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Weakly Special Classes of Modules

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Abstract In the development of Theory Radical of Rings, there are two kinds of radical constructions. The first radical construction is the lower radical construction and the second one is the upper radical construction. In fact, the class π of all prime rings forms a special class and the upper radical class $U(\pi)$ of π forms a radical class which is called the prime radical. An upper radical class which is generated by a special class of rings is called a special radical class. On the other hand, we also have the class ρ of all semiprime rings which is weakly special class of rings. Moreover, we can construct a special class of modules by using a given special class of rings. This condition motivates the existence of the question how to construct weakly special class modules by using a given weakly special class of rings. This research is a qualitative research. The results of this research are derived from fundamental axioms and properties of radical class of rings especially on special and weakly special radical classes. In this paper, we introduce the notion of a weakly special class of modules, a generalization of the notion on a special class of modules based on the definition of semiprime modules. Furthermore, some properties and examples of weakly special classes of modules are given. The main results of this work are the definition of a weakly special class of modules and their properties.

Keywords Prime Module, Semiprime Module, Special Class of Rings, Special Class of Modules, Weakly Special Class of Rings, Weakly Special Class of Modules

1. Introduction

In this paper, we consider only associative rings, but do not require them to be commutative or to have an identity. Let A be a ring. An ideal P of A is called a prime ideal of A if for any two ideals I and J of A, $IJ \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$. Moreover, an ideal K of A is called a semiprime ideal of A if for any I ideal of $I^2 \subseteq K$ implies $I \subseteq K$. The ring A is a semiprime ring if the zero ideal 0 is a semiprime ideal. An ideal L of A such that $L \subset A$ is called a maximal ideal of A if A does not contain any proper ideal I of A satisfying $L \subset I \subset A$. A nonzero ideal I of A is called a minimal ideal of A if $I \neq 0$ and A does not contain any ideal J of A such that $0 \neq J \subset I$. A nonzero ideal K of A is called an essential ideal if $K \cap I \neq 0$ for every nonzero ideal I of A and this is denoted by $K \triangleleft \circ A$.

A ring A is called a prime essential ring if A is a semiprime ring and every prime ideal of A is an essential ideal [4]. The important consequences of the existence of prime essential rings were given in [3, 4]. An $A-$ module M is said to be a prime module if there exists elements $n \in M$ and $a \in A$ such that $an \neq 0$ and if $m \in M$ and $J \triangleleft A$ are such that $Jm = 0$, then $m = 0$ or $JM = 0$. Furthermore, an $A-$ module M is said to be a faithful module if $(0 : M)_A = \{a \in A \mid aM = 0\} = 0$. Moreover, an $A-$ module M is called a simple module if there exist elements $r \in A, m \in M$ satisfying $rm \neq 0$ and the module M has only the trivial submodules, 0 and M itself [5]. A submodule N of M is said to be a semiprime submodule of M if for any $L \triangleleft A$ and every submodule P of M, $L^2 P \subseteq N$ implies $LP \subseteq M$. Furthermore, the $A-$ module M is called a semiprime module over a ring A if 0 is a semiprime submodule of M [1]. In fact, every prime module is semiprime. However, the converse is not generally true. We shall follow the Amitsur-Kurosh definition of a radical class. Let γ be any collection of rings. The class γ of rings forms a radical class if γ has the following three properties [5]:

1. The class γ is closed under homomorphism, that is, if $A \in \gamma$, then $A/I \in \gamma$ for every $I \triangleleft A$. In other words, the class γ contains all images of every ring in γ under any ring homomorphisms,

2. Let A be any ring. If we define $\gamma(A)=\sum\{I \triangleleft A \mid I \in \gamma\}$, then $\gamma(A) \in \gamma$,
3. For any ring A, the ring factor $A/\gamma(A)$ has no nonzero ideal in γ.

