Some Inequalities for N-Times Differentiable s -Convex and tgs-Convex Functions

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Abstract The aim of this study is to present some inequalities for n -times differentiable functions. These inequalities are associated with the perturbed trapezoid inequality. n th derivatives of absolute values of the considered functions are s -convex and tgs-convex. In previous studies, some inequalities for the classes of twice differentiable functions which are convex, s -convex and tgs-convex have been introduced. We expand twice differentiable functions to n -times differentiable functions. Finally, some applications are given to verify new inequalities proposed in this study.

Keywords Convex Function, s -Convex, tgs-Convex, Perturbed Trapezoid Inequalities

1. Introduction

The definition of convexity has been frequently considered by many authors. Let $\varphi: I \rightarrow \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$. Then, the function φ is said to be convex on I if

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) \quad (1)$$

satisfies for all $x, y \in I$ and $t \in [0, 1]$ [1]. For numerical integration, the trapezoid inequality is introduced as

$$\left| \int_a^b \varphi(x) dx - \frac{1}{2}(b-a)(\varphi(a) + \varphi(b)) \right| \leq \frac{1}{12} M_2 (b-a)^3$$

where $\varphi: [a, b] \rightarrow \mathbb{R}$ is supposed to be twice differentiable on the interval (a, b) , with the second derivative bounded on (a, b) by $M_2 = \sup_{x \in (a, b)} |\varphi''(x)| < +\infty$ [2-8].

Recently, Dönmez Demir and Şanal have introduced some perturbed trapezoid inequalities for n -times differentiable convex functions. In this study, we suggest some new inequalities for the classes of n -times differentiable functions which are s -convex and tgs-convex. For the class of convex functions, many well-known inequalities are constructed [15]. One of most important is Hermit-Hadamard inequality. The mentioned inequality is defined as follows. Let φ be a convex function on $[a, b] \subseteq \mathbb{R}$ where $a \neq b$. Then,

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \quad (2)$$

Definition 1.1: Let be $\varphi: [0, \infty) \rightarrow \mathbb{R}$. Then, the function φ is s -convex in the second sense if

$$\varphi(tx + (1-t)y) \leq t^s \varphi(x) + (1-t)^s \varphi(y) \quad (3)$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some constant $s \in (0, 1]$. Generally, K_s^2 describes the class of s -convex functions in the second sense [5].

For $s=1$, s -convexity is reduced to ordinary convexity of functions defined on $[0, \infty)$.

Definition 1.2: [15] Let φ be a tgs-convex function defined by $\varphi: I \rightarrow \mathbb{R}$. Then

$$\varphi(tx + (1-t)y) \leq t(1-t)[\varphi(x) + \varphi(y)] \quad (4)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If $(-\varphi)$ is tgs-convex, φ is tgs-concave.

Theorem 1.1: [5] We assume that $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense where $s \in (0, 1]$ and $a, b \in [0, \infty)$, $a < b$. If $\varphi \in L_1([0, 1])$, then the following inequalities are provided:

$$2^{s-1} \varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{s+1}. \quad (5) \quad \text{Here, } \left(\frac{1}{s+1}\right) \text{ is the best possible constant.}$$

Theorem 1.2 (Grüss inequality): [16] Suppose $\varphi, \gamma: [a, b] \rightarrow \mathbb{R}$ are integrable, $k \leq \varphi(x) \leq l$ and $m \leq \gamma(x) \leq n$ ($k, l, m, n \in \mathbb{R}$ are constants), for all $\forall x \in [a, b]$. Then

$$\left| \frac{1}{b-a} \int_a^b \varphi(x) \gamma(x) dx - \frac{1}{b-a} \int_a^b \varphi(x) dx \frac{1}{b-a} \int_a^b \gamma(x) dx \right| \leq \frac{1}{4} (l-k)(n-m) \quad (6)$$

The above inequality is held where $\frac{1}{4}$ is the best constant.

