we can see, the solution of a binomial equation can be represented on a complex plane as a regular n -gram.

The equation in the form $x^{2n} + x^n - z = 0, x, z \in \mathbb{C}$ is termed trinomial. To solve it, we use substitution $y = x^n$ and subsequently get a quadratic equation in the form $y^2 + y - z = 0$ with the roots y_1, y_2 . Then, we get a pair of binomial equations $x^n - y_1 = 0, x^n - y_2 = 0$.

With reciprocal equations, we consider an algebraic equation in the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ while for $a_k = a_{n-k}, k = 0, 1, 2, \dots, n$ we get a reciprocal equation of the first degree, and for $a_k = -a_{n-k}, k = 0, 1, 2, \dots, n$ we get a reciprocal equation of the second degree. A reciprocal equation is characterized by symmetry in its coefficients. Each reciprocal equation of the second degree has a root $x_0 = 1$ [7]. After dividing it by a binomial $(x - 1)$, we get a reciprocal equation of the first degree. Each odd reciprocal equation of the first degree has a root $x_0 = 1$. After dividing it by a binomial $(x - 1)$, we get an even reciprocal equation of the first degree. And, for each reciprocal equation of the first degree, the substitution $y = x + \frac{1}{x}$ helps us create a new half degree polynomial. If the algebraic equation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ with integer

coefficients has a rational solution, it has the form of $x = \frac{p}{q}$, while $p, q \in \mathbb{Z}$, $p, q \neq 0$ and at the same time $(p, q) = 1$. Then, the root is determined by $p|a_n$ and $q|a_0$.

The knowledge of algebraic structures and solutions of algebraic equations is suitable for the solution of geometric tasks using a compass and ruler. Next, we will use this knowledge to show the Euclidean solvability and non-solvability of selected geometric problems in the structure of triangles. Therefore, it is necessary to reshape the geometric problems into algebraic equations.

3. Construction of Triangles

In this section, we address some selected tasks in the work [8], which are selected by the author with the aim to construct the triangles using the method of algebraic geometry. The tasks present the parameters necessary to construct the triangles, the triangles are constructible (in an Euclidean or approximated way) and the algebraic solutions are aiming at an equation of a higher than third degree. The constructions are implemented in the GeoGebra environment. After each solution with the geometric method a solution with the algebraic geometry method is presented.

Task 1. Construct a triangle if $|AB| = a$, $|\sphericalangle BAC| = \alpha$ and the radius r of the circumcircle (a, α, r) is known.

Solution. The initial situation is that we have a sketched angle α , which is defined by the point A and arms p, q . The center of the incircle is moving on the circle $k_1(A, r)$. Then we determine the moving circle $k_{r1}(S, r)$, $S \in k_1$. Then, to determine the set of points characteristic of the solved constellation of parameters, we will determine the auxiliary moving point B_1 , which is defined as an intersection of the arm p of angle α and the circle k_{r1} . Subsequently, we determine the circle $k_{a1}(B_1, a)$ from the point B_1 . Then, the curve, which is specific to the side and circumcircle parameters, is determined by the point C_i , $C_i \in k_{a1} \cap k_{r1}$. It is true for the sought point C that it is located at the intersection of the specified curve and arm q of the angle α .

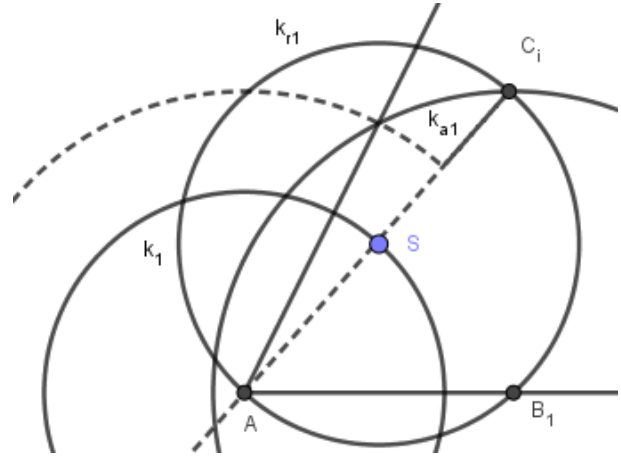


Figure 1. When applying the procedure from the searched-for set of points of the given properties, we get the curve that meets the desired conditions and shows how the radius of the circumcircle, size of the opposite side to the angle and the angle itself are related

As shown in Figure 1, the curve used is composed of a circular arc and a straight line segment (if the point B is to the right of the point A). On this basis, we can determine that the task either has no solution, i.e. the intersection of the arm q and curve does not exist, or it is in the section formed by the circular curve. The second option is that the solution exists, but the arm q coincides with the part of the curve determined by the straight line. In this case, the task does not have enough specific parameters and therefore it has no solution based on the specified parameters.

When solving the tasks with algebraic geometry, we use the relationship resulting from the sinus formula $2r = \frac{a}{\sin \alpha}$. However, based on that relationship, we see that the specified data are not linearly independent, and we cannot identify the other data necessary to construct the sought triangle.

At first glance, the task looks very standard. We have three pieces of data that appear to be independent. When using the geometric method, we can describe the conditions necessary to construct the sought triangle. On the other hand, the use of algebraic geometry requires us to use the sinus formula that connects the data in a single formula, but it lacks the necessary information for the required construction.