Furthermore, a radical class γ is hereditary if γ contains all ideals I of all the rings in γ. A hereditary radical class γ containing the class $N_0=\{R \mid R^n=0$ for some $n \in Z^+\}$ of all nilpotent rings is called a supernilpotent radical. On the other hand, let μ be a class of rings consisting of prime rings (respectively, semiprime rings). The class μ is called a special (respectively, weakly special) class if μ is hereditary and if R contains an essential ideal $I \in \mu$, then $R \in \mu$. The definition of a weakly special class is the main contribution of this work. In 1993, [3] showed that the class of all prime essential rings can be applied to determine whether a radical is special. Some properties of special classes of rings were also described in [9]. Let μ be a special class of rings. A ring A is called a $*_\mu-$ ring if $A \in \mu$ and A does not contain a nonzero proper ideal which belongs to μ. The class of all $*_\mu-$ rings will be denoted by $*_\mu$. Special classes μ which generate radical classes coinciding with the radical generated by $*_\mu$ were given in [9]. Moreover, a ring A is called a subdirectly irreducible ring if $\bigcap\{I \neq 0 \mid I \triangleleft A\} \neq 0$. In other words, the intersection of all nonzero ideals of A is nonzero, otherwise, if this intersection is zero, the ring A is called a subdirectly reducible ring. It follows from Theorem 2 in [8] that every prime essential ring is a subdirectly reducible ring. Some properties of generalizations of prime essential rings were also given in [8]. Furthermore, as in [5], for any A, let Σ_A denote the class of modules M defined over the ring A satisfying $AM \neq 0$, and let $\Sigma=\bigcup\Sigma_A$.

Now, let $\ker(\Sigma_A)=\bigcap\{(0:M)_A \mid M \in \Sigma_A\}$ and we consider the class Σ might satisfy the following conditions:

(M1) If I is an ideal of A and M belongs to $\Sigma_{A/I}$, then M belongs to Σ_A.

(M2) Let A be a ring and let I be an ideal of A such that $(0:M)_A$ contains I, then M belongs to Σ_A if and only if M belongs to $\Sigma_{A/I}$.

Moreover, it follows from [7] that for every ring A, the class Σ forms a special class of modules if Σ has the property (M1) and property (M2), and satisfies the following conditions:

(SM3) If $M \in \Sigma_A, B \triangleleft A$ and there exists an element $r \in B$ and $m \in M$ such that $rm \neq 0$, then $M \in \Sigma_B$,

(SM4) Let A be a ring and let B be any ideal of A. If $M \in \Sigma_B$, then $BM \in \Sigma_A$.

In the general case, a class ρ of modules which consist of prime modules is said to be special if the conditions (M1), (M2), (SM3), and (SM4) hold for ρ. On the other hand, a prime ring A is said to be a $*-$ ring if A is a prime ring and there is no nonzero proper ideal I such that A/I is prime. The definition of $*-$ ring was introduced by Korolczuk (France-Jackson) in her paper [6]. A significant contribution of the existence of $*-$ rings to the development of the Radical Theory of Rings can be found in Theorem 1 in [6]. Let μ be any special class of rings. Then the class $U(\mu)=\{A \mid$ there is no proper ideal I of A satisfying A/I belongs to $\mu\}$ of rings forms a radical class of rings and the upper radical class $U(\mu)$ is called a special radical class. The upper radical class $U(\pi)$ of the class π of all prime rings will be called the prime radical class and it is denoted by β. It follows from [6] that for a nonzero $*-$ rings A, the smallest special radical $\hat{L}_A$ containing A forms a special atom, that is, the smallest special radical which properly contains the prime radical β. A hereditary radical class γ is called a supernilpotent radical if γ contains the prime radical β. Therefore, every special radical is supernilpotent. A supernilpotent radical γ is called a supernilpotent atom if γ is the smallest supernilpotent radical which properly contains the prime radical β. In fact, every nonzero $*-$ ring generates a supernilpotent atom since the smallest supernilpotent radical $\bar{L}_A$ containing a nonzero $*-$ ring A is a supernilpotent atom [2]. Moreover, [11] showed that there exists a prime essential rings which generates a special atom.

