

Information Correlation in a 2×2 Game and an Extension of Purification Rationale

Rongyu Wang*

¹Independent Researcher (PhD Economics, University of Edinburgh, 2016), Qingdao, 266072, China

Received February 6, 2020; Revised March 11, 2019; Accepted March 19, 2020

Copyright ©2020 by authors, all rights reserved. Authors agree that this article remains permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

Abstract In this paper, we study a 2×2 Bayesian entry game with correlated private information. The distribution of private information is modelled by a symmetric joint normal distribution. Therefore, the correlation coefficient of the private information distribution reflects the degree of dependence of players' private information. Under such specification, players' private information can be correlated flexibly, which is not confined to the typical additive specification of private payoff shocks or private information by Carlson and van Damme (1993), where the private information is correlated due to the common payoff shock. In our game, if the private information is correlated, we find that given the variances of the private information, there exists a restriction on the degree of correlation of players' private information that allows the game can be solved by cutoff strategies. Specifically, given the variances of the private information, if players' private information in strategic substitutes (strategic complements) Bayesian games are positively (negatively) correlated, the range of correlation coefficient that allows the game can be solved by cutoff strategies is restricted so that if the correlation is out of the range, the game cannot be solved by cutoff strategies. Alternatively, given positive (negative) correlation of private information, the value of variances that allows a strategic substitutes (strategic complements) Bayesian games can be solved by cutoff strategies are restricted within certain range. If the value of variances fall out of the range, the Bayesian game cannot be solved by cutoff strategies. However, given negative (positive) correlation of players' private information in strategic substitutes (strategic complements) Bayesian games, in which the Bayesian games can always be solved by cutoff strategies, we prove that as the variances converge to zero, all pure strategy Bayesian Nash equilibria of the perturbed games converge to the respective Nash equilibria of the corresponding strategic substitutes (strategic complements) complete information games. Based on the result, we conclude that the purification rationale proposed by Harsanyi (1973) can be extended to games with dependent perturbation errors that follow a symmetric joint normal distribution if the correlation coefficient is positive for the strategic complements games or negative for the strategic substitutes games.

Keywords Correlated Private Information, Purification Rationale

1 Introduction

This paper develops a simple model of firm entry with correlated private information in a 2-player static game. This game is symmetric. In the game, after observing their respective private payoff shocks, two firms simultaneously decide whether to enter a market. The private payoff shocks are statistically correlated, and the correlation coefficient of players' joint type distribution measures the degree of information correlation. That is, there are common and idiosyncratic components of each payoff shock, and each firm only observes its own aggregate shock without knowing its component. An example of this situation is two firms that produce complementary inputs entering a local market, which is a strategic complements setting. Another example is two firms producing the same good competing for the same market, which is a strategic substitutes setting. Each firm expects its private payoff shocks of entry to be correlated with the other firm's, because the shocks depend on certain common factors of the market.

The game is solved by a cutoff strategy, which is defined as if a player's private payoff shock is above a threshold value, they choose entry, or vice versa. By solving the game, we find a critical value of the correlation coefficient. For correlation coefficients below (above) this critical value for a strategic complements (strategic substitutes) game, a cutoff strategy cannot be used to solve the game.¹ This result is determined by the normality of the joint prior distribution and

¹In this paper, we combine some statements of both strategic complements games and strategic substitutes games together. This sentence is an example. Specifically, for statements about strategic substitutes games, the relevant parts are replaced by the contents in parentheses.

the definition of the cutoff strategy. The intuition is that if the correlation coefficient is smaller (greater) than this critical value for a strategic complements (strategic substitutes) game, the expected payoff function is no longer monotonic with respect to the player’s own strategies, which contradicts the definition of either strategic complements games or strategic substitutes games. In such a situation, cutoff strategies cannot be used to solve the game.

The incomplete information entry game can be viewed as a perturbed game of a complete information entry game. According to Harsanyi (1973)’s purification rationale, if the perturbation errors on each player’s payoff are independent, a Bayesian Nash equilibrium exists that will converge to the mixed strategy equilibrium as perturbation errors tend to zero. In our game, we specify that the variances of the perturbation-error distribution converge to zero, as the process that uncertainty of perturbed games vanishes. We find that, for the strategic complements complete information games if the perturbation errors are negatively correlated, or for the strategic substitutes complete information games if the perturbation errors are positively correlated, there does not exist a Bayesian game that can be solved by the cutoff strategy as perturbation errors tend to zero. Hence, Harsanyi’s purification rationale cannot be applied to this situation. The intuition is that by assuming the variances of both players’ type distributions are identical, for negative information correlation in the strategic complements game or the positive correlation in the strategic substitutes game, there exists a critical value of variances, below which the expected payoff function is not monotonic with respect to a player’s own private payoff shock. The non-monotonic expected payoff functions contradict the definition of either strategic complements games or strategic substitutes games, and hence in such situations, cutoff strategy cannot be used to solve the games. Therefore, for negative information correlation in the strategic complements game or positive information correlation in the strategic substitutes game, only if the variances are above the cutoff value, the game can be solved by cutoff strategies.

However, if the information correlation is positive for the strategic complements games or negative for the strategic substitutes games, the purification rationale is still applicable. We find that in these situations, the Bayesian games that are supposed to converge to the complete information game as the perturbation errors degenerate to zero exist, and during the process, the pure-strategy Bayesian Nash equilibrium will converge to the corresponding Nash equilibrium of the underlying complete information game. Therefore, we extend Harsanyi’s purification rationale to dependent perturbation-error situations.

The rest of this paper proceeds as follows. Section 2 presents the game and some crucial properties caused by information correlation. Section 3 explains how purification rationale can be extended to games with dependent perturbation errors. Section 4 concludes this paper.

2 The Game

Consider a 2-player entry game. Each player has two choices, activity or entry (hereafter, 1), or inactivity (hereafter, 0). Each firm makes its own decision after observing its private payoff shock. Then, both firms implement their decisions, which can be observed by each other. The active firm will enter the market. If both firms are active, a coordination (competition) will happen between them and the profit D if the opponent chooses to be active is strictly greater (smaller) than the profit M if the opponent chooses to be inactive. At the end of the period, both firms collect their respective payoffs. The inactive firm gets payoff zero and the active firm obtains the deterministic payoff D or M plus its private payoff shock. Therefore, according to the definition of strategic complements games and strategic substitutes games (Fudenberg and Tirole, 1991), if $D > M$, the game is a strategic complements game, and if $M > D$, the game is a strategic substitutes game. It is assumed that the private payoff shocks are subject to a bivariate normal distribution $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma, \varsigma^*, \rho)$. In this paper, we use ‘*’ to denote variables of the opponent. It is always assumed that $\varsigma = \varsigma^*$ to ensure that the game is symmetric. The strategic form of this game is depicted as follows:

		Firm i^*	
		inactive (0)	active (1)
Firm i	inactive (0)	0	$M + \varepsilon^*$
	active (1)	$M + \varepsilon$	$D + \varepsilon^*$

Table 1: The incomplete information entry game

Firms adopt cutoff strategies: if payoff shock ε is above a threshold value $\bar{\varepsilon}$, a player chooses to be active, or vice versa. Therefore, the interim belief that the opponent plays out given payoff shock ε is given by $\sigma(x^*, \varepsilon) = \int_{-\infty}^{x^*} f(\varepsilon^*|\varepsilon)d\varepsilon^*$, where $f(\varepsilon^*|\varepsilon)$ is the conditional density of ε^* given ε . $\sigma_{x^*}(x^*, \varepsilon)$ is the first-order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to x^* , and $\sigma_{\varepsilon}(x^*, \varepsilon)$ is the first order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to ε . It is found that $\sigma_{x^*}(x^*, \varepsilon) > 0$, $\sigma_{\varepsilon}(x^*, \varepsilon) < 0$ if $\rho > 0$, and $\sigma_{\varepsilon}(x^*, \varepsilon) > 0$ if $\rho < 0$. $\sigma_{\varepsilon}(x^*, \varepsilon) = 0$ at $\rho = 0$. So given a player’s own payoff shock ε , if the opponent’s cutoff strategy becomes higher, then the belief that the opponent chooses being inactive will increase. Given the opponent’s strategy, if the correlation coefficient is positive, a high payoff shock of a player indicates that probably the opponent also gets a high payoff shock; thus, the belief that the opponent chooses being inactive decreases. Given the opponent’s strategy, if the correlation coefficient is negative, a high payoff shock of a player indicates that probably the opponent gets a negative payoff shock; hence, the belief that the opponent chooses being inactive increases. If the correlation coefficient equals 0, a player’s own payoff shock does not have any impact on their belief of the opponent’s behaviour. Therefore, firm i ’s expected payoff of entry can be written as

$$\begin{aligned}\mathbb{E}\Pi(x^*, \varepsilon) &= \sigma(x^*, \varepsilon)(M + \varepsilon) + (1 - \sigma(x^*, \varepsilon))(D + \varepsilon) \\ &= \sigma(x^*, \varepsilon)M + (1 - \sigma(x^*, \varepsilon))D + \varepsilon\end{aligned}\quad (1)$$

Equation (1) indicates that a player's expected payoff is composed of two parts: the payoff induced by strategic uncertainty, $\sigma(x^*, \varepsilon)M + (1 - \sigma(x^*, \varepsilon))D$, and the realised payoff shock, ε . If $\rho \geq 0$ for the strategic complements game, or if $\rho \leq 0$ for the strategic substitutes game, given ρ , M , D , ς^2 and ς^{*2} , both parts are non-decreasing with respect to ε . Intuitively, if both firms' private payoff shocks are positively (negatively) correlated, a high payoff shock ε for one firm would on average imply a high (low) payoff shock ε^* for the opponent, which provides an incentive that encourages the player to be active in the strategic complements (strategic substitutes) game. Therefore, the expected payoff should be non-decreasing with respect to ε for $\rho \geq 0$ for the strategic complements game and for $\rho \leq 0$ for the strategic substitutes game. Thus, for a positively (negatively) correlated private information situation in the strategic complements (strategic substitutes) game, the cutoff strategy can always be applied.