Task 2. Construct a triangle if $|BC| = a$, $|AS_a| = t_a$ is known, S_a is the center of the side BC , and ρ is the radius of the incircle (a, t_a, ρ) .

Solution. We begin the task by the known dimension of the side $|BC| = a$. Consequently, since we know the median of this side, we can place the point $S_a, S_a \in BC, |BS_a| = |S_aC| = \frac{a}{2}$ on it. Furthermore, the median t_a is represented by the circle $k_{ta}(S_a, t_a)$. We know that the center of the incircle S_1 moves on a straight line p parallel to the base, and we know that $|pBC| = \rho$. Furthermore, we know that the sides of the triangle belong to the circle $k_{\rho 1}(S_1, \rho)$ with the touching straight lines q, s . Then, it is true for the point A_1 that $A_1 \in q \cap s$, and when moving the circle $k_{\rho 1}$, we get the curve that is characteristic of all the triangles, in which we know one side and the radius of the incircle. Then, the sought point A is the intersection of this curve and the circle k_{ta} .

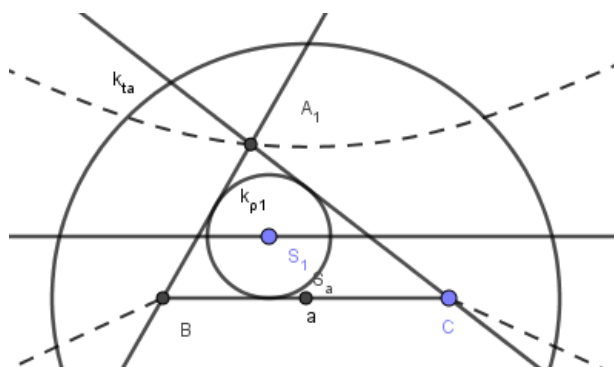


Figure 2. The construction of the triangle uses a curve, which was created based on the parameters of the side $|BC| = a$ and the radius of the incircle

Using the curve (Figure 2), we can judge the solvability conditions. We see that two solutions are possible in the upper half-plane. While two solutions are also possible in the lower half-plane, they are symmetrical with respect to the midpoint of the line segment BC . Or, there is a single solution in the upper half-plane and no solution in the lower half-plane.

When using algebraic geometry, our solution is based on the situation that the median divides the triangle into two triangles with the same area because the base is divided in half and the triangles have a base of the same length. The height of each triangle is the same because it is defined by the same point. Then we can determine the equations for area using Heron's formula [9]

$$S = \sqrt{\frac{1}{2}(a + 2c + 2t_a)(c + t_a)(a - c + 2t_a)(a + 2c - t_a)},$$

$$S = \sqrt{\frac{1}{2}(a + 2b + 2t_a)(b + t_a)(a - b + 2t_a)(a + 2b - t_a)},$$

and a formula for the calculation of triangle area, knowing the radius of the incircle $2S = \rho(a + b + c)$. Then we put

together the equations:

$$\rho^2(a + b + c)^2 = 2(a + 2c + 2t_a)(c + t_a)(a - c + 2t_a)(a + 2c - t_a),$$

$$\rho^2(a + b + c)^2 = 2(a + 2b + 2t_a)(b + t_a)(a - b + 2t_a)(a + 2b - t_a).$$

By rearranging the equations, we get a cubic equation for the calculation of the selected side, which makes our task easier because we know the two sides and the median to one of them.

Using the geometric method, we focus on determining the curve, which is characteristic of constructing the point of the triangle if we know one of its sides and the radius of the incircle. The curve is relatively easy to approximate, but it cannot be constructed in an Euclidean way, and so the solution is only approximate. On the other hand, we have compiled the relations arising from the calculation of the triangle when using the algebraic geometry method. By adjusting these relationships, we points out that the task aims at the construction of a cubic equation. Based on the cubic equation, we know that the task is not solvable with Euclidean means. As mentioned above, the rearrangement of algebraic formulas adds to the difficulty of the task using this method. The fact that the geometric method resulted in a pair of symmetric solutions to the axis of the side a corresponds to the solution using algebraic geometry where the similarity of expressions for the sides b, c may result in a confusion, thereby leading us to a solution that is identical with the geometrical result.

Task 3. Construct a triangle if $|BC| = a, |BS_b| = t_b$ is known, S_b is the center of the side BC , and ρ is the radius of the incircle (a, t_b, ρ) .

Solution. We begin the task by the fact that we know the length of the side $|BC| = a$. Consequently, since we know the median on the opposite side, we know that the center of the side b lies on the circle $k_{tb}(B, t_b)$. Then we can determine the properties of the point A based on this median. We determine the curve with the point A in a way that we freely choose the presumed center of the side b , so that $S_{b1} \in k_{tb}$. We determine the half-line CS_{b1} and circle $k_{b1}(S_{b1}, |CS_{b1}|)$. Then, the sought curve is determined by the point $A_{1tb}, A_{1tb} \in CS_{b1} \cap k_{b1}$. The second curve that determines the point A is obtained moving the incircle with center S_1 on the straight line p parallel to the base, while we know that $|p, BC| = \rho$. Furthermore, we know that the sides of the triangle belong to the circle $k_{\rho 1}(S_1, \rho)$ with the touching straight lines q, s . Then, it is true for the point $A_{1\rho}$ that $A_{1\rho}, A_{1\rho} \in q \cap s$, and when moving the circle $k_{\rho 1}$, we get the curve that is characteristic of all the triangles, in which we know one side and the radius of the incircle. The sought point A is an intersection of these curves (Figure 3).