Let R and S be rings, and let $V={}_R V_S$ and $W={}_S W_R$ be an $R-S-$ bimodule and an $S-R-$ bimodule, respectively. The $4-$tuple (R,V,W,S) is said to be a Morita context if the set $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ of all 2×2 matrices in which the entries satisfy $a_{11} \in R, a_{12} \in V, a_{21} \in W$, and $a_{22} \in S$ forms a ring under matrix addition and matrix multiplication. This definition can be considered if the maps $V \times W \to R$ and $W \times V \to S$ exist [5]. Let γ be any radical class of rings. The radical class γ of rings is said to be a normal radical if $V\gamma(S)W$ is contained in the largest ideal $\gamma(R)$ of R contained in γ for every $4-$tuples (R,V,W,S) which is a Morita context. We already know about the normal class of modules introduced in [7] as the generalization of a normal class of rings. We shall follow [7] to learn the concept of the definition of a normal class of modules. Let the class $\rho(A)$ be the class of prime $A-$ modules and $\rho=\bigcup\rho(A)$, the union extending over all rings A. The class ρ is called a normal class of modules if ρ satisfies condition M2 and for every $4-$tuple (R,V,W,S) which is a ring of Morita context and the context module $E=\begin{pmatrix} M \\ N \end{pmatrix}$ is such that if $n\in N$ satisfies $Vn=0$, then $n=0$ and $N=WM$ and $SN\neq 0, M\in\rho(R)$ implies $N\in\rho(R)$.

Moreover, let ρ be any special classes of modules. It follows from [7] that the class $\mu=\{A\mid A$ has a faithful module in $\rho(A)\}$ of rings, where $\rho(A)$ is the class of prime $A-$ modules, forms a special class of rings. Conversely, for every special class μ, the class $\rho=\bigcup\rho(A)$ of all prime $A-$ modules, where $\rho(A)=\{M\mid M$ is a prime $A-$ module and the factor ring $A/(0:M)_A$ belongs to $\mu\}$ forms a special class of modules [7].

The primeness and the semiprimeness of a ring (respectively, a module) are important in the development of modern algebra especially in the development of the Radical Theory of Rings and Modules. In the ring theory, we have the class of all prime rings which is the largest special class of rings and the class of all semiprime rings which is the largest weakly special class of rings. Some subsets of these classes generate special classes of rings and weakly special classes of rings, respectively. The class of all prime essential rings is contained in the class of all semiprime rings and it can be used as a tool in determining the speciality of any radical class of rings. Hence, the existence of prime essential rings is very important. We will give an alternative definition of prime essential rings in view of the module theory so we can work on the module theory.

In fact, some properties of semiprime modules are also valid for prime modules. Furthermore, some properties of the class of all prime modules extending over any rings have been found in [7], the class of all prime modules forms a special class of modules. In general, it is very important to know how to construct weakly special classes of modules and investigate their properties so that we can clarify some properties which are valid for special classes of modules and which also hold for weakly special class of modules.

2. Results and Discussion

We already have a necessary and sufficient condition for any ring A to have a prime module over itself. This property can be found in [5]. Furthermore, in the general case, we show that this property is also valid for the existence of semiprime modules. A necessary and sufficient condition for a ring A to have a semiprime module over itself is given below.

Theorem 2.2. Given a ring A and let $I\lhd A$. Then there exists a semiprime $A-$module M such that the annihilator $(0:M)_A=I \Leftrightarrow I$ is a semiprime ideal of A.

Proof. Let A be a ring such that there is an ideal $I\lhd A$, and let M be a semiprime module over the ring A such that $(0:M)_A=I$. Suppose $J\lhd A$ satisfies $J^2\subseteq I$. Since $(0:M)_A=I$, $J^2P=0$ for every submodule P of M. By the semiprimeness of M, $JP=0$ for every submodule P of M. Therefore, $J\subseteq I$. This means that I is a semiprime ideal of the ring A. Conversely, let I be a semiprime ideal of A such that $(0:M)_A=I$. Let $J\lhd A$ such that $J^2P=0$ for every submodule P of M. Therefore, $J^2\subseteq I$. Since I is a semiprime ideal of $A, J\subseteq I=(0:M)_A$. Hence $JP=0$. In other words, M is a semiprime $A-$module.

