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Stochastic Decomposition Result of an Unreliable Queue with Two Types of Services

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Abstract The single server queue with two types of heterogeneous services with generalized vacation for unreliable server have been extended to include several types of generalizations to which attentions has been paid by several researchers. One of the most important results which deals with such types of models is the "Stochastic Decomposition Result", which allows the system behaviour to be analyzed by considering separately distribution of system (queue) size with no vacation and additional system (queue) size due to vacation. Our intention is to look into some sort of united approach to establish stochastic decomposition result for two types of general heterogeneous service queues with generalized vacations for unreliable server with delayed repair to include several types of generalizations. Our results are based on embedded Markov Chain technique which is considerably a most powerful and popular method wisely used in applied probability, specially in queueing theory. The fundamental idea behind this method is to simplify the description of state from two dimensional states to one dimensinal state space. Finally, the results that are derived is shown to include several types of generalizations of some existing well- known results for vacation models, that may lead to remarkable simplification while solving similar types of complex models.

Keywords Stochastic Decomposition, Heterogenous Service, Generalized Vacation, Markov Chain

1 Introduction

Vacation models are characterized by the fact that the idle time of the server may be utilized for some other secondary jobs, for instance to serve the customers in other systems. Allowing the server to take vacations makes the queueing models more realistic and flexible in studying real world queueing situations. Applications of such models arise naturally in call centres with multitask employees, telecommunication and computer networks, production and quality control problems etc. Literature on vacation models is growing very rapidly and it has become the subject matter of current research due to its numerous applications in many real life situations. Instead of reviewing those again, it will be more convenient to refer the readers to survey papers by Doshi (1986), Teghem (1986), Medhi (1997) and also the monograph of Takagi (1990) for more information as well as complete set of references.

One of the most remarkable result that concerned with these types of models is "Stochastic Decomposition Result", which allows the system behaviour to be analyzed by considering separately distribution of system (queue) size with no vacation and additional system (queue) size due to vacation. The decomposition property for $M/G/1$ queue has been observed by Graver (1962), Cooper (1970), and Furhmann (1984). However, for generalized vacations the decomposition result was first established by Fuhrmann and Cooper (1985). Shanthikumar (1988) relaxed some of the assumptions considered by Fuhrmann and Cooper (1985). Harris and Marchal (1988) extended Fuhrmann and Cooper (1985) to the state dependent case. The distributional form of Little's law for decomposition result has been discussed by Keilson and Servi (1990). Doshi (1990) obtained the decomposition formula for virtual workload process. Miyazawa (1994) discussed certain generalizations of decomposition formula for the single server queue with two levels of service rates, which include vacation models as special cases.

Recently, considerable attention have been paid to study $M/G/1$ type of queueing system, in which the server provides two types of general heterogeneous service and optional repeated service policy. In such a model an arriving unit has an option to choose any one of the two types of services before the service starts. Further, if an unit is not satisfied by the service then it has an option to repeat the service once more (repeated service policy). Madan et al. (1994) first investigated such type of model, where they introduced the notion of repeated service. Recently, Choudhury and Kalita (2017) generalized this type of model for unreliable queueing system, where they introduced the concept of server's breakdown and delayed repair. More recently Kalita and Choudhury (2018, 2019[a,b]) generalized this type of batch arrival unreliable queueing model under different vacation policies.

Although some aspects of these types of unreliable vacation models been discusssed separately for batch arrival queueing system. still some questions are yet to be addressed about their theoretical structure i.e. stochastic decomposition property. Hence in this paper an attempt has been made to derive stochastic decomposition result for such types $M/G/1$ unreliable queue with generalized vacations with two types of general heterogeneous services under repeated service policy in order to unify the results of class of these types of vacation models for unreliable server with delayed repair.

The rest of the paper is organised as follows: The mathematical model is described shortly in section 2 . Section 3 deals with the queue size distribution at a departure epoch in order to establish "Stochastic Decomposition"for such types of models. Embedded Markov Chain technique is applied to establish "Stochastic Decomposition Result". Some applications of "Stochastic Decomposition Results"for specific models have been demonstrated in section 4.

2 Mathematical Model

We consider an M/G/1 queueing system with two types of heterogeneous services, where arrivals occur according to Poisson Process with rate 'λ' . There is a single server which provides two types of heterogeneous services to each unit on First Come First Service (FCFS) basis. The service mechanism is such that each unit has an option to select either the first type of service (FTS) denoted by B_1 with probability p_1 or second type of service (STS) denoted by B_2 with probability p_2 such that $p_1+p_2=1$.