However, if ρ is negative for the strategic complements game, or if ρ is positive for the strategic substitutes game, then given all parameter values, the payoff induced by strategic uncertainty $\sigma(x^*, \varepsilon)M + (1 - \sigma(x^*, \varepsilon))D$ is decreasing with respect to ε . Thus, whether the expected payoff $\mathbb{E}\Pi(x^*, \varepsilon)$ is monotonically increasing with respect to ε depends on the trade-off between the payoff induced by strategic uncertainty and by the realized payoff shock. For negative (positive) value of ρ in the strategic complements (strategic substitutes) game, if one firm draws a high payoff shock, it can be expected that its opponent draws a low (high) payoff shock, and hence, it is highly probable that the opponent chooses being inactive (active), which provides strategic disincentives for the firm to choose entry in the strategic complements (strategic substitutes) context. It is also known that ε itself is a part of the payoff and it incentivizes entering. Therefore, whether the firm will choose to be active essentially depends on the trade-off between the two contrasting effects.

If the correlation between players' private information is loose, i.e. ρ is slightly negative (positive) in the strategic complements (strategic substitutes) game, it can be deduced that the positive incentive generated by a high value of ε dominates its negative impact, and hence, in total, its expected payoff should increase with respect to ε . However, if the correlation coefficient between the players' private information is tight, i.e. ρ is close to -1 (1) for the strategic complements (strategic substitutes) game, then it can be reasonably expected that the strategic disincentive induced by the realization of a high value of ε will be strong, and hence, a high payoff shock does not necessarily bring a high expected payoff $\mathbb{E}\Pi(x^*, \varepsilon)$. In fact, it is found that there exists a unique boundary $\tilde{\rho} < 0$ ($\tilde{\rho} > 0$) in the strategic complements (strategic substitutes) discrete game such that if $\rho \geq \tilde{\rho}$ ($\rho \leq \tilde{\rho}$), given the opponent's cutoff

strategy $x^* \in \mathbb{R}$, the expected payoff $\mathbb{E}\Pi(x^*, \varepsilon)$ is increasing with respect to ε , but if $\rho < \tilde{\rho}$ ($\rho > \tilde{\rho}$), the expected payoff is no longer monotonic; such feature contradicts the definition of either strategic complements games or strategic substitutes games, and hence in such a situation cutoff strategy cannot be used to solve the game (see Appendix B). Therefore, given D , M , ς^2 and ς^{*2} , a player can legitimately use a cutoff strategy to play the game if and only if $\rho \in [\tilde{\rho}, 1)$ in the game and $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$ for the strategic complements game, or $\rho \in (-1, \tilde{\rho}]$ in the game and $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$ for the strategic substitutes game.² Thus, for each player, there exists a boundary of ρ and for the value of ρ above (below) the boundary value for the strategic complements (strategic substitutes) game, a cutoff strategy can be used to solve the game. Due to the assumption $\varsigma = \varsigma^*$, the boundary for both players are the same, i.e. $\tilde{\rho} = \tilde{\rho}^*$, and therefore, this boundary defines the range of ρ for which a cutoff strategy can be used to solve the game. This result is formally given by the following proposition:

Proposition 1 (Restriction of Applying a Cutoff Strategy to Solve the Game for Information Correlation) : Supposing $D > M$ and $\varsigma^* = \varsigma$, a cutoff strategy can be applied to solve the game if and only if $\rho \in [\tilde{\rho}, 1)$, where $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$. Supposing $M > D$ and $\varsigma^* = \varsigma$, a cutoff strategy can be applied to solve the game if and only if $\rho \in (-1, \tilde{\rho}]$, where $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$.

Proof: See Appendix. ■

$\pi = 3.14\dots$ is the ratio of a circle's circumference to its diameter. Given $\rho \in [\tilde{\rho}, 0)$ for $D > M$ or $\rho \in (0, \tilde{\rho}]$ for $M > D$, and an $x^* \in \mathbb{R}$, if $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε , it indicates that

$$\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*, \varepsilon)(M - D) + 1 \geq 0$$

for all $x^* \in \mathbb{R}$; hence,

$$1 \geq \sigma_\varepsilon(x^*, \varepsilon)(D - M)$$

Because $\sigma_\varepsilon(x^*, \varepsilon) = -\rho f(x^*|\varepsilon)$ (see Appendix A), the above inequality can be written as

$$1 \geq -\rho f(x^*|\varepsilon)(D - M)$$

and hence

$$f(x^*|\varepsilon) \leq \frac{1}{-\rho(D - M)} \quad (2)$$

As ς increases, the variance of the distribution $f(\cdot|\varepsilon)$, which equals $\varsigma^2(1 - \rho^2)$, increases, and hence the density

²For the opponent, $\tilde{\rho}^* = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}}$ in the strategic complements game and $\tilde{\rho}^* = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}}$ in the strategic substitutes game.

function flattens.³ Particularly, the maximum value of $f(x^*|\varepsilon)$, which equals $\frac{1}{\sqrt{2\pi(1-\rho^2)\varsigma}}$ and is taken at the mean $x^* = \rho\varepsilon$, decreases. Hence, (2) is easier to be satisfied and it is more certain that at the given value of ρ , $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff increase with respect to ε should be broadened as ς increases, and accordingly, $\tilde{\rho}$ decreases if $D > M$, and $\tilde{\rho}$ increases if $M > D$.

If $D - M$ ($M - D$) decreases for the strategic complements (strategic substitutes) game, the RHS of (2) increases. Hence, (2) is easier to be satisfied, and it is more certain that at the given value of ρ , $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff of entry increase with respect to ε should be broadened as $D - M$ ($M - D$) decreases, and accordingly, $\tilde{\rho}$ decreases (increases).

Given the opponent's cutoff strategy $x^* \in \mathbb{R}$, a firm's cutoff best response $g(x^*)$ is determined by $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. That is,

$$\sigma(x^*, g(x^*))(M - D) + D + g(x^*) = 0$$

It is found that $g(x^*) \in [-D, -M]$ ($g(x^*) \in [-M, -D]$) for the strategic complements (strategic substitutes) game because as long as $D > M$ ($M > D$), the maximum (minimum) of $\sigma(x^*, \varepsilon)(M - D) + D$ equals D , where $\sigma(x^*, \varepsilon) = 0$, and the minimum (maximum) of $\sigma(x^*, \varepsilon)(M - D) + D$ equals M , where $\sigma(x^*, \varepsilon) = 1$. Given the joint normal distribution, we obtain the best response function in its reverse form:

$$x^* = \rho \frac{\varsigma^*}{\varsigma} g(x^*) + \varsigma^* \sqrt{1 - \rho^2} \Phi^{-1}\left(\frac{D + g(x^*)}{D - M}\right) \quad (3)$$

where $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution. Then, we can get the derivative of $g(x^*)$ with respect to x^* as follows.

$$g'(x^*) = -\frac{\sigma_{x^*}(x^*, g(x^*))(M - D)}{\sigma_\varepsilon(x^*, g(x^*))(M - D) + 1} \quad (4)$$

Assume $\varsigma^2 = \varsigma^{*2}$ is always held. For simplicity, in the following, we specify ς^2 and ς^{*2} as the same variable. From Proposition 1, it has been known that if and only if $\rho \geq -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $D > M$ or $\rho \leq \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $M > D$, a cutoff strategy can be used to solve the game. Equivalently, it implies a restriction on the variance:

$$\varsigma^2 \geq \frac{\rho^2(D - M)^2}{2\pi(1 - \rho^2)} \text{ for } \rho < 0$$

The inequality indicates that given $D > M$ and $\rho < 0$, there exists a lower bound of ς^2 , which is denoted by $\tilde{\varsigma}^2$; hence, $\tilde{\varsigma}^2 = \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)}$ and $\tilde{\varsigma} = \sqrt{\tilde{\varsigma}^2}$. Given variances below this lower bound in the case of $D > M$ and $\rho < 0$, the game cannot be solved using a cutoff strategy. The intuition for this result is similar to the intuition of Proposition 1. Let us recall that

$$\begin{aligned} \frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} &= \sigma_\varepsilon(x^*, \varepsilon)(M - D) + 1 \\ &= \frac{\rho(D - M)}{\varsigma\sqrt{2\pi(1 - \rho^2)}} \exp\left(-\frac{1}{2}\left(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1 - \rho^2}}\right)^2\right) + 1 \end{aligned}$$

From the above expression, it can be seen that if $D > M$ and $\rho \geq 0$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} > 0$ for all $\varsigma \in (0, +\infty)$. For $\rho < 0$, $\tilde{\varsigma}$ exists and it makes $\min \frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \frac{\rho(D-M)}{\tilde{\varsigma}\sqrt{2\pi(1-\rho^2)}} + 1 = 0$.

Therefore, for $\rho < 0$, if $\varsigma \geq \tilde{\varsigma}$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ for all $x^* \in \mathbb{R}$. For $\rho < 0$, if $\varsigma < \tilde{\varsigma}$, $\mathbb{E}\Pi(x^*, \varepsilon)$ is no longer monotonic with respect to ε . This situation parallels the property of expected payoff function with $\rho < \tilde{\rho}$ given ς and $D > M$ (see Appendix B). Therefore, by assuming $\varsigma = \varsigma^*$, given $D > M$ and ρ , a player can legitimately use a cutoff strategy to play the game if and only if $\varsigma \in [\tilde{\varsigma}, +\infty)$ for $\rho < 0$ or $\varsigma \in (0, +\infty)$ for $\rho \geq 0$.

In the strategic substitutes game, it has been proven that if and only if $\rho \leq \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (M-D)^2}}$, a cutoff strategy can be used to solve the game. Equivalently, it also implies a restriction on the variance to ensure that the game can be solved by a cutoff strategy:

$$\varsigma^2 \geq \frac{\rho^2(D - M)^2}{2\pi(1 - \rho^2)} \text{ for } \rho > 0 \quad (5)$$

This inequality indicates that given $M > D$ and $\rho > 0$, there exists a lower bound of ς^2 , which is denoted by $\tilde{\varsigma}^2$ and $\tilde{\varsigma} = \sqrt{\frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)}}$. For variances below this lower bound, the game cannot be solved by a cutoff strategy. For $\rho \leq 0$, a cutoff strategy is still applicable for all $\varsigma^2 \in (0, +\infty)$ because for $\rho \leq 0$, the relationship $\rho \leq 0 \leq \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (M-D)^2}} = \tilde{\rho}$ always holds for all $\varsigma^2 \in (0, +\infty)$.