As consequences of Theorem 2.2, we therefore have the following results.

Theorem 2.3. Let R be any ring. The following conditions are equivalent

i. The ring R is a semiprime ring
ii. The ring R has a faithful semiprime module.

Proof.
(i$\Rightarrow$ ii)
Assume R is a semiprime ring. Clearly, the singleton $\{0\}$ of R is a semiprime ideal of R. It follows from Theorem 2.2 that there is an $R-$ module M such that M is a semiprime module with $(0:M)_R=0$. This means that M is a faithful semiprime $R-$module.
(ii$\Rightarrow$ i)
Conversely, suppose R is a ring such that there exists a semiprime $R-$ module M such that $(0:M)_R=0$. It follows from Theorem 2.2 that 0 is a semiprime ideal of R. Therefore, R is a semiprime ring.

Theorem 2.4. Let A be a ring. The following statements are equivalent

i. The ring A is prime essential ring
ii. The ring A has a nonzero faithful semiprime module and for every nonzero ideal I of A, I has no faithful prime module.

Proof. Obvious.

The existence of special classes of rings motivated the study of special classes of modules. In this notion, we would like to introduce the weakly special class of modules.

Definition 2.5. Let $\sigma(A)$ be a class consistings of semiprime $A-$ modules and $\sigma=\bigcup\sigma(A)$, the union extending over all rings A. Then σ is called a weakly special class of modules if σ satisfies the conditions (M1), (M2), (SM3), and (SM4).

The followings theorems show that weakly special classes of modules exist.

Theorem 2.6. Given any ring A, define $\sum_A$ to be the class of all semiprime $A-$ modules. Then the class $\sum=\bigcup\sum_A$ is a weakly special class of modules.

Proof. Let $I=(0:M)_A=0$ and let M be a semiprime A/I – module. We would like to show that M is a semiprime A – module. Let $J \lhd A$ and let N be any submodule of M such that $J^2N = 0$. Since $J \lhd A, (J+I)/I \lhd A/I$ and $J^2N = 0$, the following property holds $((J+I)/I)^2 N = 0$. Moreover, by the semiprimeness of M over $A/I, ((J+I)/I)N = 0$. This implies $JN = 0$. This gives $(J/I)N = 0$. Therefore, M is a semiprime A – module.

Let M be a semiprime A – module and let $I \lhd A$ such that $I \subseteq (0:M)_A$. Let $J/I \lhd A/I$ and N be a submodule of M satisfying $(J/I)^2 N = 0$, so consequently $(J/I)^2 N = 0$ if and only if $J^2N = 0$. By the semiprimeness of M, $JN = 0$. This gives $(J/I)N = 0$. Therefore, M is a semiprime A/I – module.

Let $M \in \sum_A$ and let $B \lhd A$. Suppose $J \lhd B$ and let N be any submodule of M such that $J^2N = 0$. It follows from Andrunakievich's Lemma that $\langle J \rangle^3 \subset J$, where $\langle J \rangle$ is the ideal of A generated by J. Therefore, $\langle J \rangle^6 = (\langle J \rangle^3)^2 \subset J^2$, consequently $\langle J \rangle^6 N = 0$. By the semiprimeness of M over the ring $A, \langle J \rangle^3 N = 0$. This condition implies $\langle J \rangle^4 N = 0$. Moreover, $\langle J \rangle^2 N = 0$. Hence, $\langle J \rangle N = 0$. It follows that $JN \subseteq \langle J \rangle N = 0 \Rightarrow JN = 0$. Thus $M \in \sum_B$.

Let $M \in \sum_B, B \lhd A$. We would like to show that $BM \in \sum_A$. In other words, that $(0:BM)_A$ is a semiprime ideal of A. Let $J \lhd A$ such that $JB \neq 0$ and $J^2 \subseteq (0:BM)_A$. Moreover, $JB \subseteq J$ and $JB \subseteq B$.

