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Lightlike Hypersurfaces of an Indefinite Kaehler Manifold with an $(\ell,\, m)$-type Connection

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Abstract Jin [1] defined an $(\ell,\, m)$-type connection on semi-Riemannian manifolds. Semi-symmetric non-metric connection and non-metric ϕ-symmetric connection are two important examples of this connection such that $(\ell,\, m) = (1,\, 0)$ and $(\ell,\, m) = (0,\, 1)$, respectively. In semi-Riemannian geometry, there are few literatures for the lightlike geometry, so we expose new theories for non-degenerate submanifolds in semi-Riemannian geometry. The goal of this paper is to study a characterization of a (Lie) recurrent lightlike hypersurface M of an indefinite Kaehler manifold with an $(\ell,\, m)$-type connection when the charateristic vector field is tangnet to M. In the special case that an indefinite Kaehler manifold of constant holomorphic sectional curvature is an indefinite complex space form, we investigate a lightlike hypersurface of an indefinite complex space form with an $(\ell,\, m)$-type connection when the charateristic vector field is tangnet to M. Moreover, we show that the total space, the complex space form, is characterized by the screen conformal lightlike hypersurface with an $(\ell,\, m)$-type connection. With a semi-symmetric non-metric connection, we show that an indefinite complex space form is flat.

Keywords Compound Non-symmetric Non-metric Connection, Lightlike Hypersurface, Indefinite Kaehler Manifold, Indefinite Complex Space Form

1 Introduction

A linear connection $\bar\nabla$ on a semi-Riemannian manifold $(\bar M, \bar g)$ is called an $(\ell,\, m)$-*type connection* [1] if $\bar\nabla$ and its torsion tensor $\bar T$ satisfy

$$(\bar\nabla_{\bar X}\bar g)(\bar Y,\bar Z) = -\ell\{\theta(\bar Y)\bar g(\bar X,\bar Z) + \theta(\bar Z)\bar g(\bar X,\bar Y)\} - m\{\theta(\bar Y)\bar g(J\bar X,\bar Z) + \theta(\bar Z)\bar g(J\bar X,\bar Y)\}, \tag{1.1}$$

$$\bar T(\bar X,\bar Y) = \ell\{\theta(\bar Y)\bar X - \theta(\bar X)\bar Y\} + m\{\theta(\bar Y)J\bar X - \theta(\bar X)J\bar Y\}, \tag{1.2}$$

where ℓ and m are two smooth functions on $\bar M$, J is a tensor field of type $(1,1)$ and θ is a 1-form associated with a smooth unit vector field ζ, which is called the *characteristic vector field* of $\bar M$, by $\theta(\bar X) = \bar g(\bar X, \zeta)$. Throughout this paper, we denote by $\bar X$, $\bar Y$ and $\bar Z$ the smooth vector fields on $\bar M$.

Two special cases are important for both the mathematical study and the applications to physics: (1) In case $(\ell,\, m) = (1,\, 0)$: This connection $\bar\nabla$ becomes a semi-symmetric non-metric connection. The notion of semi-symmetric non-metric connection was introduced by Ageshe-Chafle [2, 3] and later, studied by several authors [4]. (2) In case $(\ell,\, m) = (0,\, 1)$: $\bar\nabla$ becomes a non-metric ϕ-symmetric connection such that $\phi(\bar X,\bar Y) = \bar g(J\bar X,\bar Y)$. The notion of the non-metric ϕ-symmetric connection was introduced by this author [5, 6].

Remark 1.1. *Denote by $\widetilde\nabla$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar M,\bar g)$ with respect to $\bar g$. By directed calculations, we see that a linear connection $\bar\nabla$ on $\bar M$ is an $(\ell,\, m)$-type connection if and only if $\bar\nabla$ satisfies*

$$\bar\nabla_{\bar X}\bar Y = \widetilde\nabla_{\bar X}\bar Y + \theta(\bar Y)\{\ell\bar X + mJ\bar X\}. \tag{1.3}$$

The objective of study in this paper is lightlike hypersurfaces of an indefinite Kaehler manifold $M = (\bar M, \bar g, J)$ with an $(\ell,\, m)$-type connection subject to the conditions that (1) the tensor field J, defined by (1.1) and (1.2), is identical with the indefinite almost complex structure tensor J of $\bar M$ and (2) the characteristic vector field ζ of $\bar M$ is tangent to M. In this paper, we set $(\ell, m) \neq (0, 0)$ and we shall assume that ζ is unit spacelike, without loss of generality.