The intuition of the existence of $\tilde{\varsigma}^2$ for $M > D$ and $\rho > 0$ is similar to the intuition for $D > M$ and $\rho < 0$. Recall that if $M > D$, for $\rho \leq 0$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} > 0$ for all $\varsigma \in (0, +\infty)$. For $\rho > 0$, there exists $\tilde{\varsigma}$ such that $\min \frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \frac{\rho(D-M)}{\tilde{\varsigma}\sqrt{2\pi(1-\rho^2)}} + 1 = 0$ for all $x^* \in \mathbb{R}$.

Therefore, if $\rho > 0$, for $\varsigma \geq \tilde{\varsigma}$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ for all $x^* \in \mathbb{R}$. For $\varsigma < \tilde{\varsigma}$ if $\rho > 0$, this situation parallels that of $\rho > \tilde{\rho}$ given $\varsigma = \varsigma^*$ in the strategic substitutes game. In this situation, $\mathbb{E}\Pi(x^*, \varepsilon)$ is no longer monotonic with respect to ε , which essentially contradicts the definition of strategic complements games and strategic substitutes games (see Appendix B). Therefore, by assuming $\varsigma = \varsigma^*$, given $M > D$ and ρ , a player can use a cutoff strategy to play the game if and only if $\varsigma \in [\tilde{\varsigma}, +\infty)$ for $\rho > 0$ or $\varsigma \in (0, +\infty)$ for $\rho \leq 0$.

³The density function $f(\cdot|\varepsilon)$ refers to $f(\varepsilon^*|\varepsilon)$. For the explicit expression, please refer to Appendix A.

Proposition 2 (Restriction of Applying a Cutoff Strategy to Solve the Game for Variances) : Assume $\varsigma = \varsigma^*$. Given $D > M$ and $\rho \in (-1, 1)$, a player can use a cutoff strategy to solve the game if and only if $\varsigma \in [\tilde{\varsigma}, +\infty)$ for $\rho < 0$ or $\varsigma \in (0, +\infty)$ for $\rho \geq 0$. Given $M > D$ and $\rho \in (-1, 1)$, a player can use a cutoff strategy to solve the game if and only if $\varsigma \in [\tilde{\varsigma}, +\infty)$ for $\rho > 0$ or $\varsigma \in (0, +\infty)$ for $\rho \leq 0$. In either situation, $D > M$ or $M > D$, $\tilde{\varsigma} = \sqrt{\zeta^2}$, where $\zeta^2 = \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)}$. \square

3 An Extension of Purification Rationale

Now consider the following complete information entry game:

		Firm i^*	
		inactive (0)	active (1)
Firm i	inactive (0)	0	M
	active (1)	0	D

Table 2: The complete information entry game

Assume $D > 0 > M$. The game has three equilibria, $(0, 0)$, $(1, 1)$ and $(\frac{M}{M-D}, \frac{M}{M-D})$, where $\frac{M}{M-D}$ is the probability to choose being active. Correspondingly, assume $M > 0 > D$. The game also has three equilibria, $(0, 1)$, $(1, 0)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$, where $\frac{D}{D-M}$ is the probability to choose being active. The game shown in Table 1 is the perturbed game of this complete information game. In the following, for simplicity, we call the game shown in Table 2 as the complete information entry game, and the game shown in Table 1 as the perturbed entry game.

Harsanyi (1973) proposed a purification rationale for the play of mixed strategy equilibria. According to Harsanyi (1973), suppose that a player has some small private propensity to choose being active or being inactive, and this propensity is independent of the payoff specification. However, this information is not known to the other player at all. Then, the behaviour of such player will look as if they are randomizing between their actions to the other player. Because of the private payoff perturbation, the opponent will not in fact be indifferent to their actions, but will almost always choose a strict best response. Harsanyi’s purification theorem showed that all equilibria of almost all complete information games are the limit of pure strategy equilibria of perturbed games where players have independent small private payoff shocks.

Note that, in Harsanyi’s purification theorem, he specifies that the uncertainty of perturbed games vanishes in scale. That is, a constant η times the perturbation error ε , and let $\eta \rightarrow 0$. But in our game, we use an alternative approach to model the process that the uncertainty of perturbed games vanishes. That is, to let the variances of the perturbation-error distribution

converge to zero. Here we make a clarification. For Harsanyi’s (1973) purification rationale, it literally describes the idea that every Nash equilibrium of a complete information game can always be approached by a pure strategy Bayesian Nash equilibrium of a perturbed game. For Harsanyi’s (1973) purification theorem, it further requires that the uncertainty of perturbed games vanishes in scale.

Following Morris’ (2008) approach to decomposing Harsanyi’s purification theorem, we can correspondingly decompose Harsanyi’s purification rationale into two parts. The ‘purification’ part, where all equilibria of the perturbed game are essentially pure, and the ‘approachability’ part, where every equilibrium of a complete information game is the limit of equilibria of such perturbed games. For the first part, both Harsanyi’s purification rationale and Harsanyi’s purification theorem use the assumption of sufficiently diffuse independent payoff shocks. For our 2×2 games, the purification rationale indicates that provided that $\rho = 0$, all pure-strategy Bayesian Nash equilibria of the perturbed game obtained by using cutoff strategies (see Table 1) will finally converge to a Nash equilibrium of the complete information game (see Table 2).

However, what will be the situation if we relax the purification rationale by assuming the perturbation errors are dependent? Will Harsanyi (1973)’s purification rationale be still held for dependent payoff shocks?

Carlsson and van Damme (CvD, Appendix B, 1993) compare their global game model with Harsanyi’s model. CvD’s game is identical to our game shown in Table 1 when $D > M$. Both are symmetric and strategic complements. The only difference is that in their game the ε of our game is additively decomposed into a common shock and an idiosyncratic shock χ , i.e. $\varepsilon = \theta + \chi$. θ and χ are independent and both follow a normal distribution. CvD’s additivity specification of ε has been widely used in economics literature, especially in the research of coordination games, for example in Morris and Shin (2005), Hellwig and Veldkamp (2009) and Myatt and Wallace (2012). We denote μ_θ and μ_χ as the mean of θ and χ , respectively, and ς_θ^2 and ς_χ^2 as the variances of θ and χ . Therefore, $\varepsilon \sim N(\mu_\theta + \mu_\chi, \varsigma_\theta^2 + \varsigma_\chi^2)$, where $\mu_\theta + \mu_\chi = 0$ and $\varsigma_\theta^2 + \varsigma_\chi^2 = \varsigma^2$. Thus, ε and ε^* are correlated due to the common payoff shock, i.e. $\rho = \frac{\varsigma_\theta^2}{\varsigma_\theta^2 + \varsigma_\chi^2}$. In contrast, in our games, ε and ε^* can be dependent or correlated in any way, and due to the normal distribution specification, correlation coefficient ρ can reflect the dependence relation between ε and ε^* , rather than a simple correlation relation between the two shocks.

By specifying $\varsigma_\theta^2 \neq 0$ and $\varsigma_\chi^2 \rightarrow 0$, their model is the global game, and a unique equilibrium will be selected as $\rho \rightarrow 1$. However, CvD’s work cannot show whether Harsanyi’s (1973) purification rationale can be extended to perturbed games with correlated perturbation errors. It is because CvD’s model requires that $\varsigma_\theta^2 + \varsigma_\chi^2 \rightarrow 0$, but due to the additive error structure $\varepsilon = \theta + \chi$, as $\varsigma_\theta^2 + \varsigma_\chi^2 \rightarrow 0$, $\rho = \frac{\varsigma_\theta^2}{\varsigma_\theta^2 + \varsigma_\chi^2}$ changes as well and $\rho \rightarrow 1$. Therefore, CvD’s framework cannot isolate

ρ 's impact on the game as the perturbation errors ε and ε^* degenerate to a constant 0.

In last section of this work, we see that by assuming $D > M$ ($M > D$) and $\rho \geq 0$ ($\rho \leq 0$), the games for $\varsigma^2 \in (0, +\infty)$ can be solved by cutoff strategies. The game closest to the complete information entry game is the Bayesian game, where ς and $\varsigma^* \rightarrow 0$. If $\varsigma = \varsigma^*$, the best response function in its reverse form is given by

$$x^* = \rho g(x^*) + \varsigma \sqrt{1 - \rho^2} \Phi^{-1}\left(\frac{D + g(x^*)}{D - M}\right)$$

where $g(x^*) \in [-D, -M]$ and $x^* \in \mathbb{R}$. Therefore, as ς and $\varsigma^* \rightarrow 0$,

$$g(x^*) = \frac{1}{\rho} x^*$$

where $x^* \in [-\rho D, -\rho M]$ if $D > M$ and $\rho > 0$. Let us recall the following definition equation of $g(x^*)$:

$$\begin{aligned} \mathbb{E}\Pi(x^*, g(x^*)) &= \sigma(x^*, g(x^*))(M - D) + D + g(x^*) \\ &= \Phi\left(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}\right)(M - D) + D + g(x^*) = 0 \end{aligned}$$

As $\varsigma \rightarrow 0$, if $x^* > -\rho M$, $g(x^*) = -M$, and if $x^* < -\rho D$, $g(x^*) = -D$ (see Appendix D). Therefore, the best response function of the Bayesian games with ς and $\varsigma^* \rightarrow 0$ for all $x^* \in \mathbb{R}$ and $\rho > 0$ is given by

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho} x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M \end{cases}$$

The intuition of the piecewise expression of $g(x^*)$ as ς and $\varsigma^* \rightarrow 0$ is as follows. Supposing $D > M$, if the opponent i^* is expected to adopt a very high (low) cutoff strategy, it implies that player i expects that i^* is more likely to choose being inactive (active). In a strategic complements context, players always tend to match their action strategies, and hence as a best response, i will adopt the highest (lowest) cutoff strategy that can be achieved to indicate that the player also prefers being inactive (active). This highest (lowest) strategy is $-M$ ($-D$).⁴

Assuming $\varsigma = \varsigma^*$, as ς and $\varsigma^* \rightarrow 0$, the likelihood of the mean of the distribution of the opponent's payoff shock given a player's own payoff shock increases, while the likelihood of the payoff shocks at both sides of the distribution around the mean decreases, because the variance of the conditional payoff shock distribution, $\varsigma^2(1 - \rho^2)$, degenerates. Suppose the

⁴The contents in parentheses in this paragraph describe the intuition of the best response function for the strategic complements game as well.

payoff shock that makes player i indifferent to entry or being inactive equals $g(x^*)$, where reasonably $g(x^*) \in [-D, -M]$ for $D > M$ or $g(x^*) \in [-M, -D]$ for $M > D$, then the mean of the opponent's payoff shock distribution is $\rho g(x^*)$, which happens with a very high likelihood as ς and $\varsigma^* \rightarrow 0$.