Hence

$$(JB)^2 \subseteq J^2 \text{ and } (JB)^2 \subseteq B^2.$$

Thus,

$$(JB)^2 \subseteq J^2 \cap B^2 \subseteq J^2 \cap B \subseteq (0:BM)_A \cap B = (0:M)_B.$$

This implies that $(JB)^2 \subseteq (0:M)_B$. Since $(0:M)_B$ is a semiprime ideal of $A, JB \subseteq (0:M)_B$. This means that $JBM = 0$, consequently, $J \subseteq (0:BM)_B$. So, we may deduce that BM is a semiprime A – module.

In fact, we already have seen some examples of special classes of modules which can be accessed in [5]. A different example of a weakly special class of modules is described below.

Theorem 2.7. Every special class of modules is a weakly special class of modules.

Proof. Let $\sum$ be a special class of modules. Since every prime module is semiprime, the class $\sum$ of prime modules consists of semiprime modules. It follows from the speciality of $\sum$ that $\sum$ satisfies the conditions (M1), (M2), (SM3), and (SM4). Hence, the class $\sum$ of semiprime modules is a weakly special class.

Moreover, Theorem 2.7 clearly shows that every special class of modules is naturally a weakly special class of modules. This condition is equivalent with the property stating that every special class of rings is a weakly special class of rings.

In fact, the essential closure $*_k$ of the class $*$ of all $*$– rings forms a special class of rings. Hence, the class $\rho* = \bigcup \rho*(A)$, where $\rho*(A) = \{M \mid M$ is a prime A– module and $A/(0:M)_A \in *_k\}$, forms a special class of modules [10]. Let A be any ring and let M be an A– module. The A– module is called a $*p$– module if M is a prime A– module and M has no nonzero proper prime submodule. Some properties of $*p$– modules were given in [10]. Moreover, the following theorem shows that every nonzero $*p$– module is contained in $\rho*(A)$.

Theorem 2.8. Every $*p$– module is contained in $\rho*(A)$.

Proof. Let A be a ring and let M be a $*p$– module over the ring A. Then it follows from the definition of $*p$– module that M is a prime A– module and M has no nonzero proper prime A– submodule. Since M is a prime A– module, Proposition 3.14.16 in [5] shows that the annihilator $(0:M)_A$ of M is a prime ideal of A. Therefore, the factor ring $A/(0:M)_A \in \pi$, where π is the class of all prime rings. Suppose $A/(0:M)_A$ is not in the class $*$ of all $*$– rings. Then $A/(0:M)_A$ has a nonzero proper homomorphic image, say $A/B \cong (A/(0:M)_A)/((0:M)_A/B)$. Then, B is a prime ideal of A such that $(0:M)_A \subset B$. Moreover, BM is a nonzero proper prime A– submodule of M, contrary to the assumption that M is a $*p$– module. Hence, $A/(0:M)_A \in * \subset *_k$. This means that M is contained in the special class of modules generated by the essential closure $*_k$ of the class $*$ of all $*$– rings.

Another natural example of a weakly special class of modules is described in the following result.

Theorem 2.9. Every normal class of modules is a weakly special class of modules.

Proof. It follows from Proposition 2 in [7] that every normal class of modules forms a special class of modules. Moreover, it follows from Theorem 2.7 that every normal class of modules is weakly special.

On the other hand, some other properties of $*p$– Modules can be found in [10].

Theorem 2.10. If σ is a weakly special class of modules, then the class $\mu_\sigma = \{R \mid R$ has a faithful module in $\sigma(R)\}$ of rings is a weakly special class of rings.

Proof. We will show that μ_σ consists of semiprime rings. Let $R \in \mu_\sigma$. Then R has a faithful module in the class $\sigma(R)$ of all semiprime $R-$ modules. It follows from Theorem 2.3 that R is semiprime. Therefore, μ_σ consists of semiprime rings. Now let $0 \neq I \lhd R \in \mu_\sigma$. Since R is a semiprime ring, so is I. It means that I has a faithful module in the class $\sigma(I)$ of all semiprime $I-$ modules, so $I \in \mu_\sigma$. Now suppose R is a ring such that there exists $I \lhd \circ R$ with $I \in \mu_\sigma$. This means that I has a faithful module in $\sigma(I)$, say M. Then, $(0:M)_R = 0$ and $M \in \sigma(I)$. Since $M \in \sigma(I), IM \in \sigma(R)$. Let $x \in (0:IM)_R$. Then $xIM = 0$. Since $(0:M)_R = 0, IM \neq 0$, and consequently $xIM = 0$ implies $x = 0$. This means that IM is a faithful $R-$ module. Hence, R has a faithful module in $\sigma(R)$. So, we may deduce that μ_σ is a weakly special class of rings.

