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Adomian Decomposition Method with Modified Bernstein Polynomials for Solving Nonlinear Fredholm and Volterra Integral Equations

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Abstract Many different problems in mathematics, physics, engineering can be expressed in the form of integral equations. Among these are diffraction problems, population growth, heat transfer, particle transport problems, electrical engineering, elasticity, control, elastic waves, diffusion problems, quantum mechanics, heat radiation, electrostatics and contact problems. Therefore, the solutions which are obtained by the mathematical methods play an important role in these fields. The most two basic types of integral equations are called Fredholm (FIEs) and Volterra (VIEs). In many instances, the ordinary and partial differential equations can be converted into Fredhom and Volterra integral equations that are solved more effectively. We aim through this research to present an improved Adomian decomposition method based on modified Bernstein polynomials (ADM-MBP) to solve nonlinear integral equations of the second kind. We introduced efficient method, constructed on modified Bernstein polynomials. The formulation is developed to solve nonlinear Fredholm and Volterra integral equations of second kind. This method is tested for some examples from nonlinear integral equations. Maple software was used to obtain the solutions of these examples. The results demonstrate reliability of the proposed method. Generally, the proposed method is very convenient to apply to find the solutions of Fredholm and Volterra integral equations of second kind.

Keywords Adomian Decomposition Method (ADM), Modified Bernstein Polynomials (MBP), Fredholm Integral Equations, Volterra Integral Equations.

1. Introduction

Fredholm and Volterra family of integral equations play an important role in mathematics, physics, engineering and others. Therefore, many techniques and numerical methods were used to solve these types of equations. Among these are optimal homotopy asymptotic method [1, 2], homotopy perturbation method [3, 4], homotopy analysis method [5], Adomian Decomposition Method [6], barycentric interpolation collocation methods [7], quadrature method based on multivariate Bernstein polynomials [8], integral mean value theorem [9] and Newton-Kantorovich-quadrature method [10]. Other studies can be found on [11-14].

ADM was firstly introduced by G. Adomian [15] and applied it to solve the nonlinear problems in various fields (see [16-18]). Recently, this classical method is modified and developed to improve the accuracy of the results. Hosseini [19] proposed a new improved ADM via Chebyshev polynomials for solving linear and non-linear models. Wazwaz [20] presented the modified ADM to solve linear and nonlinear operators. Song and Wang [21] established a new improved ADM and applied it to fractional differential equations. Qasim and AL-Rawi [22] used Bernstein polynomials to modify the ADM to solve differential equations. The ADM is improved with orthogonal polynomials [23] and with Laguerre polynomials [24] for solving differential equation. In addition, several improvements were also considered on ADM (see [25- 28]).

The main objective of this research is to introduce a new analytical treatment for nonlinear Fredholm and Volterra integral equations using improved ADM-MBP.

2. Definitions and Basic Concepts

To introduce our research we will need the following basic definitions, concepts and results of integral equations and Bernstein polynomials.

Definition 2.1

The integral equation in $g(x)$ is of the form:

$$g(x)=f(x)+\lambda\int_{u(x)}^{h(x)}k(x,t)g(t)dt, \quad (1)$$

where $u(x)$ and $h(x)$ may be both fixed, variables, or mixed, λ is a constant, $k(x,t)$ is the kernel, $f(x)$ is given function and $g(x)$ is unknown. If $u(x)$ and $h(x)$ are fixed, then it is called a Fredholm. If at least one limit is a variable, then it is called a Volterra.

Definition 2.2

The Bernstein basis polynomials of degree m are defined by

$$B_{i,m}(x)=\binom{m}{i}x^{i}(1-x)^{m-i}, \quad (2)$$

where $0\le i\le m,\quad 0\le x\le 1$, and $\binom{m}{i}==\frac{m!}{i!(m-i)!}$.

Definition 2.3

A linear combination of Bernstein basis polynomials

$$B_m(x)=\sum_{i=0}^{m}B_{i,m}(x)\beta_i, \quad (3)$$

is called the Bernstein polynomials, where β_i are the Bernstein coefficients.