Let G_i be the total time taken to complete the i^{th} type of service. The service cycle can be more specifically written as,

$$B = \begin{cases} G_1, & \text{with probability } p_1 \\ G_2, & \text{with probability } p_2 \end{cases}$$

Thus, G_i total i^{th} type of service time random variable is assumed to follow general law of distribution with distribution function (d.f) $F_{G_i}(x)$, Laplace Stieltjes Transform (LST) $F^*_{G_i}(\theta)=\int_0^\infty e^{-\theta x}dF_{G_i}(x)$.
Therefore, LST of the total service time after selection of a service can be written as,

$$F^*_B(\theta)=p_1F^*_{G_i}(\theta)+p_2F^*_{G_i}(\theta) \tag{1}$$

where $F^*_B(\theta)=\int_0^\infty e^{-\theta x}dF_B(x)$ is the LST of B and $F_B(x)$ is the d.f of the total service time random variable B.
As soon as either type of service is completed by an unit, such an unit has further option to repeat the same type of service denoted by S_i once only with probability q_i or leave the system with probability $(1-q_i)$ for i=1,2. Hence, the total service time required by an unit to complete i^{th} type of service which may be called modified service time for i=1,2 is given by,

$$G_i = \begin{cases} B_i+S_i, & \text{with probability } q_i \\ B_i, & \text{with probability } (1-q_i) \end{cases}$$

Also, it is assumed that the service time and repeated service time random variable follow general probability law of distribution with d.f $F_{B_i}(x)$ and $F_{S_i}(x)$ respectively and LST $F^*_{B_i}(\theta)=\int_0^\infty e^{-\theta x}dF_{B_i}(x)$ and $F^*_{S_i}(\theta)=\int_0^\infty e^{-\theta x}dF_{S_i}(x)$ respectively with first moment μ_{b_i} and μ_{s_i} respectively.

Clearly, the LST of the modified service time random variable of the total service time G_i is given by,

$$F^*_{G_i}(\theta)=(1-q_i)F^*_{B_i}(\theta)+q_iF^*_{S_i}(\theta)F^*_{B_i}(\theta) \tag{2}$$

Now, utilizing (2) in (1), we have

$$F^*_B(\theta)=\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{S_i}(\theta)\}p_iF^*_{B_i}(\theta) \tag{3}$$

This type of queueing model is known as two type of heterogenous service queue with optional repeated service policy. In Kendall's notation the model of this type is denoted by $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1$ type queue and the same was investigated by Madan et. al. (2004). Recently, Choudhury and Kalita (2017) generalized this type of model by intoducing the concept of server's breakdown and delay processes, in which, where when the server is working with sevice or repeated service (RS), it may breakdown at any time for a short iterval of time i.e. server's life times are generated by exogenous Poisson process with rate α_1 for FTS or FTRS (i.e. First Type of Repeated service) and α_2 for STS or STRS (i.e. Second type of repeated Service). As soon as breakdowm occur, the server is sent for repair during which time it stops providing service to arriving units and wait for repair to start, which we may refer to as waiting period of the server. We define this waiting time as delay time. The delay time D_i of the server for i^{th} type of service or repeated service follow general law of distribution with d.f $D_i(y)$ and LST $D_i{}^*(\theta)=\int_0^\infty e^{-\theta y}dD_i(y)$ and first moment $\bar{d}_i$ for i=1,2. Similarly, the repair time 'R_i 'of the server for i^{th} type of service or repeated service is assumed to follow general law of distribution with d.f. $R_i(y)$, LST $R_i{}^*(\theta)=\int_0^\infty e^{-\theta y}dR_i(y)$ and first moment $\bar{r}_{i,j}$ for i=1,2. Immediately after the server is fixed (i.e. repaired), the server is ready to start its remaining service to the customers in both types of service or repeated service and in this case the service times are cummulative which we may reffered to as generalized service time.
Now, if we define H_i as generalized service time for i^{th} type of service, $H_i(x)$ and $H_i{}^*(\theta)=\int_0^\infty e^{-\theta x}dH_i(x)$ as its d.f. and LST respectively for i=1,2, then we have

$$H_i{}^*(\theta)=\sum_{n=0}^{\infty}\int_0^\infty D_i{}^{(n)}(\theta)R_i{}^{(n)}(\theta)\left[\frac{(\alpha_i x)^n}{n!}\right]e^{-\theta x}e^{-\alpha_i x}dF_{G_i}(x)=F^*_{B_i}(\theta+\alpha_i(1-D_i{}^*(\theta)R_i{}^*(\theta))) \tag{4}$$