In symmetric games, no matter whether the game exhibits strategic complements or strategic substitutes, if a player is expected to be indifferent to being active or being inactive, the opponent will also adopt a strategy such that the opponent is also indifferent to entry or being inactive as a best response. Thus, the opponent i^* will choose a strategy x^* indicating indifference to their own action choices.

Therefore, based on the analysis from the previous two paragraphs, given $g(x^*)$ between $-M$ and $-D$, i expects that the payoff shock that is most likely to happen for i^* is $\rho g(x^*)$. Because at $g(x^*)$, i is indifferent to either action choice, as a best response, at $\rho g(x^*)$, i^* will also be indifferent to either action choice. Therefore, i^* 's strategy x^* should be equal to $\rho g(x^*)$ when ς and $\varsigma^* \rightarrow 0$ if $g(x^*) \in [-D, -M]$ for $D > M$ or $g(x^*) \in [-M, -D]$ for $M > D$. Obviously, this intuition applies to both the strategic complements and strategic substitutes cases.

Because the game is symmetric, for the strategic complements game, the equilibria can be described by the intersection points between $g(x^*)$ and the 45° line. Specifically, if $D > 0 > M$, there are three equilibria (intersection points): $(-M, -M)$, $(-D, -D)$ and $(0, 0)$ (see Figure 1-1).⁵ As ς and $\varsigma^* \rightarrow 0$, the payoff shocks ε and ε^* converge to 0. Therefore, given cutoff strategy equilibrium $(-M, -M)$, since $-M > 0$, both players always choose action 0 in this equilibrium. Given cutoff strategy equilibrium $(-D, -D)$, since $-D < 0$, both players always choose action 1 in this equilibrium. Given strategy $(0, 0)$, the equilibrium belief $\sigma(0, 0)$ equals $\frac{D}{D-M}$ given any value of $\rho \in (0, 1)$. Thus, in this situation, $\sigma(0, 0)$ is independent of ρ and it is always equal to the unconditional probability of player i^* choosing action 0. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of the game expressed in the form of action strategies are given by $(0, 0)$, $(1, 1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $D > 0 > M$. Similarly, if $0 > D > M$ or $D > M > 0$, the equilibrium cutoff

⁵The intuition of the cutoff strategy equilibrium is that given $D > 0 > M$, a player can expect that the opponent either chooses being active or inactive. If a player expects the opponent to choose entry, the player will get payoff D if they also choose entry. Thus, the player will adopt a cutoff strategy $-D$. As the best response, the opponent will adopt a strategy $-D$.

In contrast, if a player expects the opponent to choose being inactive, then the player will get payoff M if they choose to enter. Thus, the player will adopt a cutoff strategy $-M$. As the best response, the opponent will adopt a strategy $-M$.

If a player expects the opponent is indifferent to being active or being inactive, it indicates that irrespective of what value ε^* is, the expected payoff of entry for opponent i^* is equal to 0. Therefore, player i^* 's cutoff strategy is equal to 0. Hence, given $D > 0 > M$, player i will adopt a strategy 0 as a best response. Therefore, another cutoff strategy equilibrium ς and $\varsigma^* \rightarrow 0$ is $(0, 0)$.

strategies are $(-M, -M)$ or $(-D, -D)$ respectively, which imply the action strategies $(0, 0)$ or $(1, 1)$ (see Figures 1-2 and 1-3).⁶ These equilibria are exactly equal to the corresponding equilibria of the games with $\varsigma = \varsigma^* = 0$ and $0 > D > M$ or with $\varsigma = \varsigma^* = 0$ and $D > M > 0$. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of the perturbed games finally converge to the equilibria of the underlying complete information game.

Therefore, if $D > M$ and perturbation errors ε and ε^* follow a joint normal distribution, all equilibria of the complete information entry games are the limit of pure-strategy Bayesian Nash equilibria of perturbed games where players have non-negatively dependent perturbation errors.

However, if $D > M$ and $\rho < 0$, then ζ^2 arises. In the previous section, we have shown that if and only if $\zeta^2 \in [\zeta^2, +\infty)$, the Bayesian games can be solved by cutoff strategies. If $\zeta^2 \in (0, \zeta^2)$, the Bayesian games that can be solved by cutoff strategies do not exist due to the violation of the definition of the cutoff strategy concept, as we have exhibited in Section 2. Therefore, the sequence of such perturbed Bayesian games that are supposed to converge to the complete information game does not exist. Hence, the ‘approachability’ part of the purification rationale cannot be satisfied, and so the purification rationale cannot be applied in this situation. Therefore, in the strategic complements games ($D > M$), if and only if $\rho \geq 0$, Harsanyi’s purification rationale is still applicable.

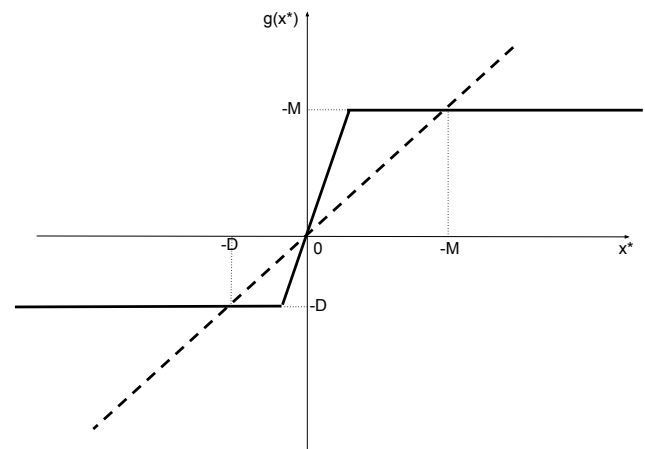
Extending purification rationale in the strategic substitutes game where $M > D$ is similar to extending it in the strategic complements game discussed above. If $M > D$ and $\rho \leq 0$, games that can be solved by cutoff strategies exist for all $\zeta^2 \in (0, +\infty)$. Since the equilibria of a game are solutions of the equation system composed of both players’ best response functions, a small perturbation of the equation system will result in a nearby equilibrium. The most closet game is the game with ς and $\varsigma^* \rightarrow 0$. Again, the best response function is given by the following piecewise function:

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho}x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M \end{cases}$$

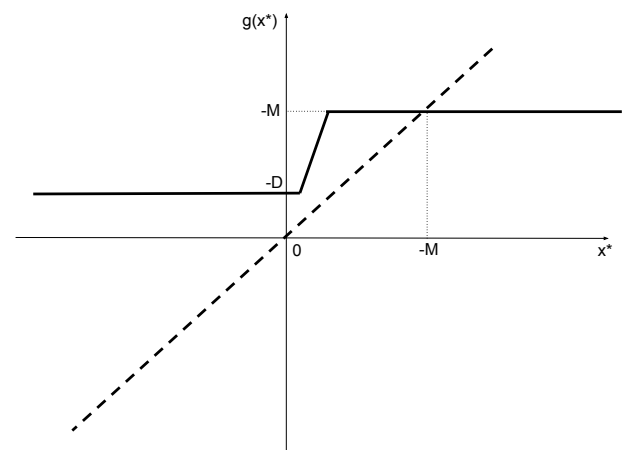
where $x^* \in \mathbb{R}$ and $\rho < 0$.

⁶The intuitions of these cutoff strategy equilibria are as follows. Suppose $0 > D > M$. As ς and $\varsigma^* \rightarrow 0$, it is very likely that each player will choose being inactive. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -M$ since $M + \varepsilon > 0$ and M is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, the cutoff strategy equilibrium $(-M, -M)$ exists in this situation.

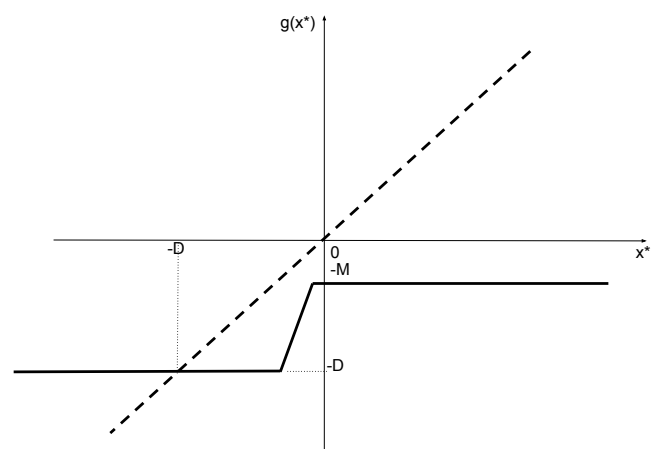
Similarly, suppose $D > M > 0$. As ς and $\varsigma^* \rightarrow 0$, it is very likely that each player will choose being active. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -D$ since $D + \varepsilon > 0$ and D is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, in this situation, we have the cutoff strategy equilibrium $(-D, -D)$.



1-1: $D > 0 > M$



1-2: $0 > D > M$



1-3: $D > M > 0$

Figure 1. The solid curve represents $g(x^*)$ as ς and $\varsigma^* \rightarrow 0$. The dashed line represents the 45° line. The intersection points between $g(x^*)$ and the 45° line are the equilibria of the game with ς and $\varsigma^* \rightarrow 0$. If $D > 0 > M$, there are three equilibria, $(-M, -M)$, $(-D, -D)$ and $(0, 0)$. If $0 > D > M$, there is a unique equilibrium $(-M, -M)$. If $D > M > 0$, there is a unique equilibrium $(-D, -D)$.