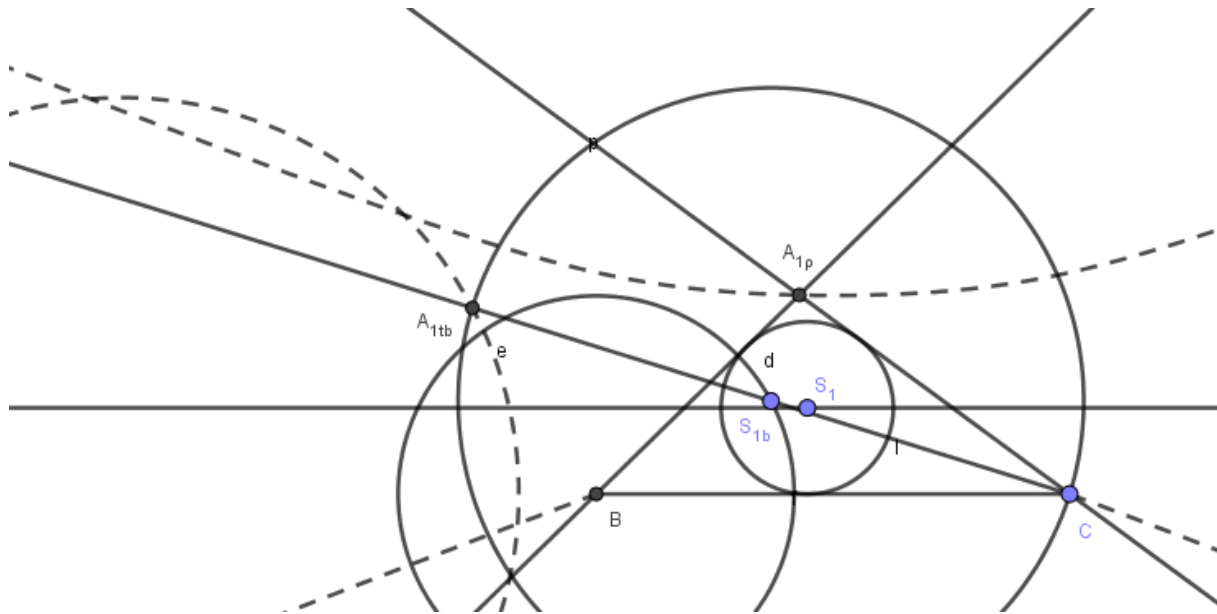


Figure 3. Construction of the triangle uses two curves whose intersections are the sought points. The first curve is determined by the side $|AC| = a$ and the radius of the incircle. The second curve is determined by the properties of the median t_b and side $|BC| = a$.

There are three intersections of the curves, and the intersection of the curves in the lower half-plane has no geometric significance for the task solved. The other two intersections of curves offer a pair of different solutions. Using the point we found, we will complete the construction by knowing the side b , which simplifies the task into one with the known angle α , side $|BC| = a$ and median on the side $b (t_b)$.

When using algebraic geometry, we make use of the fact that the median divides the triangle into two triangles with the same area, with each triangle having half the area of the original one. We can then use the equality of area of the resulting triangles using Heron's formula, and the relationship for the calculation of the area of a triangle (knowing the circumference and radius of the incircle):

$$S = \sqrt{(2a + b + 2t_a)(2t_a + b)(2a + 2t_a - b)(2a + b)},$$

$$S = \sqrt{(2c + b + 2t_b)(c + 2t_b)(2a - c + 2t_b)(2a + c)},$$

$$2S = \rho(a + b + c).$$

Thence we get:

$$\rho^2(a + b + c)^2 = 2(2a + b + 2t_a)(2t_a + b)(2a + 2t_a - b)(2a + b),$$

$$\rho^2(a + b + c)^2 = (2c + b + 2t_b)(c + 2t_b)(2a - c + 2t_b)(2a + c).$$

The first equation helps us express the relationship for the side c and where it is in the square. The expression for the side c can then be used in the second equation in order to determine the side b . With these rearrangements, we can make an equation of the sixth degree. Using the point we found, we will complete the construction by knowing the length of the side b , which simplifies the task into one

with the known angle α , side $|AC| = b$ and median t_b on the side b .

The solution shows us that when using the methods of algebraic geometry, we have to identify a pair of curves which are characteristic of the features of the sought point. The first curve is determined by the characteristics resulting from the median, and the second curve is determined by the properties of the incircle in the triangle. For comparison, the solution implementing the methods of algebraic geometry highlights the problems that occur when rearranging algebraic expressions. The subsequent solution of the sixth degree equation is not solvable algorithmically, which means that the actual algebraic solution requires the knowledge of numerical approximation methods [10].

Task 4. Construct a triangle if $|\sphericalangle BAC| = \alpha, |BS_{b1}| = t_b$ is known, S_{b1} is the center of the side AC , and ρ is the radius of the incircle (α, t_b, ρ) .