Lemma 1.1: [9] Let $\varphi: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ where n is an even number. If $\varphi^{(n)} \in L[a, b]$, then the following equality is obtained:

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \\ & + \frac{-(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \\ & = \frac{(b-a)^n}{2.n!.a_n} \times \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \end{aligned} \quad (7)$$

Theorem 1.3 (Minkowski Inequality): [3] Let φ^p , γ^p and $(\varphi + \gamma)^p$ be integrable functions on $[a, b]$. If $p > 1$, then

$$\left[\int_a^b |\varphi(x) + \gamma(x)|^p dx \right]^{\frac{1}{p}} \leq \left[\int_a^b |\varphi(x)|^p dx \right]^{\frac{1}{p}} + \left[\int_a^b |\gamma(x)|^p dx \right]^{\frac{1}{p}} \quad (8)$$

Similarly, if $p > 1$ and $a_k, b_k > 0$, then Minkowski sum inequality is expressed as

$$\left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}} \quad (9)$$

If the sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are proportional, the inequality is provided. We will use the following notations and relations throughout this article. Let us consider as $I = (0, \infty) \subset \mathbb{R} = (-\infty, +\infty)$ and $a, b \in I$ with $0 < a < b$ and $\varphi^{(n)} \in L[a, b]$ and

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}, \quad H(a, b) = \frac{2ab}{a+b}, \quad L(a, b) = \frac{b-a}{\ln b - \ln a}, \quad a \neq b,$$

$$L_p = L_p(a, b) = \begin{cases} a & ; a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & ; a \neq b \end{cases}$$

are the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, generalized p -logarithmic mean for $a, b > 0$, respectively.

In this study, we introduce some results related to the perturbed trapezoid inequality and apply them for some special means for real numbers.

2. Main Results

Theorem 2.1: Let $s \in (0, 1]$ and $\varphi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ where n is even number. If $|\varphi^{(n)}|$ is s -convex on $[a, b]$, then the inequality in the following holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ & + \frac{-(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & \left. + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times [|\varphi^{(n)}(a)| + |\varphi^{(n)}(b)|] \times \left[\sum_{k=0}^n \frac{|a_k|}{k+s+1} + \sum_{k=0}^n \frac{|a_k| \Gamma(k+1) \Gamma(s+1)}{k+s+2} \right] \end{aligned} \quad (10)$$

Proof: From Lemma 1.1 and Definition 1.1, it is concluded that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ & - \frac{(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & - \frac{(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & \left. + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \\ & = \left| \frac{(b-a)^n}{2.n!.a_n} \times \left\{ \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \right\} \right| \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times \left\{ \int_0^1 |a_n t^n + \dots + a_1 t + a_0| \left[|\varphi^{(n)}(ta + (1-t)b)| + |\varphi^{(n)}(tb + (1-t)a)| \right] dt \right\} \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times \left\{ \int_0^1 |a_n t^n + \dots + a_1 t + a_0| \left[t^s |\varphi^{(n)}(a)| + (1-t)^s |\varphi^{(n)}(b)| + t^s |\varphi^{(n)}(b)| + (1-t)^s |\varphi^{(n)}(a)| \right] dt \right\} \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times [|\varphi^{(n)}(a)| + |\varphi^{(n)}(b)|] \times \left\{ \int_0^1 [|a_n| t^n + \dots + |a_1| t + |a_0|] [t^s + (1-t)^s] dt \right\} \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times [|\varphi^{(n)}(a)| + |\varphi^{(n)}(b)|] \times \left[\sum_{k=0}^n \frac{|a_k|}{k+s+1} + \sum_{k=0}^n \frac{|a_k| \Gamma(k+1) \Gamma(s+1)}{k+s+2} \right] \end{aligned}$$

Theorem 2.2: Let $s \in (0, 1]$ and $\varphi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$,

and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ where n is even number. If the mapping $|\varphi^{(n)}|^q$ is s -convex on $[a, b]$, then we obtain

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
& + \frac{-(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
& + \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
& + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\
& + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \Bigg| \\
& \leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}} \right] \\
& \times \left\{ \left[\frac{|\varphi^{(n)}(a)|^q}{s+1} + \frac{|\varphi^{(n)}(b)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} + \left[\frac{|\varphi^{(n)}(b)|^q}{s+1} + \frac{|\varphi^{(n)}(a)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} \right\}
\end{aligned} \tag{11}$$

Proof: Using Lemma 1.1, Definition 1.1 and Hölder's integral inequality and Minkowski's integral inequality, we construct