Definition 2.4

The m th Bernstein polynomial for $f(x)$ defined by

$$B_m(f)=\sum_{i=0}^{m}B_{i,m}(x)f\left(\frac{i}{m}\right), \quad (4)$$

Note that here for each function $f:[0,1]\to R$, we have

$$\lim_{m\to\infty}B_m(f)=f(x). \quad (5)$$

Remark 2.1

The derivatives of the m th degree Bernstein polynomials are given by

$$\frac{d}{dx}B_{i,m}(x)=m\left(B_{i-1,m-1}(x)-B_{i,m-1}(x)\right). \quad (6)$$

It should be noted that the $2k$ th order derivative $f^{(2k)}$ has been shown by [29]

$$B_m^f(x)=f(x)+\sum_{a=2}^{2k-1}\frac{f^{(a)}(x)}{a!m^a}T_{m,a}(x)+O\left(\frac{1}{m^k}\right), \quad (7)$$

where

$$T_{m,a}(x)=\sum_k(k-mx)^a\binom{m}{k}x^k(1-x)^{m-k}. \quad (8)$$

3. Description of the Modified Technique

In order to apply the new modified technique to nonlinear integral equation, we rewrite Eq. (1) as

$$g(x)=f(x)+\lambda\int_{u(x)}^{h(x)}k(x,t)N(g(t))dt, \quad (9)$$

where $N(g(t))$ is a nonlinear function.

By Adomian polynomials, $N(g(t))$ is given by

$$N(g)=\sum_{n=0}^{\infty}A_n(g_1,g_2,\dots,g_n), \quad (10)$$

where $A_n=\frac{1}{n!}\frac{d^n}{d\gamma^n}\left[N\left(\sum_{i=0}^{\infty}\gamma^i g_i\right)\right], n=0,1,2,\dots$

Using Eqs. (4), (6) and (7), the modified Bernstein series can be obtained

$$f(x)=\sum_{i=0}^{m}\binom{m}{i}x^i(1-x)^{m-i}f\left(\frac{i}{m}\right)-\sum_{a=2}^{2k-1}\frac{\left(\frac{d^a}{dx^a}\right)B_{i,m}(x)}{a!m^a}T_{m,a}(x). \quad (11)$$

Now, we can obtain the solution $g(x)$ by

$$g(x)=\sum_{j=0}^{\infty}g_j(x). \quad (12)$$

Substituting Eqs. (10), (11) and (12) into Eq. (9) gives

$$\sum_{j=0}^{\infty}g_j(x)=\sum_{i=0}^{m}\binom{m}{i}x^i(1-x)^{m-i}f\left(\frac{i}{m}\right)-\sum_{a=2}^{2k-1}\frac{\left(\frac{d^a}{dx^a}\right)B_{i,m}(x)}{a!m^a}T_{m,a}(x)+\lambda\int_{u(x)}^{h(x)}k(x,t)A_n dt \quad (13)$$

And we have

$$g_0(x) = \sum_{i=0}^{m} \binom{m}{i} x^i (1-x)^{m-i} f\left(\frac{i}{m}\right) - \sum_{a=2}^{2k-1} \frac{\left(\frac{d^a}{dx^a}\right) B_{i,m}(x)}{a!\,m^a} T_{m,a}(x)$$

$$g_{n+1}(x) = \lambda \int_{u(x)}^{h(x)} k(x,t) A_n dt, n = 0,1,2,\dots \quad (14)$$

The ADM-MBP solution can be obtained as

$$g_m(x) = \sum_{j=0}^{\infty} g_j(x). \quad (15)$$

Theorem 3.1

The series (15) of Eq. (1) converges if $\exists \alpha \in [0,1)$ and $\|g_{n+1}\| = \alpha \|g_n\|$ such that $\|g_0\| < \infty$.

Proof. Let $S_n = g_1 + g_2 + \dots + g_n$. For $m \ge n$, we shall prove that S_n is a Cauchy sequence

$$\|S_m - S_n\| = \left\| \sum_{i=0}^{m} yi - \sum_{i=0}^{n} yi \right\| = \left\| \sum_{i=n+1}^{m} y_i \right\|.$$

We know that

$$\|S_m - S_n\| = \|(S_{n+1} - S_n) + (S_{n+2} - S_{n+1}) + \dots + (S_m - S_{m-1})\|$$

$$\le \|S_{n+1} - S_n\| + \|S_{n+2} - S_{n+1}\| + \dots + \|S_m - S_{m-1}\|$$

$$\le \alpha^{n+1}\|g_0\| + \alpha^{n+2}\|g_0\| + \dots + \alpha^m \|g_0\|$$

$$\le \left(\alpha^{n+1} + \alpha^{n+2} + \dots + \alpha^m\right)\|g_0\|$$

$$\le \alpha^{n+1}\left(1 + \alpha + \dots + \alpha^{m-n-1}\right)\|g_0\|$$

$$\le \alpha^{n+1}\left(\frac{1-\alpha^{m-n}}{1-\alpha}\right)\|g_0\|$$

$$\le \left(\frac{\alpha^{n+1}}{1-\alpha}\right)\|g_0\| \quad (\text{Since } \alpha \in [0,1) \Rightarrow 1-\alpha^{m-n} < 1)$$

But $\|g_0\| < \infty$, i.e. $\|S_m - S_n\| \to 0$ as $n \to 0$. So the series converges.