Solution. The angle α is defined by the point A and arms defined by the half-line. The incircle $k_\rho(S, \rho)$ is inscribed into this angle. Let us select a point S_{b1} on the half-line p . From the point S_{b1} , we select the circle $k_{1tb}(S_{b1}, t_b)$. Then we get the point $B_1, B_1 \in q \cap k_{1tb}$. Then, we draw a touch straight line a_1 from the point B_1 to the circle k_ρ . Further, from the point S_{b1} we select the circle $k_{1b/2}(S_{b1}, |AS_{b1}|)$ and determine the point $C_{1A}, C_{1A} \in p \cap k_{1b/2}$. Next, we determine the circle $k_{1a}(B_1, |B_1C_{1A}|)$ by which we determine the point $C_{1B}, C_{1B} \in k_{1a} \cap a_1$. The point C_{1B} defines the curve, which is characteristic of the point C under the defined conditions, and it does not lock the point on the angle arm α , i.e. the half-line p . The sought point C can be determined as an intersection of the curve with the arm of

angle α (half-line p) and the curve under consideration (Figure 4).

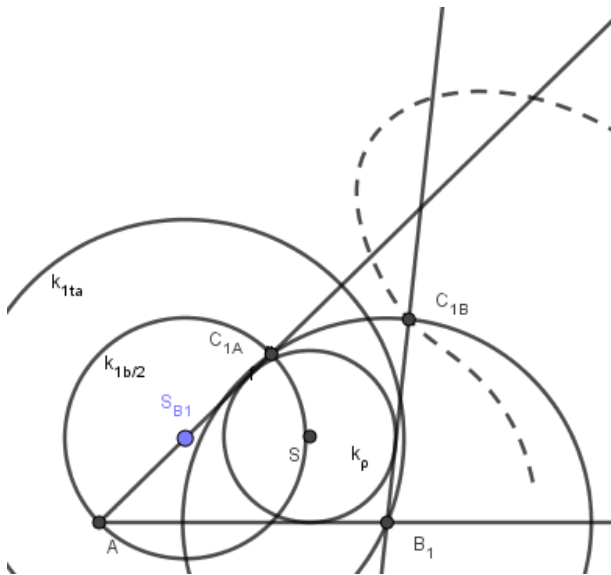


Figure 4. To solve the task, we use a curve, which is determined by the properties of the incircle inscribed into the triangle and the median t_b . Subsequently, the intersection of this curve with the angle arm determines the sought point of the triangle

Before we begin to solve the task by algebraic geometry, we must realize that the median divides the triangle into two triangles with the same area. Therefore, we will be looking for the relations associated with triangle area. We will use the Heron's formula for the triangles created by median division. Next, we use the calculation of the triangle area using the angle of the sides, and the formula for the calculation of the triangle area knowing the radius of the incircle and the perimeter of the triangle:

$$4S = \sqrt{(2a + 2t_b + b)(2t_b + b)(2a + b)(2a + 2t_b)},$$

$$4S = \sqrt{(2a + 2t_b + c)(2t_b + c)(2a + c)(2a + 2t_b)},$$

$$2S = b \cdot c \cdot \sin \alpha, \quad 2S = (a + b + c)\rho.$$

By comparing $2S = 2S$ we get the expression for $b = \frac{(a+c)\rho}{c \cdot \sin \alpha - \rho}$, which we put into a rearranged equation $4S = 4S$, i.e. the relation $(2a + 2t_b + b)(2t_b + b)(2a + b) = (2a + 2t_b + c)(2t_b + c)(2a + c)$. Then

$$(2a + 2t_b + c)(2t_b + c)(2a + c) = \left(2a + 2t_b + \frac{(a+c)\rho}{c \cdot \sin \alpha - \rho}\right) \left(2t_b + \frac{(a+c)\rho}{c \cdot \sin \alpha - \rho}\right) \cdot \left(2a + \frac{(a+c)\rho}{c \cdot \sin \alpha - \rho}\right).$$

Then we can express the side a as a solution of the quadratic equation. Then we determine the side c by substituting the continuously expressed expressions into the equation

$$4(a + b + c)^2 \rho^2 = (2a + 2t_b + b)(2t_b + b)(2a + b)(2a + 2t_b).$$

By gradually resolving this system, we can change the task into a triangle construction task if we know all three

sides.

It follows from the comparison of methods that in both cases it is necessary to resort to approximative methods. In the case of the geometric method, we approximate the curve determined by the properties of the points which belong to it. In the case of the algebraic geometry method, we follow from three relations for the triangle area, which we use to determine a series of four nonlinear equations with four unknowns. When using the algebraic geometry method, we are facing a difficulty in rearranging the algebraic expressions such as the multiplication of polynomials, which complicates the calculation. Additionally, the solution of the system of nonlinear equations has no algorithmic solution, and the most appropriate method is the gradual substitution. This means that we gradually express the unknowns from the various equations, to finally get one equation of the fourth degree.

Task 5. Construct a triangle if $|\angle BAC| = \beta, |BS_b| = t_b$ is known, S_b is the center of the side AC , and o is the perimeter of the triangle (β, t_b, o) .

Solution. We know the angle β , which is defined by the point B , and the arms defined by the half-lines p, q . On the half-line p we determine a line segment $|BA_1| = o$ and circle $k_{tb}(B, t_b)$, on which lies the center of the side b . We select the point $C_1, C_1 \in BA_1$ and then determine the circle $k_{1b}(C_1, |C_1A_1|)$. Then $A_{1i(j)}, A_{1i(j)} \in q \cap k_{1b}$, and we get two points. Subsequently $S_{1bi(j)} \in C_1A_{1i(j)}$ and $|C_1S_{1bi(j)}| = |S_{1bi(j)}A_{1i(j)}|$. The point $S_{1bi(j)}$ determines the curve on which lies the point S_b as the center of side b , and the intersection of this curve with the circle k_{tb} determines the center of the side b .