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
& + \frac{-(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
& + \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
& + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\
& + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \Bigg| \\
& = \left| \frac{(b-a)^n}{2.n!.a_n} \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \right| \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)| dt + \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)| dt \right] \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \\
& \times \left[\left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi^{(n)}(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi^{(n)}(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right]
\end{aligned} \tag{12}$$

$$\leq \frac{(b-a)^n}{n!|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}} \right] \\ \times \left\{ \left[\frac{|\varphi^{(n)}(a)|^q}{s+1} + \frac{|\varphi^{(n)}(b)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} + \left[\frac{|\varphi^{(n)}(b)|^q}{s+1} + \frac{|\varphi^{(n)}(a)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} \right\}$$

such that $\frac{1}{p} + \frac{1}{q} = 1$. Considering the convexity of $|\varphi^{(n)}|^q$, then one obtains

$$\int_0^1 |\varphi^{(n)}(ta + (1-t)b)|^q dt \leq \int_0^1 \left[t^s |\varphi^{(n)}(a)|^q + (1-t)^s |\varphi^{(n)}(b)|^q \right] dt = \frac{|\varphi^{(n)}(a)|^q}{s+1} + \frac{|\varphi^{(n)}(b)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \quad (13)$$

$$\int_0^1 |\varphi^{(n)}(tb + (1-t)a)|^q dt \leq \int_0^1 \left[t^s |\varphi^{(n)}(b)|^q + (1-t)^s |\varphi^{(n)}(a)|^q \right] dt = \frac{|\varphi^{(n)}(b)|^q}{s+1} + \frac{|\varphi^{(n)}(a)|^q \times \Gamma(s+1)}{\Gamma(s+2)} \quad (14)$$

Using the Minkowski inequality, we have

$$\left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} \leq \left(\int_0^1 |a_n|^p t^{np} dt \right)^{\frac{1}{p}} + \dots + \left(\int_0^1 |a_1|^p t^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 |a_0|^p dt \right)^{\frac{1}{p}} \\ = \frac{|a_n|}{(np+1)^{\frac{1}{p}}} + \dots + \frac{|a_1|}{(p+1)^{\frac{1}{p}}} + |a_0| = \sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}} \quad (15)$$

Substituting (13), (14) and (15) to (12), then (11) is obtained.

Theorem 2.3: Let $s \in (0,1]$ and $\varphi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ where n is even number. If the mapping $|\varphi^{(n)}|^p$ is s -convex on $[a, b]$, then the inequality in the following holds:

$$\left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ \left. + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \quad (16)$$

$$\leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\sum_{k=0}^n \frac{|a_k|}{k+1} \right]^{1-\frac{1}{p}} \times \left\{ \left[\left(\sum_{k=0}^n \frac{|a_k|}{k+s+1} \right) |\varphi^{(n)}(a)|^p + \left(\sum_{k=0}^n \frac{|a_k| \times \Gamma(s+1) \times \Gamma(k+1)}{\Gamma(k+s+2)} \right) |\varphi^{(n)}(b)|^p \right]^{\frac{1}{p}} + \left[\left(\sum_{k=0}^n \frac{|a_k|}{k+s+1} \right) |\varphi^{(n)}(b)|^p + \left(\sum_{k=0}^n \frac{|a_k| \times \Gamma(s+1) \times \Gamma(k+1)}{\Gamma(k+s+2)} \right) |\varphi^{(n)}(a)|^p \right]^{\frac{1}{p}} \right\}.$$

Proof: Using Lemma 1.1, Definition 1.1 and power mean integral inequality, one obtains

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ & + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \Big| \\ & = \left| \frac{(b-a)^n}{2.n!.a_n} \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \right| \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \left\{ \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)| dt + \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)| dt \right\} \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| dt \right)^{1-\frac{1}{p}} \\ & \times \left\{ \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)|^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)|^p dt \right)^{\frac{1}{p}} \right\} \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \left(|a_n| \int_0^1 t^n dt + \dots + |a_1| \int_0^1 t dt + \int_0^1 |a_0| dt \right)^{1-\frac{1}{p}} \\ & \times \left\{ \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| (t^s |\varphi^{(n)}(a)|^p + (1-t)^s |\varphi^{(n)}(b)|^p) dt \right)^{\frac{1}{p}} + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| (t^s |\varphi^{(n)}(b)|^p + (1-t)^s |\varphi^{(n)}(a)|^p) dt \right)^{\frac{1}{p}} \right\} \end{aligned}$$

$$\leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\sum_{k=0}^n \frac{|a_k|}{k+1} \right]^{1-\frac{1}{p}} \times \left\{ \left[\left(\sum_{k=0}^n \frac{|a_k|}{k+s+1} \right) \left| \varphi^{(n)}(a) \right|^p + \left(\sum_{k=0}^n \frac{|a_k| \times \Gamma(s+1) \times \Gamma(k+1)}{\Gamma(k+s+2)} \right) \left| \varphi^{(n)}(b) \right|^p \right]^{\frac{1}{p}} + \left[\left(\sum_{k=0}^n \frac{|a_k|}{k+s+1} \right) \left| \varphi^{(n)}(b) \right|^p + \left(\sum_{k=0}^n \frac{|a_k| \times \Gamma(s+1) \times \Gamma(k+1)}{\Gamma(k+s+2)} \right) \left| \varphi^{(n)}(a) \right|^p \right]^{\frac{1}{p}} \right\}.$$