2 Preliminaries

Let $(\bar M, \bar g, J)$ be an indefinite Kaehler manifold equipped with an $(\ell,\, m)$-type connection $\bar\nabla$ and a Levi-Civita connection $\widetilde{\nabla}$, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure such that

$$J^2 = -I,\ \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}),\ (\widetilde{\nabla}_{\bar{X}} J)\bar{Y} = 0. \quad (2.1)$$

By direct calculation from (1.3) and $(2.1)_3$, we see that

$$\begin{aligned}(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \ell\{\theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X}\} \\ + m\{\theta(\bar{Y})\bar{X} + \theta(J\bar{Y})J\bar{X}\}.\end{aligned} \quad (2.2)$$

Let (M, g) be a lightlike hypersurface of $\bar{M}$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. It is known that the normal bundle $TM^\perp$ of M is a subbundle of the tangent bundle TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is a non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. It is known [7] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bunde $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$\begin{aligned}T\bar{M} &= TM \oplus tr(TM) \\ &= \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).\end{aligned} \quad (2.3)$$

In case the vector field ζ is tangent to M. If ζ belongs to $Rad(TM)$, then

$$\zeta = a\xi, \qquad 1 = \bar{g}(\zeta, \zeta) = a^2 g(\xi, \xi) = 0, \quad a \in F(M).$$

It is a contradiction. Thus ζ does not belong to $Rad(TM)$. This result enables one to choose a screen distribution $S(TM)$ which contains ζ. Thus we consider lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type connection and a screen distribution $S(TM)$ which contains ζ.

Denote by X, Y and Z the smooth vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulae of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2.4)$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \quad (2.5)$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \quad (2.6)$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad (2.7)$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators and τ is a 1-form.

Due to [7, Section 6.2], for a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, such that $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. It follow that $J(TM^\perp) \oplus J(tr(TM))$ is a subbundle of $S(TM)$, of rank 2. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to the indefinite almost complex structure J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{TM^\perp \oplus_{orth} J(TM^\perp)\} \oplus_{orth} D_o.$$

In this case, the decomposition of TM is reduced to

$$TM = D \oplus J(tr(TM)). \quad (2.8)$$

Consider two null vector fields U and V and two 1-forms u and v such that

$$\begin{aligned}U = -JN,\ V = -J\xi, \\ u(X) = g(X, V),\ v(X) = g(X, U).\end{aligned} \quad (2.9)$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \quad (2.10)$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying J to (2.10) and using (2.1) and (2.9), we have

$$F^2 X = -X + u(X)U. \quad (2.11)$$

As $u(U) = 1$ and $FU = 0$, the set $(F,\ u,\ U)$ defines an indefinite almost contact structure on M and F is called the *structure tensor field* of M.

3 $(\ell,\ m)$-type connections

Using (1.1), (1.2), (2.4) and (2.10), we obtain

$$\begin{aligned}(\nabla_X g)(Y,Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ - \ell\{\theta(Y)g(X, Z) + \theta(Z)g(X, Y)\} \\ - m\{\theta(Y)\bar{g}(JX, Z) + \theta(Z)\bar{g}(JX, Y)\},\end{aligned} \quad (3.1)$$

$$\begin{aligned}T(X, Y) =& \ell\{\theta(Y)X - \theta(X)Y\} \\ &+ m\{\theta(Y)FX - \theta(X)FY\},\end{aligned} \quad (3.2)$$

$$B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \quad (3.3)$$

where T is the torsion tensor with respect to ∇ and η is a 1-form such that $\eta(X) = \bar{g}(X, N)$. From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfy

$$B(X, \xi) = 0, \qquad B(\xi, X) = 0. \quad (3.4)$$

The local second fundamental forms are related to their shape operators by

$$\begin{aligned}&B(X, Y) = g(A^*_\xi X, Y) + mu(X)\theta(Y), \\ &\bar{g}(A^*_\xi X, N) = 0,\end{aligned} \quad (3.5)$$

$$\begin{aligned} C(X, PY) =&g(A_{N}X, PY)\\ &+\{\ell\eta(X)+mv(X)\}\theta(PY), \\ \bar{g}(A_{N}&X, N) = 0. \end{aligned} \tag{3.6}$$

As $S(TM)$ is non-degenerate, from $(3.4)_2$ and (3.5), we obtain

$$A^*_\xi \xi = 0. \tag{3.7}$$

Applying $\bar{\nabla}_X$ to $(2.9)_{1,\,2,\,4}$ and (2.10), we have

$$\begin{aligned} B(X, U) &= u(A_{N}X) + m\theta(U)u(X)\\ &= C(X, V) + m\{\theta(U)u(X)\\ &- \theta(V)v(X)\} - \ell\theta(V)\eta(X), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \nabla_X U = F(A_{N}X) + \tau(X)U\\ + \theta(U)\{\ell X + mFX\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \nabla_X V = F(A^*_\xi X) - \tau(X)V\\ + \theta(V)\{\ell X + mFX\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} (\nabla_X F)Y = u(Y)A_{N}X - B(X, Y)U\\ + \ell\{\theta(FY)X - \theta(Y)FX\}\\ + m\{\theta(Y)X + \theta(FY)FX\}. \end{aligned} \tag{3.11}$$

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an $(\ell,\, m)$-type connection such that ζ is tangent to M. Then (1) A^*_ξ is self-adjoint, and (2) B is symmetric on TM if and only if $m = 0$.*

Proof. (1) From (3.3) and (3.5), we see that $g(A^*_\xi X, Y) = g(X, A^*_\xi Y)$. Thus A^*_ξ is self-adjoint. (2) If $m = 0$, then B is symmetric by (3.3). Conversely, if B is symmetric, then, taking $X = V$ and $Y = U$ to (3.3), we get $m\theta(V) = 0$. Also, taking $X = \zeta$ and $Y = U$ to (3.3) and using $m\theta(V) = 0$, we have $m = 0$. □

4 Some results

Definition 1. *The structure tensor field F of M is said to be recurrent [8] if there exists a 1-form ϖ on TM such that*

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is said to be recurrent if it admits a recurrent structure tensor field F.

Theorem 4.1. *There exist no recurrent lightlike hypersurface of an indefinite Kaehler manifold with an $(\ell,\, m)$-type connection such that ζ is tangent to M.*

Proof. From the above definition and (3.11), we get

$$\begin{aligned} \varpi(X)FY = u(Y)A_{N}X - B(X, Y)U\\ + \ell\{\theta(FY)X - \theta(Y)FX\}\\ + m\{\theta(Y)X + \theta(FY)FX\}, \end{aligned}$$

Taking the scalar product with N to this and using $(3.6)_2$, we have

$$\begin{aligned} \varpi(X)v(Y) = \{\ell\eta(X) + mv(X)\}\theta(FY)\\ - \{\ell v(X) - m\eta(X)\}\theta(Y). \end{aligned} \tag{4.1}$$

Replacing Y by ξ to this and using the facts that $F\xi = -V$ and $\theta(\xi) = 0$, we obtain $\{\ell\eta(X) + mv(X)\}\theta(V) = 0$. It fellows that

$$\ell\theta(V) = 0, \qquad m\theta(V) = 0.$$

Replacing Y by V to (4.1) and using the last equation, we obtain $\varpi = 0$.