Similarly, if we define H_i^R as generalized service time for i^{th} type of repeated service $H_i^R(x)$ and $H_i^{R^*}(\theta)=\int_0^\infty e^{-\theta x}dH_i^R(x)$ as its d.f and LST respectively for i=1,2, then we have

$$H_i^{R^*}(\theta)=F^*_{S_i}(\theta+\alpha_i(1-D_i{}^*(\theta)R_i{}^*(\theta))) \tag{5}$$

Now, LST of the total generalized service time provided by the server to a unit denoted by 'A 'can be written as,

$$F^*_A(\theta)=\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\theta+\alpha_i(1-D_i{}^*(\theta)R_i{}^*(\theta)))\}p_iF^*_{S_i}(\theta+\alpha_i(1-D_i{}^*(\theta)R_i{}^*(\theta))) \tag{6}$$

where $F^*_A(\theta)$ is the LST of A so that the utilization factor of the system is given by,

$$\rho=\sum_{i=1}^{2}\{1+\alpha_i(\bar{d}_i+\bar{r}_i)\}p_i(\rho_{b_i}+q_i\rho_{s_i})$$

where $\rho_{b_i}=\lambda\mu_{b_i}$ and $\rho_{s_i}=\lambda\mu_{s_i}$.
This type of model is known as $M/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1$ type of queue for unreliable server with delayed repair and studied recently by Choudhury and Kalita (2017). Now, for further development of such a type of model, we introduce the notion of generalized vacation. Infact, the notion of generalized vacation was first introduced by Fuhrmann and Cooper's for an $M/G/1$ type of model in order to generalize results of many existing vacation models of $M/G/1$ type. However, in our existing model, it is assumed that the services given in both types of services is non-primitive i.e., once selected for generalized service (FTS or STS) or generalized repeated service (i.e. FTRS or STRS) an unit is served to service completion continously (i.e. busy period). As soon as busy period ends, vacation period of the server begins. A vacation period may contain a number of vacations. The vacation period begins and ends according to well defined rules, which may depend on either the current or past evaluation of the system. Further, a vacation period can be terminated by a condition depending on arrival process during that vacation period.
Moreover, it is assume that generalized service time and generalized repeated service time random variables are mutually independent of each other and that of arrival process. Also, it is assumed that each generalized service time and generalized repeated service time are independent of sequence of vacation periods that precede that generalized service time or generalized repeated service time. A queueing system that satisfies these properties is called $M/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1$ unreliable queueing system subject to delayed repair and generalized vacation.

3 Stochastic Decomposition Result

In this section an attempt will be made to derive the Stochastic Decomposition Result for our model. Accordingly, we state the main result in the following Theorem 1.

Theorem 1: *Under the stability condition $\rho < 1$, we have*

$$\pi(z) = \frac{(1-\rho)[1-\gamma(z)]\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))\right]}{\gamma'(1)\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))-z\right]}$$

where, $\beta_i(z) = \Psi(z)+\alpha_i(1-R_i^*(\Psi(z))D_i^*(\Psi(z))$ for i=1,2; $\Psi(z) = \lambda(1-z)$ and $\gamma(z)$ is the Probability Generating Function (PGF) of the queue size distribution at busy period initiation epoch and $\pi(z)$ is the PGF of the queue size distribution at a departure epoch.

Proof Let t_m be the time of m^{th} service completion epoch i.e. we are considering the epochs at which total generalized service by an unit expires. Then the sequence $X_n = N(t_n+0)$ (where N(t) represents number of units in the system at time instant t) form a Discrete Time Markov Chain (DTMC) which is embedded Markov renewal process of a continuous time Markov Process. The sequence $\{X_n; n \geq 0\}$is a homogeneous DTMC and it is associated with the following transition,

$$X_{(n+1)} = \begin{cases} L_{(n+1)} + V_{(n+1)} - 1, if X_n = 0 \\ X_{(n)} + L_{(n+1)} - 1, if X_n > 0 \end{cases}$$

where $L_{(n)}$ is the number of units arrived during n^{th} total generalized service period and $X_{(n)}$ is the number of units arrived during n^{th} vacation period.