The proof is completed.

Theorem 2.4: Let $\varphi: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a n times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ where n is an even number. If $\left| \varphi^{(n)} \right|$ is tgs-convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ & + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \Big| \\ & \leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\left| \varphi^{(n)}(a) \right| + \left| \varphi^{(n)}(b) \right| \right] \times \left[\sum_{k=0}^n \frac{|a_k|}{(k+2)(k+3)} \right] \end{aligned} \quad (17)$$

Proof: Using Lemma 1.1 and Definition 1.2, it follows that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\ & - \frac{(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\ & + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\ & - \frac{(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\ & + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \Big| \\ & = \left| \frac{(b-a)^n}{2.n!.a_n} \times \left\{ \int_0^1 (a_n t^n + \dots + a_1 t + a_0) \left[\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a) \right] dt \right\} \right| \\ & \leq \frac{(b-a)^n}{2.n!.|a_n|} \times \left\{ \int_0^1 |a_n t^n + \dots + a_1 t + a_0| \left[\left| \varphi^{(n)}(ta + (1-t)b) \right| + \left| \varphi^{(n)}(tb + (1-t)a) \right| \right] dt \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b-a)^n}{2.n!.|a_n|} \\
&\times \left\{ \int_0^1 [a_n t^n + \dots + a_1 t + a_0] \left[t(1-t) \left| \varphi^{(n)}(a) \right| + t(1-t) \left| \varphi^{(n)}(b) \right| + t(1-t) \left| \varphi^{(n)}(b) \right| + t(1-t) \left| \varphi^{(n)}(a) \right| \right] dt \right\} \\
&\leq \frac{(b-a)^n}{2.n!.|a_n|} \times \left[\left| \varphi^{(n)}(a) \right| + \left| \varphi^{(n)}(b) \right| \right] \left\{ \int_0^1 [a_n t^n + \dots + a_1 t + a_0] [t(1-t)] dt \right\} \\
&\leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\left| \varphi^{(n)}(a) \right| + \left| \varphi^{(n)}(b) \right| \right] \times \left[\sum_{k=0}^n \frac{|a_k|}{(k+2)(k+3)} \right].
\end{aligned}$$

Then, the proof is completed.

Theorem 2.5: Let $\varphi: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a n times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ where n is an even number. If the mapping $\left| \varphi^{(n)} \right|^q$ is tgs-convex on $[a, b]$, then the inequality holds:

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
&+ \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
&+ \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
&+ \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\
&+ \left. \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \\
&\leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}} \right] \times \left[\frac{|\varphi^{(n)}(a)|^q + |\varphi^{(n)}(b)|^q}{6} \right]^{\frac{1}{q}}
\end{aligned} \tag{18}$$

Proof: Using Lemma 1.1, Definition 1.2 and Hölder's integral inequality and Minkowski's integral inequality, we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
&+ \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
&+ \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
&+ \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\
&+ \left. \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \\
&= \left| \frac{(b-a)^n}{2.n!.a_n} \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \right|
\end{aligned} \tag{19}$$

$$\begin{aligned}
 &\leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)| dt + \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)| dt \right] \\
 &\leq \frac{(b-a)^n}{2.n!.|a_n|} \\
 &\times \left[\left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi^{(n)}(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi^{(n)}(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{(k p + 1)^{\frac{1}{p}}} \right] \times \left[\frac{|\varphi^{(n)}(a)|^q + |\varphi^{(n)}(b)|^q}{6} \right]^{\frac{1}{q}}
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Using the convexity of $|\varphi^{(n)}|^q$, we have

$$\int_0^1 |\varphi^{(n)}(ta + (1-t)b)|^q dt \leq \int_0^1 [t(1-t)|\varphi^{(n)}(a)|^q + t(1-t)|\varphi^{(n)}(b)|^q] dt = \frac{|\varphi^{(n)}(a)|^q + |\varphi^{(n)}(b)|^q}{6} \quad (20)$$

$$\int_0^1 |\varphi^{(n)}(tb + (1-t)a)|^q dt \leq \int_0^1 [t(1-t)|\varphi^{(n)}(b)|^q + t(1-t)|\varphi^{(n)}(a)|^q] dt = \frac{|\varphi^{(n)}(a)|^q + |\varphi^{(n)}(b)|^q}{6} \quad (21)$$