Replacing X by ξ and V to (4.1) with $\varpi = 0$ by turns, we obtain

$$m\theta(X) = -\ell\theta(FX), \qquad \ell\theta(X) = m\theta(FX).$$

As $(\ell, m) \neq (0, 0)$, from the last two equations we obtain $(\ell^2 + m^2)\theta(X) = 0$. Taking $X = \zeta$ to this, we get $\ell^2 + m^2 = 0$. It follows that $\ell = 0$ and $m = 0$. It is a contradiction to $(\ell, m) \neq (0, 0)$. Thus we have our theorem. □

Definition 2. *The structure tensor field F of M is said to be Lie recurrent [8] if there exists a 1-form ϑ on M such that*

$$(\mathcal{L}_{X} F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_{X}$ denotes the Lie derivative on M with respect to X. F is called Lie parallel if $\vartheta = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Theorem 4.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an $(\ell,\, m)$-type connection such that ζ is tangent to M. Then we have the following three assertion ;*

(1) the structure tensor field F is Lie parallel,

(2) the 1-form τ satisfies $\tau = 0$, and

(3) A^*_ξ and A_{N} satisfy $A^*_\xi U = A^*_\xi V = 0$ and $A_{N}V = 0$.

Proof. (1) Using the above definition, (2.10), (2.11), (3.2) and (3.11), we get

$$\begin{aligned} \vartheta(X)FY = -\nabla_{FY}X + F\nabla_Y X + u(Y)A_{N}X\\ - \{B(X, Y) - m\theta(Y)u(X)\}U. \end{aligned} \tag{4.2}$$

Taking $Y = \xi$ to (4.2) and using $(3.4)_1$ and the fact that $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X. \tag{4.3}$$

Taking the scalar product with V to (4.3) and using $g(FX, V) = 0$, we have

$$u(\nabla_V X) = g(\nabla_V X, V) = 0. \tag{4.4}$$

Replacing Y by V to (4.2) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - \{B(X, V) - m\theta(V)u(X)\}U.$$

Applying F to this equation and using (2.11) and (4.4), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X.$$

Comparing this equation with (4.3), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking $X = U$ to $\nabla_V X + F\nabla_\xi X = 0$ and using (2.11) and (3.9), we get

$$F(A_N V) + \tau(V)U - A_N\xi + u(A_N\xi)U = 0.$$

Taking the scalar product with N to this equation, we obtain

$$g(A_N V, U) = 0. \tag{4.5}$$

Replacing X by V to (4.2) and using (2.11), (3.3), (3.5) and (3.10), we get

$$\begin{aligned} -F(A^*_\xi FY) + \tau(FY)V - A^*_\xi Y \\ - \tau(Y)\xi + u(Y)A_N V = 0. \end{aligned} \tag{4.6}$$

Taking the scalar product with U to (4.6) and using (3.5) and (4.5), we have

$$B(X, U) - m\theta(U)u(X) = \tau(FX). \tag{4.7}$$

From this equation and (3.8), we see that

$$u(A_N X) = \tau(FX). \tag{4.8}$$

Replacing X by U to (4.2) and using (2.11), (3.3) (3.8) and (3.9), we get

$$u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U = 0. \tag{4.9}$$

Taking the scalar product with V to (4.9) and using (4.8), we get $\tau(FY) = 0$.

Taking the scalar product with N to (4.6) and using (3.5) and $(3.6)_2$, we get

$$B(FY, U) = -\tau(Y).$$

Replacing Y by U to this and using the fact that $FU = 0$, we obtain

$$\tau(U) = 0. \tag{4.10}$$

Taking $Y = FX$ to $\tau(FY) = 0$ and using (2.11) and (4.10), we have $\tau(X) = 0$.

(3) As $\tau = 0$, from (4.7) we have $B(X, U) = m\theta(U)u(X)$. Thus

$$B(U, X) = m\theta(X). \tag{4.11}$$

Taking $X = U$ to (3.5) and using (4.11), we get $g(A^*_\xi U, X) = 0$. Using this and the fact that $S(TM)$ is non-degenerate, we have $A^*_\xi U = 0$. Replacing X by ξ to (4.3) and using (2.7), (3.7) and the fact that $\tau = 0$, we obtain $A^*_\xi V = 0$. On the other hand, replacing Y by U to (4.6) and using (4.10), we get $A_N V = A^*_\xi U$. Thus we see that $A_N V = 0$. □