Theorem 3.2

The maximum absolute error of Eq. (15) can be obtained as

$$\max_{\forall x \in J} \left| g(x) - \sum_{i=0}^{n} g_i(x) \right| \le \left(\frac{\alpha^{n+1}}{1-\alpha}\right) \max_{\forall x \in J} \|g_0\|.$$

Proof. From Theorem 3.1 for $m \to \infty$ then $S_m \to g(x)$, so

$$\|g(x) - S_n\| \le \left(\frac{\alpha^{n+1}}{1-\alpha}\right)\|g_0\|.$$

Thus

$$\max_{\forall x \in J} \left| g(x) - \sum_{i=0}^{n} g_i(x) \right| \le \left(\frac{\alpha^{n+1}}{1-\alpha}\right) \max_{\forall x \in J} \|g_0\|.$$

4. Applications

Some applications of nonlinear integral equations are solved to show the accuracy of the new modified technique.

Application 4.1

Consider the VIE given by [30]

$$g(x) = x + \int_0^x g^2(t) dt, \quad (16)$$

with analytical solution $g(x) = \tan x$. Using ADM then $N(g) = g^2(t)$ and $f(x) = x$.

Applying the Adomian polynomials to $N(g) = g^2(t)$, we have

$$A_0 = N(g_0) = g_0^2(t),$$
$$A_1 = 2g_0(t) g_1(t),$$
$$A_2 = 2g_0(t) g_2(t) + g_1^2(t),$$
$$A_3 = 2g_0(t) g_3(t) + 2g_1(t) g_2(t),$$

and so on. Using Eqs. (11) - (14) when m=6 and $k = 2$, we set

$$g_0(x) = \sum_{i=0}^{6} \binom{6}{i} x^i (1-x)^{6-i} f\left(\frac{i}{6}\right) - \sum_{a=2}^{3} \frac{\left(\frac{d^a}{dx^a}\right) B_{i,6}(x)}{a!\,6^a} T_{6,a}(x)$$

$$= x(1-x)^5 + 5x^2(1-x)^4 + 10x^3(1-x)^3 + 10x^4(1-x)^2 + 5x^5(1-x) + x^6 = x$$

$$g_1(x)=\int_0^x A_0 dt=\frac{1}{3}x^3$$

$$g_2(x)=\int_0^x A_1 dt=\frac{2}{15}x^5$$

$$g_3(x)=\int_0^x A_2 dt=\frac{17}{315}x^7$$

$$g_4(x)=\int_0^x A_3 dt=\frac{62}{2835}x^9$$

and so on. This in turn gives

$$g_m(x)=\sum_{j=0}^{\infty} g_j(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7 +\frac{62}{2835}x^9+\ldots=\tan(x). \quad (17)$$

Table 1 presents the solutions of $g_7(x)$ with $m=6$ and $k=2$.

Table 1. Solutions of Application 1.

x	Exact Solution	$g_7(x)$	Absolute Error
0	0	0	0
0.1	0.1003346721	0.1003346720	$1.458312143\ 10^{-11}$
0.2	0.2027100355	0.2027100356	$9.008369769\ 10^{-12}$
0.3	0.3093362496	0.3093362497	$8.82960703\ 10^{-12}$
0.4	0.4227932187	0.4227932187	$7.3901140\ 10^{-11}$
0.5	0.5463024898	0.5463024849	$4.96164436\ 10^{-9}$
0.6	0.6841368083	0.6841366915	$1.168913441\ 10^{-7}$
0.7	0.8422883805	0.8422866677	$1.712783529\ 10^{-6}$
0.8	1.029638557	1.0296206170	$1.793931522\ 10^{-5}$
0.9	1.260158218	1.2600117280	$1.464901213\ 10^{-4}$
1.0	1.557407725	1.5564156070	$9.92117936\ 10^{-4}$