Using the Minkowski inequality, we have

$$\begin{aligned}
 \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0|^p dt \right)^{\frac{1}{p}} &\leq \left(\int_0^1 |a_n|^p t^{np} dt \right)^{\frac{1}{p}} + \dots + \left(\int_0^1 |a_1|^p t^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 |a_0|^p dt \right)^{\frac{1}{p}} \\
 &= \frac{|a_n|}{(np+1)^{\frac{1}{p}}} + \dots + \frac{|a_1|}{(p+1)^{\frac{1}{p}}} + |a_0| = \sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}}
 \end{aligned} \quad (22)$$

Substituting (20), (21) and (22) to (19), then (18) is obtained.

Theorem 2.6: Let $\varphi: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a n -times differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ where n an even number. If the mapping $|\varphi^{(n)}|^p$ tgs-convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
 &+ \frac{-(b-a)^{n-4} [n(n-1)(n-2)a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
 &+ \frac{(b-a)^{n-3} [n(n-1)a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
 &+ \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \quad (23)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times \left[\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a) \right] \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\sum_{k=0}^n \frac{|a_k|}{k+1} \right]^{1-\frac{1}{p}} \times \left[\left| \varphi^{(n)}(a) \right|^p + \left| \varphi^{(n)}(b) \right|^p \right]^{\frac{1}{p}} \times \left[\sum_{k=0}^n \frac{|a_k|}{(k+2)(k+3)} \right]^{\frac{1}{p}}.
\end{aligned}$$

Proof: Using Lemma 1.1, Definition 1.2 and power mean integral inequality, we obtain

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b \varphi(x) dx - \frac{\varphi(a) + \varphi(b)}{2} + \dots \right. \\
& + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4]}{2.n!.a_n} \times [\varphi^{(n-4)}(a) + \varphi^{(n-4)}(b)] \\
& + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2]}{2.n!.a_n} \times [\varphi^{(n-3)}(b) - \varphi^{(n-3)}(a)] \\
& + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2]}{2.n!.a_n} \times [\varphi^{(n-2)}(a) + \varphi^{(n-2)}(b)] \\
& \left. + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0]}{2.n!.a_n} \times [\varphi^{(n-1)}(b) - \varphi^{(n-1)}(a)] \right| \\
& = \left| \frac{(b-a)^n}{2.n!.a_n} \int_0^1 (a_n t^n + \dots + a_1 t + a_0) [\varphi^{(n)}(ta + (1-t)b) + \varphi^{(n)}(tb + (1-t)a)] dt \right| \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \left\{ \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)| dt + \int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)| dt \right\} \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| dt \right)^{1-\frac{1}{p}} \\
& \times \left\{ \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(ta + (1-t)b)|^p dt \right)^{\frac{1}{p}} \right. \\
& \left. + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| |\varphi^{(n)}(tb + (1-t)a)|^p dt \right)^{\frac{1}{p}} \right\} \\
& \leq \frac{(b-a)^n}{2.n!.|a_n|} \left(|a_n| \int_0^1 t^n dt + \dots + |a_1| \int_0^1 t dt + \int_0^1 |a_0| dt \right)^{1-\frac{1}{p}} \\
& \times \left\{ \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| \left(t(1-t) |\varphi^{(n)}(a)|^p + t(1-t) |\varphi^{(n)}(b)|^p \right) dt \right)^{\frac{1}{p}} \right. \\
& \left. + \left(\int_0^1 |a_n t^n + \dots + a_1 t + a_0| \left(t(1-t) |\varphi^{(n)}(b)|^p + t(1-t) |\varphi^{(n)}(a)|^p \right) dt \right)^{\frac{1}{p}} \right\}
\end{aligned}$$

$$\leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\sum_{k=0}^n \frac{|a_k|}{k+1} \right]^{1-\frac{1}{p}} \times \left[|\varphi^{(n)}(a)|^p + |\varphi^{(n)}(b)|^p \right]^{\frac{1}{p}} \times \left[\sum_{k=0}^n \frac{|a_k|}{(k+2)(k+3)} \right]^{\frac{1}{p}}$$

Then, the proof is completed.