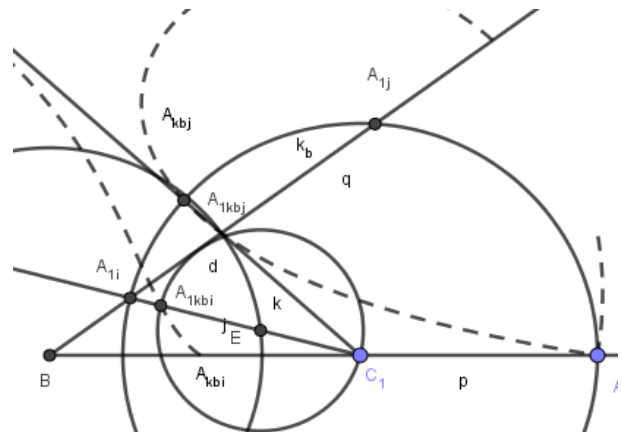


Figure 5. Based on the process and desired properties of the curve, we get two curves. For this reason, we used indexing in the construction characteristic of a curve in order to distinguish between them. Subsequently, the intersection of the envisaged curves with the arm angle determines the sought point A

Next, we determine the curve that determines the point C based on the characteristics determined by the circumference of angle α and using the median t_b . We use the selected point $C_1, C_1 \in BA_1$ and determine the circle $k_{1b}(C_1, |C_1A_1|)$. Then $A_{1i(j)}, A_{1i(j)} \in q \cap k_{1b}$, and we get two points. We select a half-line $C_1A_{1i(j)}$ and

$S_{2bi(j)} \in C_1A_{1i(j)} \cap k_{tb}$. Then we determine the circle $k_{1bs}(S_{2bi(j)}, |S_{2bi(j)}C_1S_{1bi(j)}|)$. The curve determining the sought point A is established by means of the point $A_{1bi(j)}$, which emerges as an intersection of the half-line $C_1A_{1i(j)}$ and circle k_{1bs} . Then, the intersection of the curve determined by the point $A_{1bi(j)}$ and the arm q determines the sought point A (Figure 5). Based on the points A, S_b we can clearly identify the point C and get all the points of the triangle ABC .

Before we begin to solve the task by means of algebra, we must realize that the median divides the triangle into two triangles with the same area. Therefore, we will be looking for the relations associated with triangle area. We will use the Heron's formula for the triangles created by median division. Next, we use the calculation of the triangle area using the angle between the sides. Another formula we use is the formula for the calculation of triangle angle using the lengths of the sides and their angle:

$$4S = \sqrt{(2a + b + 2t_b)(b + 2t_b)(2a + 2t_b)(2a + b)},$$

$$4S = \sqrt{(2c + b + 2t_b)(b + 2t_b)(2c + 2t_b)(2c + b)},$$

$$2S = b \cdot c \cdot \sin \alpha, \quad o = a + b + c.$$

After rearrangement, substitution and comparison, we get a system of two nonlinear equations with two unknowns, which lead to an equation of the fourth degree:

$$(a + o + c + 2t_b)(2a + 2t_b)(a + o - c) = (c + o - a + 2t_b)(2c + 2t_b)(c + o - a),$$

$$8(a + o - c + 2t_b)(o - a - c + 2t_b)(2a + 2t_b)(a + o - c) = (o - a - c)^2 c^2 \sin^2 \alpha.$$

The task solved with the geometric method is focusing on the properties resulting from the median. Thus, we determined the properties of the curves, which we used to determine the center of the side b and the sought point A . On the other hand, we used the solution with the method of algebraic geometry where we have put together a system of equations that help us determine the unknown sides, which changes the task into a triangle construction task knowing all three sides.

Task 6. Construct a triangle if $|AS_a| = t_a$ is the height to the side a , $|AS_a| = t_a$, where S_a is the center of the side AC , and o is the perimeter of the triangle (v_a, t_a, o) .

Solution. Let us select the line segment $|BA_1| = o$ on the line p . The height is represented by the parallel line $q, |p q| = v_a$. Let us select the point $C_1, C_1 \in BA_1$ and point S_{a1} where $|BS_{a1}| = |S_{a1}C_1|$. Let us select the circle $k_{ta}(S_{a1}, t_b)$ and circle $k_{b1}(C_1, |C_1A_1|)$. We get the point $A_{1k}, A_{1k} \in k_{ta} \cap k_{b1}$, which is characteristic of the curve determined for the triangle with a known circumference and median. Then, we get the sought point A as an intersection of the curve determined by the properties of the point A_{1k} and line q . Subsequently, we can determine the side $|AB| = c$, which provides us with a new parameter and simplifies the task (Figure 6).