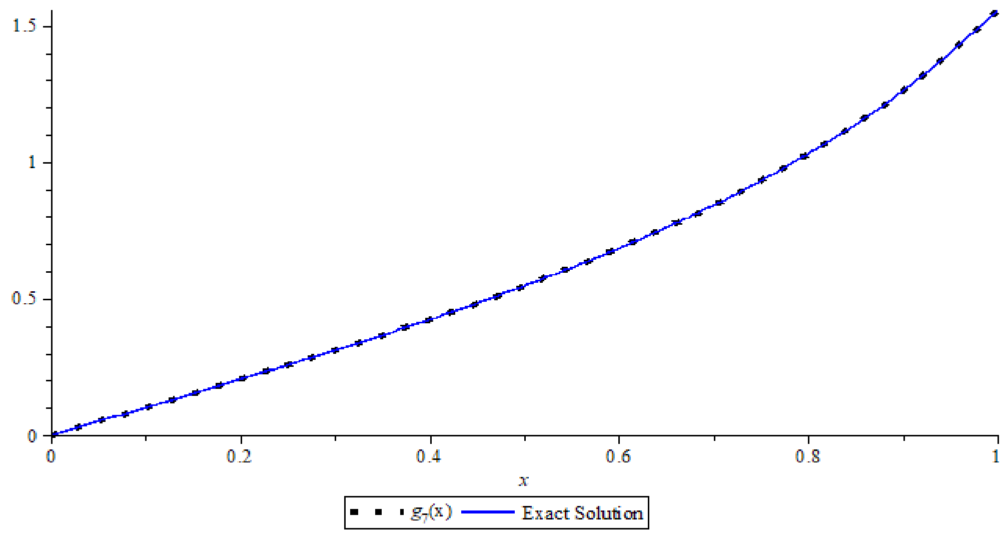

Figure 1. The numerical results obtained by g_7 (x) with m=6 and k=2

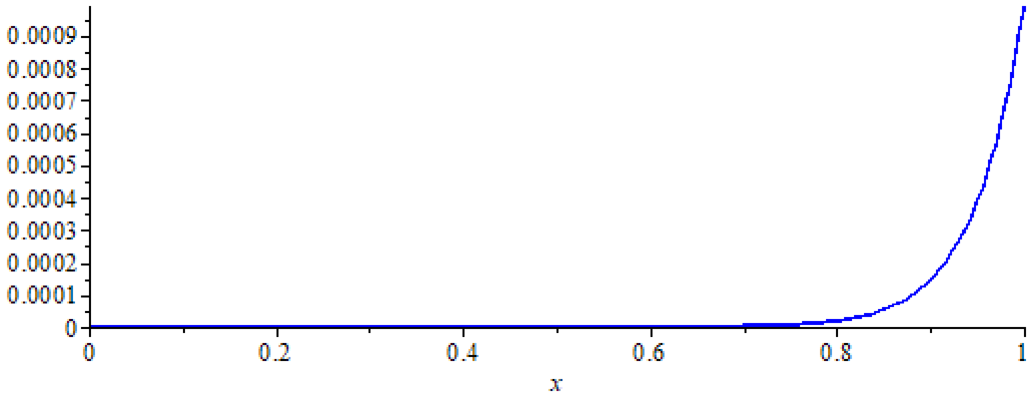

Figure 2. Absolute error between the exact and $g_7(x)$ with m=6 and k=2.

Application 4.2

Consider the FIE given by [30]

$$g(x)=\frac{5}{6}x+\int_0^1 xt^2 g^3(t)dt, \tag{18}$$

with analytical solution $g(x)=x$. Using ADM then $N(g)=g^3(t)$ and $f(x)=\frac{5}{6}x$.

Applying the Adomian polynomials to $N(g)=g^3(t)$, we have

$$A_0=N(g_0)=g_0^3(t),$$
$$A_1=3g_0^2(t)g_1(t),$$
$$A_2=3g_0^2(t)g_2(t)+3g_1^2(t)g_0(t),$$
$$A_3=g_1^3(t)+6g_0(t)g_1(t)g_2(t)+3g_0^2(t)g_3(t),$$

and so on. Using Eqs. (11) - (14) when m=6 and $k=2$, we set

$$g_0(x)=\sum_{i=0}^{6}\binom{6}{i}x^i(1-x)^{6-i}f\left(\frac{i}{6}\right)-\sum_{a=2}^{3}\frac{\left(\frac{d^a}{dx^a}\right)B_{i,6}(x)}{a!6^a}T_{6,a}(x)$$

$$=\frac{5}{6}x(1-x)^5+\frac{25}{6}x^2(1-x)^4+\frac{25}{3}x^3(1-x)^3+\frac{25}{3}x^4(1-x)^2+\frac{25}{9}x^5(1-x)+\frac{5}{6}x^6=\frac{5}{6}x$$

$$g_1(x)=\int_0^x A_0 dt=\frac{125}{1296}x$$

$$g_2(x)=\int_0^x A_1 dt=\frac{3125}{93312}x$$

$$g_3(x)=\int_0^x A_2 dt=\frac{78125}{5038848}x$$

$$g_4(x)=\int_0^x A_3 dt=\frac{107421875}{13060694016}x$$

and so on. This in turn gives

$$g_m(x)=\sum_{j=0}^{\infty}g_j(x)=\frac{5}{6}x+\frac{125}{1296}x+\frac{3125}{93312}x+\frac{78125}{5038848}x+\ldots=x. \tag{19}$$

Table 2 presents the solutions of $g_7(x)$ with $m=6$ and $k=2$.