5 Indefinite complex space forms

Definition 3. *An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that*

$$\begin{aligned} \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4}\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} \\ + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} \\ + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\}, \end{aligned} \tag{5.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Denote by $\bar{R}$ the curvature tensors of the (ℓ, m)-type connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.2) and (1.3), we see that

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})\{\ell\bar{Y} + mJ\bar{Y}\} \\ - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})\{\ell\bar{X} + mJ\bar{X}\} + \theta(\bar{Z})\{(\bar{X}\ell)\bar{Y} \\ - (\bar{Y}\ell)\bar{X} + (\bar{X}m)J\bar{Y} - (\bar{Y}m)J\bar{X}\}. \end{aligned} \tag{5.2}$$

Denote by R and R^* the curvature tensors of the induced connection ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae and (3.2), we obtain the Gauss equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{R}(X, Y)Z = R(X, Y)Z \\ + B(X, Z)A_N Y - B(Y, Z)A_N X \\ + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ - \ell[\theta(X)B(Y, Z) - \theta(Y)B(X, Z)] \\ - m[\theta(X)B(FY, Z) - \theta(Y)B(FX, Z)]\}N, \end{aligned} \tag{5.3}$$

$$\begin{aligned} R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A^*_\xi Y \\ - C(Y, PZ)A^*_\xi X \\ + \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ - \ell[\theta(X)C(Y, PZ) - \theta(Y)C(X, PZ)] \\ - m[\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)]\}\xi, \end{aligned} \tag{5.4}$$

respectively. Comparing the tangential and transversal components of the left and right terms of (5.2) and using (5.1), and (5.3), we obtain

$$\begin{aligned} R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y \\ + (\bar{\nabla}_X\theta)(Z)\{\ell Y + mFY\} \\ - (\bar{\nabla}_Y\theta)(Z)\{\ell X + mFX\} \\ + \theta(Z)\{X\ell)Y - (Y\ell)X + (Xm)FY - (Ym)FX\} \\ + \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + \bar{g}(JY, Z)FX \\ - \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ\}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &+ \{\tau(X) - \ell\theta(X)\}B(Y, Z) - \{\tau(Y) - \ell\theta(Y)\}B(X, Z) \\ &- m\{\theta(X)B(FY, Z) - \theta(Y)B(FX, Z)\} \\ &- m\{(\bar{\nabla}_X\theta)(Z)u(Y) - (\bar{\nabla}_Y\theta)(Z)u(X)\} \\ &- \theta(Z)\{(Xm)u(Y) - (Ym)u(X)\} \\ &= \frac{c}{4}\{u(X)g(FY, Z) - u(Y)g(FX, Z) \\ &\quad + 2u(Z)\bar{g}(X, JY)\}. \end{aligned} \tag{5.6}$$

Taking the scalar product with N to (5.2) such that $\bar{Z} = PZ$ and substituting (5.1), (5.3) and (5.4) into the resulting equation and using $(3.6)_2$, we get

$$\begin{aligned}
&(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad - \{\tau(X) + \ell\theta(X)\}C(Y, PZ) \\
&\quad + \{\tau(Y) + \ell\theta(Y)\}C(X, PZ) \\
&\quad - m\{\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)\} \\
&\quad - (\bar{\nabla}_X\theta)(PZ)\{\ell\eta(Y) + mv(Y)\} \\
&\quad + (\bar{\nabla}_Y\theta)(PZ)\{\ell\eta(X) + mv(X)\} \\
&\quad - \theta(PZ)\{(X\ell)\eta(Y) - (Y\ell)\eta(X) \\
&\quad + (Xm)v(Y) - (Ym)v(X)\} \\
&\quad = \frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) \\
&\quad + v(X)g(FY, PZ) - v(Y)g(FX, PZ) \\
&\quad + 2v(PZ)\bar{g}(X, JY)\}.
\end{aligned} \tag{5.7}$$

Definition 4. *A screen distribution $S(TM)$ is called totally geodesic [7] in M if $C = 0$ on a cooerinate neighborhood $\mathcal{U}$.*

Theorem 5.1. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with an $(\ell,\, m)$-type connection such that ζ is tangent to M. If one of the following four conditions is satisfied;*

(1) M Lie recurrent,

(2) U is parallel with respect to ∇,

(3) V is parallal with respect to ∇,

(4) $S(TM)$ is totally geodesic in M,

then $c = 0$ and $\bar{M}(c)$ is flat.