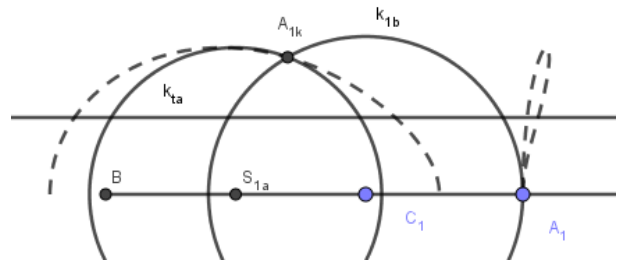


Figure 6. The curve designated by the properties resulting from the perimeter of the triangle and median t_a is determined by the point A_{1k} . Its intersection with the line q is the sought point A , which can be used to complete the construction of the sought triangle

When solving the task with algebraic geometry, it is necessary to realize the fundamental relations that can be used to build the equation. The median divides the triangle into two triangles with the same area. Based on this fact, we can put together an equation for the triangle area based on the height, and use Heron's formula:

$$2S = \sqrt{(a + 2t_a + 2b)(2t_a + b - a)(a + 2t_a - b)(a - t_a + b)},$$

$$2S = \sqrt{(a + 2t_a + 2c)(2t_a + c - a)(a + 2t_a - c)(a - t_a + c)},$$

$$2S = a \cdot v_a, \quad o = a + b + c.$$

We express $a = o - b - c$, and substitute this relation into the remaining equations:

$$(o - c + 2t_a + b)(2t_a + 2b - o + c)(o - c + 2t_a)(o - c - t_a) = (o - b - c) \cdot v_a,$$

$$(o - c + 2t_a + b)(2t_a + 2b - o + c)(o - c + 2t_a)(o - c - t_a) = (o - b + 2t_a + c)(2t_a + 2c - o + b)(o - b + 2t_a)(o - b - t_a).$$

By rearranging the equations, we can put together the cubic equations to determine the sides. The rearrangement of algebraic expressions, i.e. Multiplication of a few brackets that contain several members, is quite demanding, albeit algorithmic matter. Subsequently, once the algebraic equation of a higher degree is ready, it needs to be solved with the method of approximate calculations.

The solution of this tasks requires us to use the methods of approximate constructions. When solving the task with geometric methods, we use the properties resulting from the provided parameters. In the case of side height, we take advantage of the parallel line whose distance from the base line is determined by the side height. Another curve is the one determined by the circumference of the triangle and median of the side. We make use of the fact that the median is determined by the extreme and center of the opposite side. Subsequently, we determine the freely moving point C_1 , which we used to determine the sought curve, and that makes the point A easy to determine and effectively make the task easier because we know the side $|AB| = c$. On the other hand, we have pointed to the solution with algebraic geometry, and using the relations for triangle area we put together a system of nonlinear equations for determining the unknown sides. Using the substitution method, we

pointed out that the task aims at the construction of a cubic equation. The equation degree points to Euclidean solvability (and/or insolvability) of the task.

Task 7. Construct a triangle if v_a is the height to the side a , $|BS_b| = t_b$, where S_b is the center of the side AC , and o is the perimeter of the triangle (v_a, t_b, o)

Solution. Let us select the line segment $|BA_1| = o$ on the line p . The height is represented by the parallel line $q, |pq| = v_a$. Then we determine the circle $k_{tb}(B, t_b)$ with center of the side b . We select the point $C_1, C_1 \in BA_1$ and then determine the circle $k_{1b}(C_1, |C_1A_1|)$. Then $A_{1i(j)}, A_{1i(j)} \in q \cap k_{1b}$, and we get two points. Subsequently $S_{1bi(j)} \in C_1A_{1i(j)}$ and $|C_1S_{1bi(j)}| = |S_{1bi(j)}A_{1i(j)}|$. The point $S_{1bi(j)}$ determines the curve on which lies the point S_b as the center of side b , and the intersection of this curve with the circle k_{tb} determines the center of the side b . Next, we determine the curve that meets certain properties of the point A based on the characteristics determined by the circumference of angle α and median t_b . We use the selected point $C_1, C_1 \in BA_1$ and then determine the circle $k_{1b}(C_1, |C_1A_1|)$. Then $A_{1i(j)}, A_{1i(j)} \in q \cap k_{1b}$, and we get two points. We select the half-line $C_1A_{1i(j)}$ followed by point $S_{2bi(j)} \in C_1A_{1i(j)} \cap k_{tb}$ and determine the circle $k_{1bS}(S_{2bi(j)}, |S_{2bi(j)}C_1S_{1bi(j)}|)$. The curve with the sought point A is established by means of the point $A_{1bi(j)}$, which emerges as an intersection of the half-line $C_1A_{1i(j)}$ and circle k_{1bS} . Then, the intersection of the curve is determined by the point $A_{1bi(j)}$ and the arm q determines the sought point A (Figure 7). On the basis of the point A and point S_b we can clearly identify the point C .

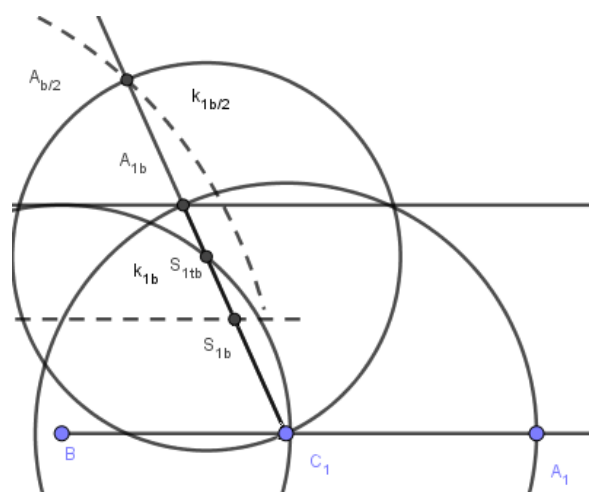


Figure 7. The picture shows a construction with two curves. The first curve is used to determine the center of the CA side. The second curve is used to determine the point A . Subsequently, if we know the triangle extreme and the center of the side, it is possible to finalize the construction