3. Some Applications

In this section, we consider the results of Section 2 to verify the new proposed inequalities.

Proposition 3.1: Let $a, b \in \mathbb{R}$, $0 < a < b$, $n > 2$ where n is an even number. Then, the inequality in the following holds:

$$\begin{aligned} & \left| \frac{1}{b-a} (e^b - e^a) - A(e^a, e^b) + \dots + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4] A(e^a, e^b)}{4!.a_n} \right. \\ & \quad + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2] (e^b - e^a)}{2.3!.a_n} \\ & \quad \left. + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2] A(e^a, e^b)}{2!.a_n} + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0] (b-a)}{2.a_n} \right| \\ & \leq \frac{A(e^a, e^b)(b-a)^n}{n!.|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{k+s+1} + \sum_{k=0}^n \frac{|a_k| \Gamma(k+1) \Gamma(s+1)}{k+s+2} \right] \end{aligned}$$

Proof: The proof is clearly obtained from Theorem 2.1 for $\varphi(x) = e^x$, $x \in \mathbb{R}$.

Proposition 3.2: Let $a, b \in \mathbb{R}$, $0 < a < b$, $n > 2$ where n is an even number. For all $p > 1$, one obtains

$$\begin{aligned} & \left| \frac{1}{b-a} (e^b - e^a) - A(e^a, e^b) + \dots + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4] A(e^a, e^b)}{4!.a_n} \right. \\ & \quad + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2] (e^b - e^a)}{2.3!.a_n} \\ & \quad \left. + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2] A(e^a, e^b)}{2!.a_n} + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0] (b-a)}{2.a_n} \right| \\ & \leq \frac{(b-a)^n}{n!.|a_n|} \times \left[\sum_{k=0}^n \frac{|a_k|}{(kp+1)^{\frac{1}{p}}} \right] \\ & \quad \times \left\{ \left[\frac{e^{aq}}{s+1} + \frac{e^{bq} \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} + \left[\frac{e^{bq}}{s+1} + \frac{e^{aq} \times \Gamma(s+1)}{\Gamma(s+2)} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

Proof: The proof is obtained from Theorem 2.2 applied for $\varphi(x) = e^x$, $x \in \mathbb{R}$.

Proposition 3.3: Let $a, b \in \mathbb{R}$, $0 < a < b$, $n > 2$ where n is an even number. Then, we obtain for all $p > 1$,

$$\begin{aligned} & \left| \frac{1}{b-a} (e^b - e^a) - A(e^a, e^b) + \dots + \frac{-(b-a)^{n-4} [n.(n-1).(n-2).a_n + \dots + 4.3.2.a_4] A(e^a, e^b)}{4!.a_n} \right. \\ & \quad + \frac{(b-a)^{n-3} [n.(n-1).a_n + \dots + 4.3.a_4 + 3.2.a_3 + 4a_2] (e^b - e^a)}{2.3!.a_n} \\ & \quad \left. + \frac{-(b-a)^{n-2} [n.a_n + \dots + 2a_2] A(e^a, e^b)}{2!.a_n} + \frac{(b-a)^{n-1} [a_n + \dots + a_1 + 2a_0] (b-a)}{2.a_n} \right| \end{aligned}$$

$$\leq \frac{(b-a)^n}{2.n!.|a_n|} \left[\sum_{k=0}^n \frac{|a_k|}{k+1} \right]^{1-\frac{1}{p}} \\ \times \left\{ \left[\sum_{k=0}^n \frac{|a_k| e^{ap}}{k+s+1} + \sum_{k=0}^n \frac{|a_k| \Gamma(s+1) \Gamma(k+1) e^{bp}}{\Gamma(k+s+2)} \right]^{\frac{1}{p}} + \left[\sum_{k=0}^n \frac{|a_k| e^{bp}}{k+s+1} + \sum_{k=0}^n \frac{|a_k| \Gamma(s+1) \Gamma(k+1) e^{ap}}{\Gamma(k+s+2)} \right]^{\frac{1}{p}} \right\}.$$

Proof: The proof is obtained from Theorem 2.3 such that $\varphi(x) = e^x$, $x \in [a, b]$.

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