Although the expression of the best response function is the same as the one for $D > M$ and $\rho > 0$, the intuitions are not exactly the same. The intuition of $g(x^*) \in [-M, -D]$ is as follows. Given that $M > D$, if the opponent i^* is expected to adopt a very high (low) strategy, it means player i expects that i^* is most likely to choose being inactive (active). In a strategic substitutes context, players always tend to mismatch their action strategies, and hence as the best response, i will adopt the lowest (highest) strategy that can be achieved to indicate the player's preference of being active (inactive). This lowest (highest) strategy is $-M$ ($-D$).⁷

Because the game is symmetric, $g(x^*)$ and $g^*(x)$ are symmetrically located around the 45° line. The equilibria are the intersection points between $g(x^*)$ and $g^*(x)$. Specifically, for $M > 0 > D$, if $M > \rho D$ and $D > \rho M$, there are three equilibria (intersection points): $(-\frac{D}{\rho}, -D)$, $(-D, -\frac{D}{\rho})$ and $(0, 0)$ (see Figure 2).⁸

As ς and $\varsigma^* \rightarrow 0$, the limits of payoff shocks ε and ε^* are always equal to 0. Therefore, given cutoff strategy equilibrium $(-\frac{D}{\rho}, -D)$, since $-\frac{D}{\rho} < 0$ and $-D > 0$, in this equilibrium, player i always chooses action 1 and player i^* always chooses action 0. Hence, the action strategy representation of this equilibrium is $(1, 0)$. In the same way, the cutoff strategy equilibrium $(-D, -\frac{D}{\rho})$ indicates the action strategy $(0, 1)$. Given cutoff strategy equilibrium $(0, 0)$, the equilibrium belief $\sigma(0, 0)$ is always equal to $\frac{D}{D-M}$ given any value of $\rho \in (-1, 0)$. Thus, $\sigma(0, 0)$ is always equal to the unconditional probability of player i^* choosing action 0. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of the game expressed in the form of action strategies are given by $(1, 0)$, $(0, 1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $M > 0 > D$. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of the perturbed games finally converge to the equilibria of the underlying complete information games.

For $M > 0 > D$ and $\rho < 0$, if $M < \rho D$ and $D < \rho M$,

⁷The contents in parentheses in this paragraph describe the intuition of the best response function for the strategic substitutes game as well.

⁸The intuition of the cutoff strategy equilibrium is that given $M > 0 > D$, a player can expect that the opponent either chooses being active or inactive. If a player expects the opponent to choose entry, the player will get payoff D if they also choose entry. Thus, the player will adopt a cutoff strategy $-D$. As the best response, the opponent will adopt a strategy $-\frac{D}{\rho}$.

In contrast, if a player expects the opponent to choose being inactive, then the player will get payoff M if they choose to enter. Thus, at least when $\varepsilon \geq -M$, the player will consider entry. However, $M > \rho D$ and hence $-M < -\rho D$, where $-\rho D$ is the entry threshold that opponent i^* expects player i to most likely adopt conditional on that i^* expects i will choose entry. Thus, if i gets a payoff shock ε such that $-M < \varepsilon < -\rho D$, the opponent expects that i will not choose entry but in fact i indeed chooses entry. Hence, a contradiction arises and i cannot adopt $-M$. Therefore, based on the opponent's belief that i will choose entry and accordingly i^* will adopt a strategy $-D$, i 's best response will be $-\frac{D}{\rho}$.

If a player expects the opponent is indifferent to being active or being inactive, it indicates that irrespective of what value ε^* is, the expected payoff of entry for i^* is equal to 0. Therefore, player i^* 's cutoff strategy is equal to 0. Hence, given $M > 0 > D$, as a best response, player i will adopt a strategy 0. Therefore, another cutoff strategy equilibrium as ς and $\varsigma^* \rightarrow 0$ is $(0, 0)$.

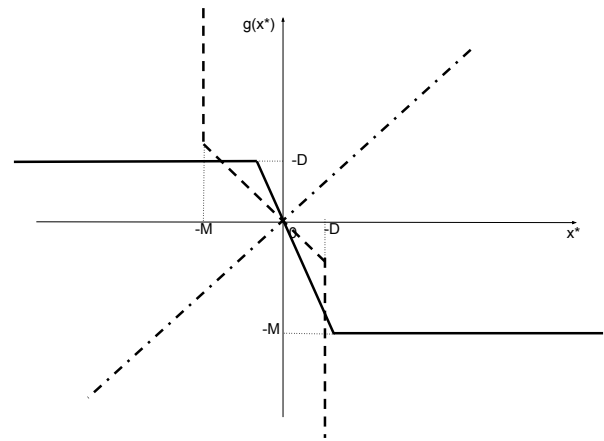


Figure 2. The solid curve represents $g(x^*)$ as ς and $\varsigma^* \rightarrow 0$. The dashed curve represents $g^*(x)$ as ς and $\varsigma^* \rightarrow 0$. The dashed-dot line represents the 45° line. The intersection points between $g(x^*)$ and $g^*(x)$ are the equilibria of the game with ς and $\varsigma^* \rightarrow 0$. For $M > 0 > D$ and $\rho < 0$, if $M > \rho D$ and $D > \rho M$, there are three equilibria $(-\frac{D}{\rho}, -D)$, $(-D, -\frac{D}{\rho})$ and $(0, 0)$.

there are three cutoff strategy equilibria (intersection points): $(-\frac{M}{\rho}, -M)$, $(-M, -\frac{M}{\rho})$ and $(0, 0)$ (see Figure 3).⁹ ¹⁰ As ς and $\varsigma^* \rightarrow 0$, the limits of payoff shocks ε and ε^* are always equal to 0. Therefore, given cutoff strategy equilibrium $(-\frac{M}{\rho}, -M)$, since $-\frac{M}{\rho} > 0$ and $-M < 0$, in this equilibrium player i always chooses action 0 and player i^* always chooses action 1. Hence, the action strategy representation of this equilibrium is $(0, 1)$. In the same way, the cutoff strategy equilibrium $(-M, -\frac{M}{\rho})$ indicates the action strategy $(1, 0)$. Given cutoff strategy equilibrium $(0, 0)$, the equilibrium belief $\sigma(0, 0)$ is equal to $\frac{D}{D-M}$ given any value of $\rho \in (-1, 0)$. Hence, it equals the unconditional probability of i^* choosing action 0. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of this

⁹For $M > 0 > D$ and $\rho < 0$, the following parameter specifications cannot be held: $M > \rho D$ and $D < \rho M$ or $M < \rho D$ and $D > \rho M$. It is because if $\rho = -1$, in either parameter specification, one inequality indicates $M + D > 0$, while the other one indicates $M + D < 0$. Obviously, the two inequalities cannot be held simultaneously.

¹⁰The intuitions of these cutoff strategy equilibria are similar to the previous case where $M > \rho D$ and $D > \rho M$. Given $M > 0 > D$, a player can expect that the opponent either chooses being active or inactive. If player i expects the opponent to choose being inactive, then the player will get payoff M if they choose entry. Thus, player i will adopt a cutoff strategy $-M$. As the best response, the opponent i^* will adopt a strategy $-\frac{M}{\rho}$.

Otherwise, if player i expects the opponent i^* to choose being active, the player will get payoff D if they choose to enter. Thus, at least $\varepsilon \geq -D$, i will consider entry. However, $D < \rho M$ and hence $-D > -\rho M$, where $-\rho M$ is the entry threshold that i^* expects i is most likely to adopt conditional on that i^* expects i will choose being inactive. Thus, if i gets a payoff shock ε such that $-D > \varepsilon > -\rho M$, the opponent will expect that i will choose being active but in fact i chooses being inactive. Hence, a contradiction arises and i cannot adopt $-D$. Therefore, based on the opponent's belief that i will choose being inactive and accordingly i^* will adopt a strategy $-M$, i 's best response will be $-\frac{M}{\rho}$.

The intuition of cutoff strategy $(0, 0)$ is the same as that in the previous case where $M > \rho D$ and $D > \rho M$.

game are given by $(1, 0)$, $(0, 1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $M > 0 > D$.

It should be noted that in the case of $M > 0 > D$, irrespective of whether $M > \rho D$ and $D > \rho M$, or $M < \rho D$ and $D < \rho M$, given M , D , and ς and $\varsigma^* \rightarrow 0$, as ρ changes, the best response function changes and the cutoff strategy equilibria, except $(0, 0)$, change as well. However, when we translate these cutoff strategies with respect to different values of ρ into action strategies, they indicate the same action strategies. For example, if $M > \rho D$ and $D > \rho M$, where $\rho < 0$, a cutoff strategy equilibrium is $(-\frac{D}{\rho}, -D)$. For different values of ρ , $(-\frac{D}{\rho}, -D)$ differs, but it always indicates the action strategy equilibrium $(1, 0)$.