Proof. (1) Applying $\bar{\nabla}_X$ to $\theta(\xi) = 0$ and using (2.7), (3.4) and (3.5), we get

$$(\bar{\nabla}_X\theta)(\xi) = \theta(A^*_\xi X). \tag{5.8}$$

Replacing Z by ξ to (5.5) and using $(3.4)_1$ and (5.8), we have

$$\begin{aligned}
R(X, Y)\xi = {} & \frac{c}{4}\{u(Y)FX - u(X)FY - 2\bar{g}(X, JY)V\} \\
& + \theta(A^*_\xi X)\{\ell Y + mFY\} - \theta(A^*_\xi Y)\{\ell X + mF\bar{X}\}.
\end{aligned}$$

In general, using the Gauss-Weingarten formulae (2.6) and (2.7) for $S(TM)$, we obtain the Codazzi equation for $S(TM)$ such that

$$\begin{aligned}
R(X, Y)\xi = {} & -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X, Y] \\
& - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\
& + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi.
\end{aligned}$$

If M Lie recurrent, then we have $\tau = 0$ and $A^*_\xi U = A^*_\xi V = 0$ by Theorem 4.2. Comparing the radical components of the last two equations, we obtain

$$\begin{aligned}
& C(Y, A^*_\xi X) - C(X, A^*_\xi Y) \\
& = \frac{c}{4}\{u(Y)v(X) - u(X)v(Y)\} \\
& + \theta(A^*_\xi X)\{\ell\eta(Y) + mv(Y)\} - \theta(A^*_\xi Y)\{\ell\eta(X) + mv(X\},
\end{aligned}$$

since $\tau = 0$. Taking $X = V$ and $Y = U$ to this equation and using the fact that $A^*_\xi U = A^*_\xi V = 0$, we obtain $c = 0$.

(2) If U is parallel with respect to ∇, then, taking the scalar product with U and U to (3.9) such that $\nabla_X U = 0$ by turns, we obtain

$$\theta(U)\{\ell v(X) - m\eta(X)\} = 0, \quad \tau(X) + \ell\theta(U)u(X) = 0,$$

respectively. From these two equations, we obtain

$$\ell\theta(U) = 0, \quad m\theta(U) = 0, \quad \tau = 0. \tag{5.9}$$

Applying $\bar{\nabla}_X$ to $(5.9)_{1,\,2}$ and using (2.4) and the fact that $\nabla_X U = 0$, we get

$$\begin{aligned}
&(X\ell)\theta(U) + \ell(\bar{\nabla}_X\theta)(U) = 0, \\
&(Xm)\theta(U) + m(\bar{\nabla}_X\theta)(U) = 0.
\end{aligned} \tag{5.10}$$

Also, taking the scalar product with N to (3.9): $F(A_{N}X) = 0$ and using (3.6), $(5.9)_{1,\,2}$ and the fact that $\nabla_X U = 0$, we obtain

$$C(X, U) = 0, \qquad (\nabla_X C)(Y, U) = 0.. \tag{5.11}$$

Taking $PZ = U$ to (5.7) and using $(5.10)_{1,\,2}$ and $(5.11)_{1,\,2}$, we obtain $c = 0$.