Table 2. Solutions of Application 2

x	Exact Solution	$g_7(x)$	Absolute Error
0	0	0	0
0.1	0.1	0.09963862376	$3.613762354\ 10^{-4}$
0.2	0.2	0.19927724750	$7.227524708\ 10^{-4}$
0.3	0.3	0.29891587130	$1.084128706\ 10^{-3}$
0.4	0.4	0.39855449510	$1.445504942\ 10^{-3}$
0.5	0.5	0.49819311880	$1.806881177\ 10^{-3}$
0.6	0.6	0.59783174260	$2.168257412\ 10^{-3}$
0.7	0.7	0.69747036640	$2.529633648\ 10^{-3}$
0.8	0.8	0.79710899010	$2.891009883\ 10^{-3}$
0.9	0.9	0.89674761390	$3.252386119\ 10^{-3}$
1.0	1.0	0.99638623760	$3.613762354\ 10^{-3}$

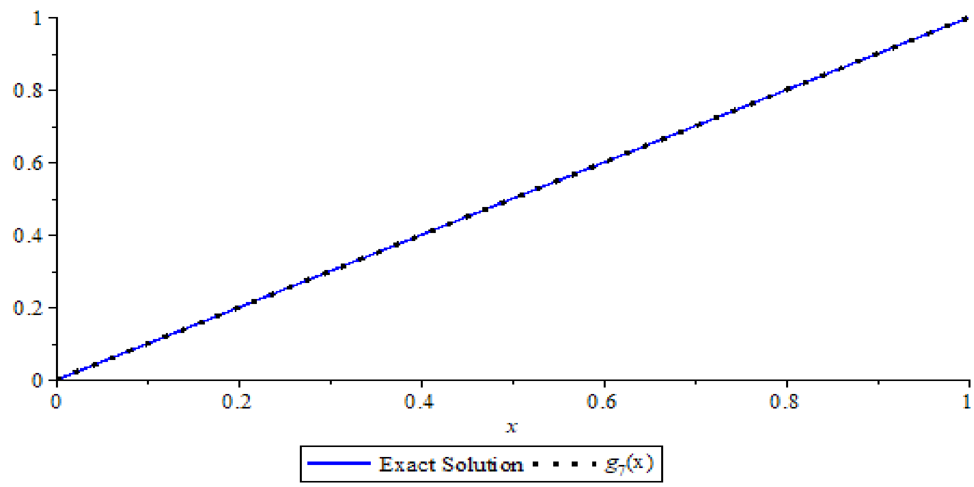

Figure 3. The numerical results obtained by g_7 (x) with m=6 and k=2.

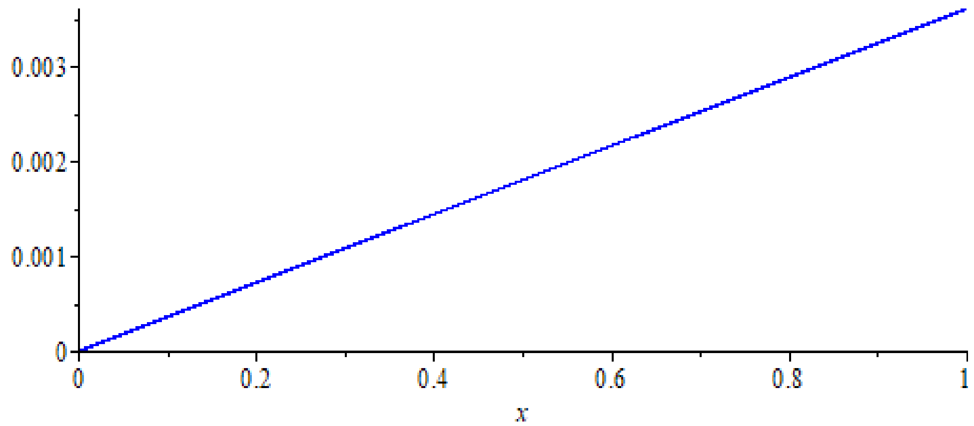

Figure 4. Absolute error between the exact and g_7 (x) with m=6 and k=2

5. Conclusions

We improved ADM based on MBP in order to solve nonlinear Fredholm and Volterra integral equations. This method is tested for some examples from nonlinear integral equations. The results demonstrate reliability of the proposed method.

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