When solving the task with algebraic geometry, it is necessary to realize the fundamental relations that can be

used to build the equation. The median divides the triangle into two triangles with the same area. Based on this fact, we can put together an equation for the triangle area based on the height, and use Heron's formula:

$$2S = \sqrt{(2a + 2t_b + b)(2t_b + b - 2a)(2a - 2t_b + b)(2a + 2t_b - b)},$$

$$2S = \sqrt{(2c + 2t_b + b)(2t_b + b - 2c)(2c - 2t_b + b)(2c + 2t_b - b)},$$

$$2S = a \cdot v_a, \quad o = a + b + c.$$

We express $b = o - a - c$, and use this relation in the remaining equations:

$$(a + 2t_b + o - c)(2t_b + o - c - 3a)(a - 2t_b + o - c)(3a + 2t_b - o + c) = a \cdot v_a,$$

$$(a + 2t_b + o - c)(2t_b + o - c - 3a)(a - 2t_b + o - c)(3a + 2t_b - o + c) = (c + 2t_b + o - a)(2t_b + o - a - 3c)(c - 2t_b + o - a)(3c + 2t_b - o + a).$$

By rearranging the equations, we can put together the cubic equations to determine the sides. Subsequently, once the algebraic equation of a higher degree is ready, it needs to be solved with the method of approximate calculations.

This task is very similar to task $(\alpha, t_b, o), (v_a, t_a, o)$. When using the geometrical method, the task is very similar to (α, t_b, o) , which was directly used in the solution. On the other hand, when using algebraic geometry, the task was similar to the task (v_a, t_a, o) , which facilitates the construction of the system of equations to solve the task.

Task 8. Construct a triangle if v_a is the height to the side a , $|BS_b| = t_b$, where S_b is the center of the side AC , and r is the radius of the circumcircle (v_a, t_b, r)

Solution. We start out by knowing that the point B is located on the line p . The height is represented by the parallel line $q, |pq| = v_a$. Then we determine the circle $k_{tb}(B, t_b)$ with center of the side b . Next, we determine the circle $k(A, r)$ and select the point S_1 on it as the center of the circumcircle. Then we select the circle $k_{r1}(S_1, r)$ and get the points $C_1, C_1 \in p \cap k_{r1}$, $A_1, A_1 \in q \cap k_{r1}$ and points S_{1itb} and S_{1jtb} as the intersections of the circle k_{r1} with the line n , $C_1A_1 \in n$. Then we determine the circle $k_{bi(j)1}(S_{1i(j)tb}, |S_{1itb}C_1|)$. The point $A_{1i(j)b}$, which it is characteristic of the curve that is characterized by the parameters t_b, r , is determined as an intersection of the line n and circle $k_{bi(j)1}$ (Figure 8). Then, the intersection of the curve determined like this and the line q defines the sought point A , which substantially simplifies the task because we already know to find the center of the circumcircle, and thus unequivocally identify the sought triangle.

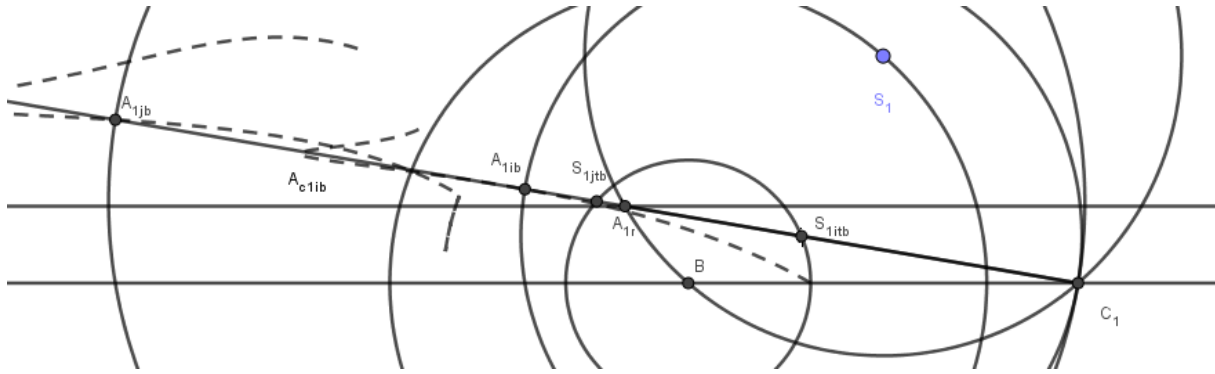


Figure 8. To construct the triangle, we use the fact that the sought point can be found as the intersection of the curve determined by the point $A_{1i(j)b}$, which we determined thanks to the median and the properties of the circumcircle. Then, by determining the point A we get the side