As a direct consequence of Theorem 2.10, we therefore have the following result.

Theorem 2.11. Let σ be class of rings which is a weakly special class of modules and let $\mu_\sigma = \{R \mid$ there exists an $R-$ module M such that $(0:M)_R = 0$ and M belongs to $\sigma(R)\}$ be the class of rings such that every member of μ_σ has a faithful module in the class $\sigma(R)$ of all $R-$ semiprime modules. Then the upper radical $U(\mu_\sigma)$ generated by the class μ_σ of rings is a supernilpotent radical class of rings

Proof. It follows from Theorem 2.10 that μ_σ is a weakly special class of rings. Therefore, $U(\mu_\sigma)$ is a supernilpotent radical class.

3. Conclusions

Based on the results, we can construct a semiprime module over a ring R if the ring R has a semiprime ideal. This condition is described in Theorem 2.2. As consequence of this property, a necessary and sufficient condition for a ring to be semiprime is having a faithful semiprime module. Moreover, the construction of a weakly special class of modules is also successfully determined by following the construction of a special class of modules which was introduced by Nicholson and Watters in [7]. A further property of $*p-$modules described in this paper is that every $*p-$module is contained in $\rho^*(A)$, the class of all $A-$ modules M such that the annihilator $(0:M)_A \in *_k$. We recommend for further investigation to the reader to find some additional properties of $*p-$ modules and to find whether any properties of special classes of modules are also valid for weakly special classes of modules.

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REFERENCES

[1] M. Behboodi. A Generalization of Baer's Lower Nilradical for Modules, J. Algebra. Appl, Vol.6, 337-353.

[2] H. France-Jackson. On Atoms of the lattice of Supernilpotent Radicals, Quaestiones Mathematicae, vol.10, 251-255.

[3] H. France-Jackson. On Prime Essential Rings, Bull. Austral. Math. Soc, Vol.47, 287-290.

[4] B. J. Gardner, P. M. Stewart. Prime Essential Rings, Proceedings of The Edinburgh Mathematical Society, Vol.34, 241-250.

[5] B. J. Gardner, R. Wiegandt. Radical Theory of Rings, Marcel Dekker Inc, New York, 2004.

[6] H. Korolczuk. A Note on The Lattice of Special Radicals, Bulletin De LAcademie Polonaise Des Sciences, vol.29, 3-4.

[7] W. Nicholson, J. Watters. Normal Radicals and Normal Classes of Modules. Glasgow Math. J Vol.30, 97-100.

[8] P. W. Prasetyo, I. E. Wijayanti, H. France-Jackson. Some properties of prime essential rings and their generalization, Proceedings of the 1st International Conference on Science and Technology, (120003)1-6, 2016.

[9] P. W. Prasetyo, I. E. Wijayanti, H. France-Jackson. Some special classes μ whose upper radical U(μ) determined by μ coincides with the upper radical U(*μ)k determined by (*μ)k. Proceedings of The 7th SEAMS UGM Int. Conference on Mathematics and Its Applications, (020013)1-4, 2016.

[10] P. W. Prasetyo, I. E. Wijayanti, H. France-Jackson. *p-Module and A Special Class of Modules Determined by The Essential Closure of the Class of All *-Rings, JP Journal of Algebra, Number Theory, and Its Applications, vol.39, 11-20.

[11] S. Wahyuni, I. E. Wijayanti, H. France-Jackson. A Prime Essential Rings That Generates a Special Atoms, Bull. Aust. Math. Soc Vol.95, 214-218.