(3) If V is parallel with respect to ∇, then, taking the scalar product with V and U to (3.10) such that $\nabla_X V = 0$ by turns, we have

$$\ell\theta(V)u(X) = 0, \quad \tau(X) = \theta(V)\{\ell v(X) - m\eta(X)\},$$

respectively. From these two equations, we obtain

$$\ell\theta(V) = 0, \quad \tau(X) = -m\theta(V)\eta(X). \tag{5.12}$$

Applying $\bar{\nabla}_X$ to $(5.12)_1$ and using (2.4) and the fact that $\nabla_X V = 0$, we have

$$(X\ell)\theta(V) + \ell(\bar{\nabla}_X\theta)(V) = 0. \tag{5.13}$$

Using (5.12), the equation (3.10) is reduced to

$$F(A^*_\xi X) + m\theta(V)\{FX + \eta(X)V\} = 0.$$

Taking the scalar product with N to this equation and using (3.5), (3.8), $(5.12)_1$ and the fact that $\nabla_X V = 0$, we obtain

$$C(X, V) = 0, \qquad (\nabla_X C)(Y, V) = 0.$$

Taking $PZ = V$ to (5.7) and using (5.13) and the last two equations, we get

$$\begin{aligned}
& - \{(Xm)\theta(V) + m(\bar{\nabla}_X\theta)(V)\}v(Y) \\
& + \{(Ym)\theta(V) + m(\bar{\nabla}_Y\theta)(V)\}v(X) \\
& = \frac{c}{4}\{\eta(X)u(Y) - \eta(Y)v(X) + 2\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $X = \xi$ and $Y = U$ to this equation, we have $c = 0$.

(4) Taking $X = \xi$ and $X = V$ to (3.8) by turns and using $(3.4)_2$, we have

$$\ell\theta(V) = 0,\ B(V, U) = -m\theta(V),\ B(U, V) = 0. \tag{5.14}$$

As $\theta(J\zeta) = 0$ and $\theta(N) = 0$, we have $\theta(F\zeta) = 0$ due to (2.10). Also we have $v(F\zeta) = 0$ and $u(F\zeta) = 0$. Taking $X = U$ and $Y = F\zeta$ to (3.3), we obtain

$$B(U, F\zeta) = B(F\zeta, U).$$

Replacing X by $F\zeta$ to (3.8) and using $(5.14)_1$, we have $B(F\zeta, U) = 0$. Thus

$$B(U, F\zeta) = 0. \tag{5.15}$$

Applying $\bar{\nabla}_U$ by $\ell\theta(V) = 0$ and using (3.10), $(5.14)_1$ and (5.15), we get

$$(U\ell)\theta(V) + \ell(\bar{\nabla}_U\theta)(V) = 0. \tag{5.16}$$

As $S(TM)$ is totally geodesic in M, (5.7) is reduced to

$$\begin{aligned}&-(\bar{\nabla}_X\theta)(PZ)\{\ell\eta(Y) + mv(Y)\} + (\bar{\nabla}_Y\theta)(PZ)\{\ell\eta(X) + mv(X)\}\\&-\theta(PZ)\{(X\ell)\eta(Y) - (Y\ell)\eta(X) + (Xm)v(Y) - (Ym)v(X)\}\\&= \frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\\&+ v(X)g(FY, PZ) - v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.\end{aligned}$$

Replacing X by ξ to this and using $(3.5)_1$ and $(3.5)_2$, we have

$$\begin{aligned}&\ell(\bar{\nabla}_Y\theta)(PZ) - (\bar{\nabla}_\xi\theta)(PZ)\{\ell\eta(Y) + mv(Y)\}\\&+ \theta(PZ)\{Y\ell - (\xi\ell)\eta(Y) - (\xi m)v(Y)\}\\&= \frac{c}{4}\{g(Y, PZ) + v(Y)u(PZ) + 2u(Y)v(PZ)\}.\end{aligned}$$

Taking $Y = U$ and $PZ = V$ to this equation by turns and using $(5.14)_{2,\,3}$ and (5.16), we obtain $c = 0$. □

Definition 5. *A lightlike hypersurface M is said to be screen conformal [9] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that*

$$C(X, PY) = \varphi B(X, PY). \tag{5.17}$$

Theorem 5.2. *Let M be a screen conformal lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with an $(\ell,\, m)$-type connection such that ζ is tangent to M. Then the vector field μ, defined by $\mu = U - \varphi V$, is an eigenvector of A^*_ξ corresponding to the eigenvector $-m\theta(V)$. If $m\theta(V) = 0$, then $c = 0$.*