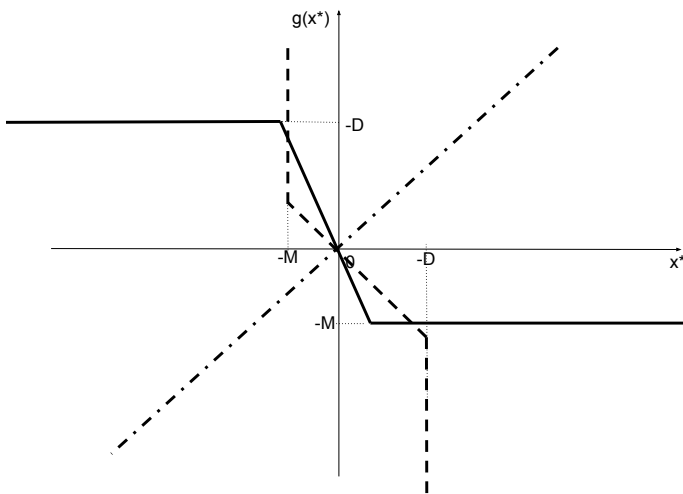
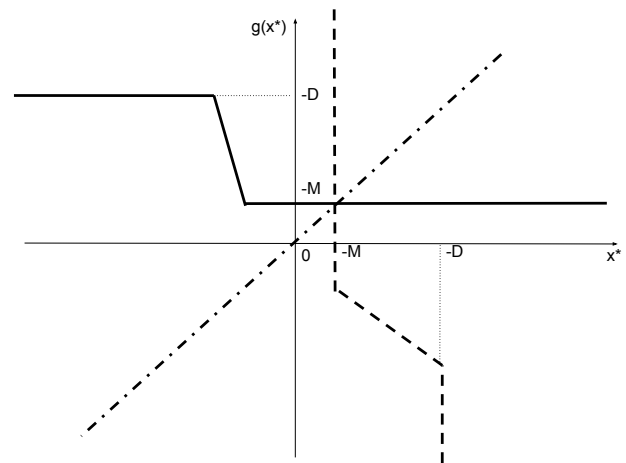
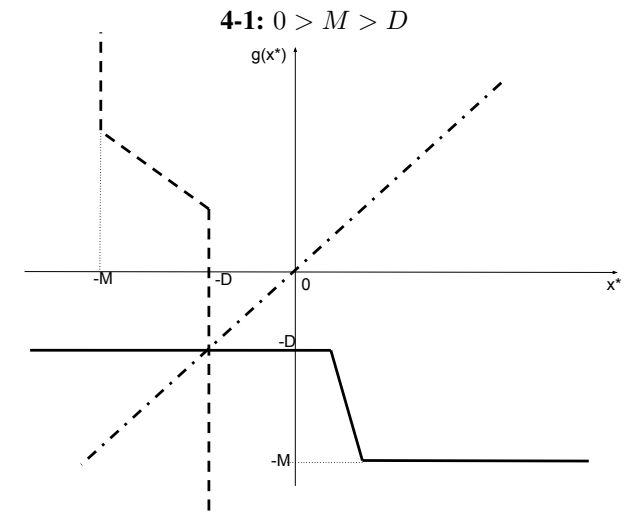


Figure 3. The solid curve represents $g(x^*)$ as ς and $\varsigma^* \rightarrow 0$. The dashed curve represents $g^*(x)$ as ς and $\varsigma^* \rightarrow 0$. The dashed-dot line represents the 45° line. The intersection points between $g(x^*)$ and $g^*(x)$ are the equilibria of the game with ς and $\varsigma^* \rightarrow 0$. For $M > 0 > D$ and $\rho < 0$, if $M < \rho D$ and $D < \rho M$, there are three equilibria $(-\frac{M}{\rho}, -M)$, $(-M, -\frac{M}{\rho})$ and $(0, 0)$.

Similarly, if $0 > M > D$ or $M > D > 0$, the equilibrium cutoff strategies are $(-M, -M)$ and $(-D, -D)$, respectively, which imply the action strategy equilibria $(0, 0)$ and $(1, 1)$ (see Figures 4-1 and 4-2).¹¹ These equilibria are exactly equal to

¹¹The intuitions of these cutoff strategy equilibria are as follows. Suppose $0 > M > D$. As ς and $\varsigma^* \rightarrow 0$, it is very likely that each player will choose being inactive. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -M$ since $M + \varepsilon > 0$ and M is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, the cutoff strategy equilibrium $(-M, -M)$ exists in this situation.

Similarly, suppose $M > D > 0$. As ς and $\varsigma^* \rightarrow 0$, it is very likely that each player will choose being active. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -D$ since $D + \varepsilon > 0$ and D is the payoff they can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, in this



4-1: $0 > M > D$
4-2: $M > D > 0$
Figure 4. The solid curve represents $g(x^*)$ as ς and $\varsigma^* \rightarrow 0$. The dashed curve represents $g^*(x)$ as ς and $\varsigma^* \rightarrow 0$. The dashed-dot line represents the 45° line. The intersection points between $g(x^*)$ and $g^*(x)$ are the equilibria of the game with ς and $\varsigma^* \rightarrow 0$. For $\rho < 0$, if $0 > M > D$, the equilibrium is $(-M, -M)$, and if $M > D > 0$, the equilibrium is $(-D, -D)$.

the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $0 > M > D$ or with $\varsigma = \varsigma^* = 0$ and $M > D > 0$. Therefore, as ς and $\varsigma^* \rightarrow 0$, the equilibria of the perturbed games finally converge to the equilibria of the underlying complete information games.

Therefore, Harsanyi's purification rationale can also be extended to perturbed games with non-positively dependent perturbation errors in a strategic substitutes context.

However, if $M > D$ and $\rho > 0$, the Bayesian games that can be solved by cutoff strategies do not exist for $\varsigma^2 \in (0, \tilde{\varsigma}^2)$. Therefore, the sequence of perturbed games that are supposed to converge to the complete information game does not exist. Since the 'approachability' requirement cannot be satisfied, situation, we have the cutoff strategy equilibrium $(-D, -D)$.

Harsanyi’s purification rationale cannot be applied in this situation.

In conclusion, for the perturbation errors that are either positively dependent in strategic complements games or negatively dependent in strategic substitutes games, as the perturbation errors degenerate to zero, the Bayesian games that are supposed to converge to the underlying complete information game exist. Supposing the perturbed games exist as variances of the prior distribution tend to 0, given the same primitives except the correlation coefficient, the best response function differs with different values of the correlation coefficient because the slope changes. Except the case of $M > 0 > D$, the value of cutoff strategy equilibria does not depend on the correlation coefficient. For the case of $M > 0 > D$, except the cutoff strategy equilibrium $(0, 0)$, all cutoff strategy equilibria differs with different values of correlation coefficient. However, in any situation, given different values of correlation coefficient, if we translate these cutoff strategy equilibria into action strategy equilibria, they represent the same action strategy equilibria given the same payoffs M and D . These action strategy equilibria are equal to the corresponding Nash equilibria of the complete information game.

Finally, we formally describe the extension of Harsanyi’s purification rationale to the normally distributed dependent perturbation-error situations in the following proposition:

Proposition 3: (An Extension of Purification Rationale):

In a 2×2 symmetric entry game, described in Table 2, all equilibria are the limit of the pure-strategy Bayesian Nash equilibria of a sequence of perturbed games described in Table 1 as $(\varsigma, \varsigma^*) \rightarrow 0$, if and only if $D > M$ and $\rho \geq 0$ or $M > D$ and $\rho \leq 0$. $(\varepsilon, \varepsilon^*)$ follows a joint normal distribution $N(0, 0, \varsigma^2, \varsigma^{*2}, \rho)$ and the perturbed games are solved by using cutoff strategies, as defined in Section 2. □

4 Conclusion

In this paper, we study a 2×2 entry game in which players’ private information are correlated. The game is symmetrically specified. Given other parameters, there exists a critical value of correlation coefficient below (above) which a cutoff strategy cannot be used to solve the game if the game exhibits strategic complements (strategic substitutes).

Provided that the game can be solved by cutoff strategy, we extend Harsanyi’s (1973) purification rationale to a dependent-perturbation error setting for both strategic complements and strategic substitutes games. In our game, the uncertainty of perturbed games vanishes as the variances of perturbation-error distribution degenerate to zero. By assuming that the perturbed games are solved by cutoff strategies and the perturbation errors follow the joint normal distribution as given in this paper, the purification rationale can be extended

to perturbed games with positively dependent perturbation errors if the complete information game exhibits strategic complements or negatively dependent perturbation errors if the complete information game exhibits strategic substitutes. If we assume that the perturbation errors are negatively dependent if the complete information game exhibits strategic complements or positively dependent if the complete information game exhibits strategic substitutes, then the ‘approachability’ part of the purification rationale cannot be satisfied, and hence, we cannot extend the purification rationale to such situations. Finally, for future research, we will study whether and how the purification with dependent randomization rationale can be applied to more general games.

5 Appendix

A Preliminaries and Glossaries of Notations

The standard Gaussian density function is denoted by $\phi(\cdot)$, and the standard Gaussian cumulative density function is denoted by $\Phi(\cdot)$. Given a Gaussian distribution $x \sim N(\mu, \varsigma^2)$, the density function is written as

$$f(x) = \frac{1}{\sqrt{2\pi\varsigma}} \exp\left(-\frac{(x - \mu)^2}{2\varsigma^2}\right) = \frac{1}{\varsigma} \phi\left(\frac{x - \mu}{\varsigma}\right)$$

The joint Gaussian distribution is denoted by $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma^2, \varsigma^{*2}, \rho)$. The density function of the bivariate Gaussian distribution is

$$f(\varepsilon, \varepsilon^*) = \frac{1}{2\pi\varsigma\varsigma^*\sqrt{1 - \rho^2}} \exp\left(-\frac{1}{2(1 - \rho^2)}\left(\frac{\varepsilon^2}{\varsigma^2} + \frac{\varepsilon^{*2}}{\varsigma^{*2}} - \frac{2\rho\varepsilon\varepsilon^*}{\varsigma\varsigma^*}\right)\right)$$

The conditional density function is

$$f(\varepsilon^*|\varepsilon) = \frac{1}{\varsigma^*\sqrt{1 - \rho^2}} \phi\left(\frac{\frac{\varepsilon^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1 - \rho^2}}\right)$$

and the conditional cumulative density function is

$$\begin{aligned} F(\bar{\varepsilon}^*|\varepsilon) &= \int_{-\infty}^{\bar{\varepsilon}^*} f(\varepsilon^*|\varepsilon) d\varepsilon^* \\ &= \int_{-\infty}^{\bar{\varepsilon}^*} \frac{1}{\varsigma^*\sqrt{2\pi(1 - \rho^2)}} \exp\left(-\frac{1}{2}\left(\frac{\frac{\varepsilon^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1 - \rho^2}}\right)^2\right) d\varepsilon^* \\ &= \int_{-\infty}^{\frac{\bar{\varepsilon}^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du \\ &= \Phi\left(\frac{\frac{\bar{\varepsilon}^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1 - \rho^2}}\right) \end{aligned}$$