To solve the task with the method of algebraic geometry, we use the expressions for the calculation of triangle area. The median divides the triangle into two triangles with half the area. Furthermore, to calculate the triangle area, we will use the height to the side, and the calculation of the triangle area can also be achieved by two sides and their angle. The angle can be linked with the radius of the incircle by means of the sine theorem. Based on these data, we can determine the system of equations from which we subsequently determine the length of the sought sides:

$$2S = \frac{2S}{\sqrt{(b + 2t_b + 2a)(2t_b + 2a - b)(b - 2t_b + 2a)(b + 2t_b - 2a)}},$$

$$2S = \frac{2S}{\sqrt{(b + 2t_b + 2c)(2t_b + 2c - b)(b - 2t_b + 2c)(b + 2t_b - 2c)}},$$

$$2S = a \cdot v_a, \quad 4S = \frac{a \cdot b \cdot c}{r}.$$

We compare the last two relations and express $b = \frac{2r}{c \cdot v_a}$. Then, by comparing the first two relations and substituting b , we get:

$$\left(\frac{2r}{c \cdot v_a} + 2t_b + 2a\right) \left(2t_b + 2a - \frac{2r}{c \cdot v_a}\right) \left(\frac{2r}{c \cdot v_a} - 2t_b + 2a\right) \left(\frac{2r}{c \cdot v_a} + 2t_b - 2a\right) = \left(\frac{2r}{c \cdot v_a} + 2t_b + 2c\right) \left(2t_b + 2c - \frac{2r}{c \cdot v_a}\right) \left(\frac{2r}{c \cdot v_a} - 2t_b + 2c\right) (b + 2t_b - 2c) = a^2 \cdot v_a^2.$$

By gradual multiplication, we would get an equation of the fourth degree for the sides a , or possibly c .

The task of constructing a triangle with the known parameters (v_a, t_b, r) is similar to task (v_a, t_b, o) and (a, α, r) at first glance. Similarity has been used even in the geometric method, in which we use the movement of the center of the circumcircle, but also in the algebraic geometry method, in which we use the relations to determine the triangle area. In the algebraic geometry method, we subsequently compile a set of equations that lead to an equation of the fourth degree. In the task with specific numerical figures, the resulting compiled equation could be solved e.g. with Euler's method even if the solution is not constructible in an euclidean way.

4. Conclusions

Since its beginnings geometry has been linked with real life, and many historical tasks required a solution, albeit not perfectly accurate. This inter alia prompts the development of approximate calculations and methods of approximate structures. This paper is focused on the approximate solutions to geometric problems related to the construction of triangles, which result in the construction of cubic and biquadratic equations when the algebraic geometry methods are used. The algebraic geometry method is very important and it results in the solution of algebraic equations. The solution of algebraic equations is closely related to algebraic structures, which resulted in the theory of solvable Euclidean tasks in geometry. Several of the above tasks are aimed at the construction of a triangle with the geometric and algebraic method. In addition, we highlight other important sets of points and characteristics that result from the data for the triangle. We point out the possibility of constructing them as geometric curves, while focusing on the point structure of the curve. We focus mainly on some non-traditional tasks in construction geometry, and the solution results in the emergence of parametric curves that are used in computational solutions irrespective of the software used. The above tasks point to the different methods of construction and lead to the development of knowledge and skills in other parts of mathematics. Given the fact that these tasks are complicated from the computational point of view, our current limitation is that the computing software keeps failing in some calculations. Although it is possible to create the intended curve, it is sometimes impossible to clearly determine the intersection of the curve with another geometric object. Therefore, in addition to using the geometric and algebraic approach to solving some specific tasks in the construction of triangles, it is also necessary to deal with the very complexity of algorithms for computational solutions to geometric problems [11].

Conflicts of Interest

The authors declare that they have no conflicts of interests.

REFERENCES

- [1] Čižmár J. (2017). *Dejiny matematiky – od najstarších čias po súčasnosť*. Bratislava: PERFECT, Slovak Republic, ISBN: 9788080468293.
- [2] Descartes R. (1954). *The Geometry of Rene Descartes (Dover Books on Mathematics)*. US: Dover Publications, Inc.
- [3] Dolotin V., Morozov, A. (2007). *Introduction to Nonlinear Algebra*. Singapore: World Scientific Publishing, ISBN: 9789812708007.
- [4] Janson S. (2010). *Roots of polynomials of degrees 3 and 4*. USA, New York: Cornell University, arXiv:1009.2373.
- [5] Brown J. W., Churchill R. V. (2013). *Complex Variables and Applications*. 9th ed. US, New York: McGraw-Hill Education, ISBN: 9780073383170.
- [6] Nickalls R. W. D. (2009). *The quartic equation: invariants and Euler's solution revealed*. In: *The Mathematical Gazette*, UK: Cambridge University Press, Vol. 93, No. 526, p. 66–75, DOI: 10.1017/S0025557200184190.
- [7] Sharma A. K. (2010). *Text Book Of Algebra*. India: Discovery Publishing House, ISBN: 9788171418961.
- [8] Bičík, L. (2013). *Algebraické řešení euklidovských nekonstruovatelných trigonometrických úloh*. PhD Thesis, Brno, Czech Republic: Faculty of Science Masaryk University.
- [9] Shahbazi Z. (2017). *An Extension of Heron's Formula*. USA, New York: Cornell University, arXiv:1705.04320.
- [10] Otto S. R., Denier J. P. (2005). *An introduction to Programming and Numerical Methods in MATLAB*. London: Springer-Verlag London Limited, ISBN: 978-185233-919-7.
- [11] Garey M. R., Johnson D. S. (1990). *Computers and Intractability – A Guide to the Theory of NP-Completeness*. NY, USA: W. H. Freeman & Co., ISBN 0716710455.