Proof. Taking $X = \xi$ to (5.17) and using $(3.4)_2$, we get $C(\xi, PZ) = 0$. Thus $C(\xi, V) = 0$. Taking $X = \xi$ to (3.8) and using $C(\xi, V) = 0$, we have

$$\ell\theta(V) = 0. \tag{5.18}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (5.7) and using (5.6), we obtain

$$\begin{aligned}&\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)\\&- \bar{\nabla}_X\theta)(PZ)\{\ell\eta(Y) + mg(Y, \mu)\}\\&+ (\bar{\nabla}_Y\theta)(PZ)\{\ell\eta(X) + mg(X, \mu)\}\\&- \theta(PZ)\{(X\ell)\eta(Y) - (Y\ell)\eta(X)\\&+ (Xm)g(Y, \mu) - (Ym)g(X, \mu)\}\\&= \frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\\&+ g(X, \mu)g(FY, PZ) - g(Y, \mu)g(FX, PZ)\\&+ 2g(PZ, \mu)\bar{g}(X, JY)\}.\end{aligned}$$

Replacing X by ξ to this equation and using $(3.4)_2$ and (3.8), we have

$$\begin{aligned}&\{\xi\varphi - 2\varphi\tau(\xi)\}B(Y, PZ) + \ell(\bar{\nabla}_Y\theta)(PZ)\\&- (\bar{\nabla}_\xi\theta)(PZ)\{\ell\eta(Y) + mg(Y, \mu)\}\\&+ \theta(PZ)\{Y\ell - \eta(Y)\xi\ell - g(Y, \mu)\xi m\}\\&= \frac{c}{4}\{g(Y, PZ) + g(Y, \mu)u(PZ)\\&+ 2g(PZ, \mu)u(Y)\}.\end{aligned} \tag{5.19}$$

Applying $\bar{\nabla}_X$ to $\ell\theta(V) = 0$ and using (3.5), (3.10) and (5.18), we obtain

$$(X\ell)\theta(V) + \ell(\bar{\nabla}_X\theta)(V) - \ell B(X, F\zeta) = 0. \tag{5.20}$$

As $\mu = U - \varphi V$ and $\ell\theta(V) = 0$, from (3.8) we have

$$B(X, \mu) = m\{\theta(U)u(X) - \theta(V)v(X)\}. \tag{5.21}$$

As $\theta(J\zeta) = 0$ and $\theta(N) = 0$, we have $\theta(F\zeta) = 0$. Also we have $v(F\zeta) = 0$ and $u(F\zeta) = 0$. Replacing Y by μ to (3.3) and using (5.21), we obtain

$$B(\mu, X) = m\{\theta(X) - \theta(V)g(X, \mu)\}. \tag{5.22}$$

Taking $X = V$ and $X = F\zeta$ to (5.22) by turns and using (5.20), we obtain

$$B(\mu, V) = 0, B(\mu, F\zeta) = 0, (\mu\ell)\theta(V) + \ell(\bar{\nabla}_\mu\theta)(V) = 0. \tag{5.23}$$

Taking $Y = \mu$ and $PZ = V$ to (5.19) and using $(5.23)_{1,\,3}$, we have

$$\varphi\{m(\bar{\nabla}_\xi\theta)(V) + \theta(V)\xi m\} = \frac{3}{8}c. \tag{5.24}$$

Replacing X by μ to (3.5) and using (5.22), we have

$$A^*_\xi\mu = -m\theta(V)\mu.$$

Thus μ is an eigenvector of A^*_ξ corresponding to the eigenvalue $-m\theta(V) = 0$. If $m\theta(V) = 0$, then, applying $\bar{\nabla}_\xi$ to this and using (3.7) and (3.10), we have

$$(\xi m)\theta(V) + m(\bar{\nabla}_\xi\theta)(V) = 0.$$

From this equation and (5.24), we obtain $c = 0$. □

Corollary 5.3. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If M is screen conformal, then $A^*_\xi\mu = 0$ and $c = 0$.*

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