We denote a player's belief function by $\sigma(x^*, \varepsilon) = F(x^*|\varepsilon)$, where ε is a player's own private information, and x^* is the opponent's expected cutoff strategy. We obtain the following results of $\sigma(x^*, \varepsilon)$:

$$\begin{aligned}\sigma(x^*, \varepsilon) &= F(x^*|\varepsilon) = \Phi\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right) \\ \sigma_{x^*}(x^*, \varepsilon) &= \frac{1}{\varsigma^* \sqrt{1-\rho^2}} \phi\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right) \\ \sigma_\varepsilon(x^*, \varepsilon) &= -\frac{\rho}{\varsigma \sqrt{1-\rho^2}} \phi\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

By assuming $\varsigma = \varsigma^*$, these expressions can be simplified into the following equations, respectively:

$$\begin{aligned}\sigma(x^*, \varepsilon) &= \Phi\left(\frac{x^* - \rho\varepsilon}{\varsigma \sqrt{1-\rho^2}}\right) \\ \sigma_{x^*}(x^*, \varepsilon) &= \frac{1}{\varsigma \sqrt{1-\rho^2}} \phi\left(\frac{x^* - \rho\varepsilon}{\varsigma \sqrt{1-\rho^2}}\right) \\ \sigma_\varepsilon(x^*, \varepsilon) &= -\frac{\rho}{\varsigma \sqrt{1-\rho^2}} \phi\left(\frac{x^* - \rho\varepsilon}{\varsigma \sqrt{1-\rho^2}}\right)\end{aligned}$$

The expected payoff function $\mathbb{E}\Pi(x^*, \varepsilon)$ is expressed as

$$\begin{aligned}\mathbb{E}\Pi(x^*, \varepsilon) &= \sigma(x^*, \varepsilon)(M + \varepsilon) + (1 - \sigma(x^*, \varepsilon))(D + \varepsilon) \\ &= \sigma(x^*, \varepsilon)(M - D) + D + \varepsilon \\ &= \Phi\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)(M - D) + D + \varepsilon \\ &= \Phi\left(\frac{x^* - \rho\varepsilon}{\varsigma \sqrt{1-\rho^2}}\right)(M - D) + D + \varepsilon\end{aligned}$$

B Restrictions for Implementing the Cutoff Strategy Concept to Solve the Game (Proofs of Proposition 1)

Lemma B1: There exists a $\tilde{\rho} \in (-1, 1)$, if $D > M$, $\forall \rho \in [\tilde{\rho}, 1)$ and $\forall x^* \in \mathbb{R}$, or if $M > D$, $\forall \rho \in (-1, \tilde{\rho}]$ and $\forall x^* \in \mathbb{R}$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$, where the equality is obtained at $\varepsilon = \frac{\varsigma}{\varsigma^*} \frac{x^*}{\rho}$ with $\rho = \tilde{\rho}$.

Proof: For all $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*, \varepsilon) = \sigma(x^*, \varepsilon)(M - D) + D + \varepsilon = \Phi\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)(M - D) + D + \varepsilon$. Therefore, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*, \varepsilon)(M - D) + 1 = \frac{\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2}\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)^2\right) + 1$. Hence, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$

is equivalent to $\frac{\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2}\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)^2\right) + 1 \geq 0$.

Therefore, the inequality $\exp\left(\frac{1}{2}\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)^2\right) \geq \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$

is the necessary and sufficient condition of $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$. Apparently, that $\rho(D - M) \geq 0$ is sufficient to make the necessary and sufficient condition hold. Therefore, that $D > M$ and $\rho \geq 0$, or $M > D$ and $\rho \leq 0$, is sufficient to guarantee $\exp\left(\frac{1}{2}\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)^2\right) > \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$, and thus $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} > 0$.

Suppose $\rho(D - M) < 0$. Then, the necessary and sufficient condition $\exp\left(\frac{1}{2}\left(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}\right)^2\right) \geq \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$ can be equivalently transformed into $\left(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}\right)^2 \geq 2(1-\rho^2) \ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$.

Therefore, under the condition $\rho(D - M) < 0$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ always holds if and only if for all $x^* \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, $\left(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}\right)^2 \geq 2(1-\rho^2) \ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$ always holds. Hence,

as long as all parameters satisfy $2(1-\rho^2) \ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} \leq 0$,

the necessary and sufficient condition always holds, and thus $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ given $\rho(D - M) < 0$. Since $\ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} = 0$

as long as $\frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} = 1$, given D, M, ς and ς^* , and

denoting the solution by $\tilde{\rho}$, we have $\tilde{\rho}^2 = \frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (M-D)^2}$. Furthermore, as long as $\rho^2 < \tilde{\rho}^2$, $2(1-\rho^2) \ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}} < 0$.

Therefore, if $D > M$ and $\tilde{\rho} \leq \rho < 0$, or if $M > D$ and $0 < \rho \leq \tilde{\rho}$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ always holds, and $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*} \frac{x^*}{\tilde{\rho}}$. Therefore, combined with the results

for $\rho(D - M) \geq 0$, it can be concluded that if $D > M$, $\forall \rho \in [\tilde{\rho}, 1)$, or if $M > D$, $\forall \rho \in (-1, \tilde{\rho}]$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ always holds, and $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*} \frac{x^*}{\tilde{\rho}}$, where

$\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $D > M$, and $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $M > D$.

Finally, the game is symmetric and hence $\varsigma = \varsigma^*$. Therefore, $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}} = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2} + (D-M)^2}} = \tilde{\rho}^*$ for $D > M$, and $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2} + (D-M)^2}} = \tilde{\rho}^*$ for $M > D$. Hence, both players have an identical range to ensure that their respective expected payoff function $\mathbb{E}\Pi(x^*, \varepsilon)$ always increases with respect to $\varepsilon \in \mathbb{R}$. ■

Proof of Proposition 1: The proof of Proposition 1 is based on the proof of Lemma B1. We denote the set of ρ that makes $\mathbb{E}\Pi(x^*, \varepsilon)$ always increase with respect to ε given x^* by $\Gamma \equiv \{\rho \mid \rho \geq \tilde{\rho} \text{ if } D > M \text{ or } \rho \leq \tilde{\rho} \text{ if } M > D\}$, where $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $D > M$ and $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (D-M)^2}}$ for $M > D$ as defined in Lemma B1. From Lemma B1, it has been known that given D and M , the necessary and sufficient condition for $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$ is $\rho \in \Gamma$. Therefore, it is certain that as long as ρ does not belong to Γ , $\mathbb{E}\Pi(x^*, \varepsilon)$ is not monotonic with respect to ε given any $x^* \in \mathbb{R}$. Equivalently, it means that for some ε , $\left(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}\right)^2 < 2(1-\rho^2) \ln \frac{-\rho(D-M)}{\varsigma \sqrt{2\pi(1-\rho^2)}}$

for $\rho \notin \Gamma$. Without loss of generality, Figure B1 geometrically gives a general description of the relation between $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ and $z(\varepsilon) = 2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ given x, M, D, ρ, ς and ς^* for all $\rho \notin \Gamma$ (see Figure B1).

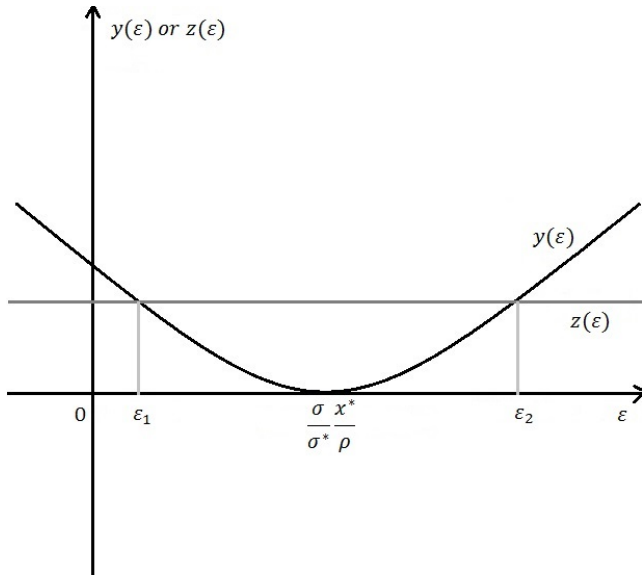


Figure B1: A geometric description of the relation between function $y(\varepsilon)$ and $z(\varepsilon)$ as long as $\rho \notin \Gamma$, where $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ and $z(\varepsilon) = 2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. There must be two intersection points which make $f(\varepsilon) = g(\varepsilon)$, and in this figure, they are denoted by ε_1 and ε_2 . The function $y(\varepsilon)$ reaches its global minimum 0 at $\varepsilon = \frac{\varsigma}{\varsigma^*} \frac{x^*}{\rho}$.

According to the quadratic structure of $y(\varepsilon)$, as long as $\rho \notin \Gamma$, there should be two solutions to solve the equation $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 = 2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. They are $\varepsilon_1 = \frac{\varsigma}{\rho\varsigma^*}x - \frac{x^*}{\rho} \sqrt{2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ and $\varepsilon_2 = \frac{\varsigma}{\rho\varsigma^*}x + \frac{x^*}{\rho} \sqrt{2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. Therefore, for $\varepsilon \leq \varepsilon_1$ or $\varepsilon \geq \varepsilon_2$, $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1 - \rho^2) \ln \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$, then $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} \geq 0$, where the equality is taken when $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$. For $\varepsilon_1 < \varepsilon < \varepsilon_2$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} < 0$. Based on these results, without loss of generality, Figure B2 geometrically gives a general description of function $\mathbb{E}\Pi(x^*, \varepsilon)$ with respect to ε given any value of $x^* \in \mathbb{R}$, for all $\rho \notin \Gamma$ (see Figure B2).

According to the Proof in Appendix C, for $\rho \notin \Gamma$, $g'(x^*) > 0$ is not held for all $x^* \in \mathbb{R}$ for $D > M$, and $g'(x^*) < 0$ is not held for all $x^* \in \mathbb{R}$ for $M > D$. Such features contradict the definition of strategic complements games and strategic substitutes games. Therefore, cutoff strategy cannot be used to solve the game for all $\rho \notin \Gamma$.¹²

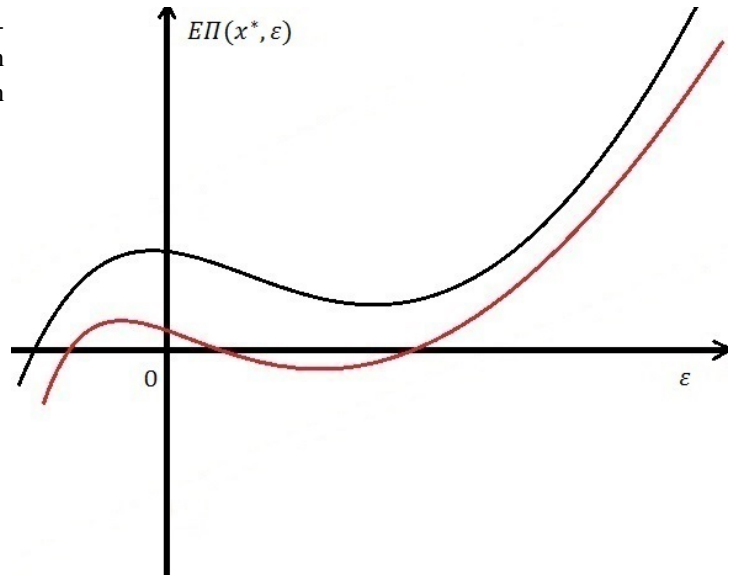


Figure B2: A general description of expected payoff function $\mathbb{E}\Pi(x^*, \varepsilon)$ with respect to ε given any value of x^* , for all $\rho \notin \Gamma$. The position of $\mathbb{E}\Pi(x^*, \varepsilon)$ depends on x^* , and $\mathbb{E}\Pi(x^*, \varepsilon)$ is always located within $[M + \varepsilon, D + \varepsilon]$ for all $x^* \in \mathbb{R}$ if $D > M$, or within $[D + \varepsilon, M + \varepsilon]$ for all $x^* \in \mathbb{R}$ if $M > D$. If $D > M$ ($M > D$), increasing x^* will bring $\mathbb{E}\Pi(x^*, \varepsilon)$ downward (upward). As long as the cutoff strategy concept is used to solve the game, for all $\rho \notin \Gamma$, given $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*, \varepsilon)$ behaves non-monotonically and it is possible that for certain value of x^* , $\mathbb{E}\Pi(x^*, \varepsilon) = 0$ has three intersections with the x -axis, which is demonstrated by the red curve. Expected payoff functions with such shapes violate the definition of strategic complements games or the definition of strategic substitutes games, and therefore for all $\rho \notin \Gamma$, cutoff strategy cannot be used to solve either the strategic complements game or the strategic substitutes game.

Besides, because for all $x^* \in \mathbb{R}$, given all primitives, the expected payoff function $\mathbb{E}\Pi(x^*, \varepsilon)$ is always located between the line $D + \varepsilon$ and $M + \varepsilon$, and if $D > M$ ($M > D$), increasing x^* will bring $\mathbb{E}\Pi(x^*, \varepsilon)$ downward (upward), and for $\rho \notin \Gamma$, this property implies that it is possible that for some value of x^* , there are two or three solutions of ε satisfying $\mathbb{E}\Pi(x^*, \varepsilon) = 0$ (see Figure B2 for example). Therefore, the set Γ not only indicate that $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$ but also characterizes the set of cutoff strategy Bayesian Nash equilibria of the symmetric strategic complements games and the symmetric strategic substitutes games. Therefore, Proposition 1 is obtained. ■

¹²If $g'(x^*) < 0$ for a strategic complements game, it indicates given the other player's strategy, a player chooses a strategy to offset the other player's payoff, which contradicts the definition of strategic complements games. If

$g'(x^*) > 0$ for a strategic substitutes game, it indicates given the other player's strategy, a player chooses a strategy to reinforce the other player's payoff, which contradicts the definition of strategic substitutes games.

C Derivation of the (Inverse) Best Response Function and Monotonicity of the Best Response Function

The best response function, $g(x^*)$, is defined to satisfy $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. Therefore, it is obtained that $\sigma(x^*, g(x^*))(M - D) + D + g(x^*) = 0$, and further $\Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}})(M - D) + D + g(x^*) = 0$. This equation can

be equivalently transformed into $\frac{D+g(x^*)}{D-M} = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}})$. Since the cumulative density function of normal distribution is invertible, we obtain $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}$. Finally, we obtain the inverse best response function $x^* = \rho \frac{\varsigma}{\varsigma} g(x^*) + \varsigma \sqrt{1 - \rho^2} \Phi^{-1}(\frac{D+g(x^*)}{D-M})$.

Still, for the definition equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, or $\Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}})(M - D) + D + g(x^*) = 0$, we differentiate this equation with respect to x^* on both sides, and obtain $\mathbb{E}\Pi'_{x^*}(x^*, g(x^*)) + \mathbb{E}\Pi'_\varepsilon(x^*, g(x^*))g'(x^*) = 0$. Therefore, $g'(x^*) = -\frac{\mathbb{E}\Pi'_{x^*}(x^*, g(x^*))}{\mathbb{E}\Pi'_\varepsilon(x^*, g(x^*))} = -\frac{\frac{\partial \mathbb{E}\Pi(x, \varepsilon)}{\partial x^*} |_{\varepsilon=g(x^*)}}{\frac{\partial \mathbb{E}\Pi(x, \varepsilon)}{\partial \varepsilon} |_{\varepsilon=g(x^*)}} = -\frac{\sigma_{x^*}(x^*, g(x^*))(M-D)}{\sigma_\varepsilon(x^*, g(x^*))(M-D)+1} = \frac{1}{\frac{\varsigma^* \sqrt{2\pi(1-\rho^2)} \exp(\frac{1}{2}(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1-\rho^2}})^2)}{M-D}} \cdot \sigma_{x^*}(x^*, g(x^*)) > 0$,

and it is known that if and only if $\rho \in \Gamma$, $\frac{\partial \mathbb{E}\Pi(x^*, g(x^*))}{\partial \varepsilon} \geq 0$ $\forall x^* \in \mathbb{R}$; hence, if $D > M$, $g'(x^*) > 0$ and if $M > D$, $g'(x^*) < 0$. Therefore, if and only if the concept of cutoff strategy Bayesian Nash equilibria can be applied to solve the game, i.e. $\rho \in \Gamma$, $g(x^*)$ globally increases for a strategic complements game, and globally decreases for a strategic substitutes game. ■

D Derivation of the Best Response Function for ς and $\varsigma^* \rightarrow 0$

Assume $\varsigma = \varsigma^*$. Suppose $D > M$ and $\rho > 0$. As shown in Section 3, as ς and $\varsigma^* \rightarrow 0$,

$$g(x^*) = \frac{1}{\rho} x^*$$

where $x^* \in [-\rho D, -\rho M]$. Let us recall the definition equation of the cutoff best response $g(x^*)$: $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}})(M - D) + D + g(x^*) = 0$. If $x^* > \rho g(x^*)$, $\lim_{\varsigma \rightarrow 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}) = \Phi(+\infty) = 1$ and hence $g(x^*) = -M$. Therefore, if $x^* > -\rho M$, $g(x^*) = -M$.

If $x^* < \rho g(x^*)$, $\lim_{\varsigma \rightarrow 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}) = \Phi(-\infty) = 0$ and hence $g(x^*) = -D$. Therefore, if $x^* < -\rho D$, $g(x^*) = -D$.

Therefore, if $D > M$ and $\rho > 0$,

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho} x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M \end{cases}$$

Suppose $M > D$ and $\rho < 0$. As shown in Section 3, as ς and $\varsigma^* \rightarrow 0$,

$$g(x^*) = \frac{1}{\rho} x^*$$

where $x^* \in [-\rho D, -\rho M]$. According to the equation $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}})(M - D) + D + g(x^*) = 0$, if $x^* > \rho g(x^*)$, $\lim_{\varsigma \rightarrow 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}) = \Phi(+\infty) = 1$ and hence $g(x^*) = -M$. Hence, if $x^* > -\rho M$, $g(x^*) = -M$.

If $x^* < \rho g(x^*)$, $\lim_{\varsigma \rightarrow 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma \sqrt{1 - \rho^2}}) = \Phi(-\infty) = 0$ and hence $g(x^*) = -D$. Therefore, if $x^* < -\rho D$, $g(x^*) = -D$.

Therefore, if $M > D$ and $\rho < 0$,

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho} x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M. \end{cases}$$

Acknowledgements

The content of this paper was presented at the 5th World Congress of the Game Theory Society, 2016, and the Asian Meeting of Econometric Society, 2016. It is organized based on section 7 of chapter 3 of my PhD thesis. I thank Tim Worrall, Andy Snell, József Sákovics, Jonathan Thomas and Subir Bose for their helpful comments and suggestions to the content of this paper. I also appreciate the efforts dedicated by Tim Worrall, Andy Snell, József Sákovics and Jonathan Thomas to my study at University of Edinburgh.

REFERENCES

- [1] Carlsson H, van Damme E (1993) Global Games and Equilibrium Selection. *Econometrica*. 61(5): 989-1018.
- [2] Fudenberg D, Tirole J (1991) *Game Theory*. (MIT Press, Cambridge, MA).
- [3] Harsanyi J (1973) Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points. *International Journal of Game Theory*. 2: 1-23.

- [4] Hellwig C, Veldkamp L (2009) Knowing What Others Know: Coordination Motives in Information Acquisition. *The Review of Economic Studies*. 76(1): 223-251.
- [5] Morris S, Shin H S (2005) Heterogeneity and Uniqueness in Interaction Games, in *The Economy as an Evolving Complex System, III*, edited by L E Blume and S N Durlauf, in Santa Fe Institute Studies on the Sciences of Complexity. (Oxford University Press).
- [6] Morris S (2008) Purification, in *The New Palgrave Dictionary of Economics* (Second Edition), edited by S N Durlauf and L E Blume. (Palgrave Macmillan, London).
- [7] Myatt D P, Wallace C (2012) Endogenous Information Acquisition in Coordination Games. *The Review of Economic Studies*. 79(1): 340-374.
- [8] Wang R (2016) Strategic Choices in Realistic Settings. PhD Thesis. University of Edinburgh.