Clearly, our Transition Probability matrix (TPM) $P = (P_{ij})$ is readily seen to be Δ_2 matrix which is a special case of $\Delta_{m,n}$ matrix introduced and studied properties of Markov Chain by Abolnikov and Dukhovny (1991). Thus, for our model Δ_2 matrix is of the following form,

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{12} & \dots \\ 0 & P_{21} & P_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

where, $P_{ij} = Pr\{X_{(n+1)} = j/X_n = i\}$ is the transition probability associated with the DTMC $\{X_n; n \geq 0\}$.

Our Δ_2 matrix differs from the $M/G/1$ type of queue in the first row only i.e. TPM associated with DTMC $\{X_n; n \geq 0\}$ has the elements.

$$P_{i,j} = \begin{cases} \{\sum_{n=1}^{j+1}\gamma_n\sum_{i=1}^{2}p_i\{(1-q_i)m_{i,j-n+1}+q_ik_{i,j-n+1}\}, \, if l = 0, \, j \geq 0 \\ \sum_{i=1}^{2}p_i\{(1-q_i)m_{i,j-l+1}+q_ik_{i,j-l+1}\}, if \, j \geq l-1, \, l \geq 1 \\ 0, \, otherwise \end{cases}$$

where, for $j \geq 0$ and for i=1,2; $m_{i,j} = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^j}{j!}dH_i(z)$ is the probability that j units arrive during i^{th} type of generalized service time 'H_i'. $k_{i,j}$ is the probability that 'j'units arrive during the time interval $[H_i + H_i^R] = \sum_{n=0}^{j} m_{i,n}c_{i,j-n}$; $c_{i,j} = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^j}{j!}dH_i^R(x)$ is the probability that j units arrive during i^{th} type of generalized repeated service time and $\gamma_j = Lim_{n->\infty}Pr\{V_{(n+1)} = j\}$ is the limiting probability that j units find the server to start a busy period.
Consequently, $\rho < 1$ is the necessary and sufficient condition for existence of steady state condition.
Next, we assume that $\rho < 1$ to guarantee that $\{X_n; n \geq 0\}$ is positive recurrent. Thus, limiting probability $\pi_j = Lim_{n->\infty}Pr\{X_n = j\}$ exist and is positive.

Then, Kolmogorov equation associated with DTMC $\{X_n; n \geq 0\}$ can be written as,

$$\pi_j = \sum_{n=1}^{j+1}(\pi_o\gamma_n + \pi_n)\left[\sum_{i=1}^{2}p_i\{(1-q_i)m_{j-n+1} + q_ik_{i,j-n+1}\}\right] \; j \geq 0 \tag{7}$$

Next, let us define the following PGFs:
$\pi(z) = \sum_{j=0}^{\infty} z^j\pi_j$, $\gamma(z) = \sum_{j=1}^{\infty} z^j\gamma_j$, $m_i(z) = \sum_{j=0}^{\infty} z^jm_{i,j}$, $k_i(z) = \sum_{j=0}^{\infty} z^jk_{i,j}$ and $c_i(z) = \sum_{j=0}^{\infty} z^jc_j$ respectively, then from equation (7) we have,

$$\pi(z) = \pi_0\gamma(z)\left[\sum_{i=1}^{2}p_i((1-q_i)m_i(z)+q_ik_i(z))z^{-1}\right] + \pi_0\gamma(z)\{\pi(z)-\pi_0\}\left\{\sum_{i=1}^{2}p_i((1-q_i)m_i(z)+q_ik_i(z))\right\} \tag{8}$$

Now, because of presence of convolution, the equation (8) can be transformed with the help of the following PGFs $m_i(z) = F^*_{B_i}(\beta_i(z))$, $k_i(z) = m_i(z)c_i(z)$ for $i = 1, 2$. Note that $c_i(z) = F^*_{S_i}(\beta_i(z))$ for i=1,2 where, $\beta_i(z) = \Psi(z) + \alpha_i(1 - R^*_i(\Psi(z))D^*_i(\Psi(z))$ and $\Psi(z) = \lambda(1 - z)$.
Therefore, from equation (8), we have

$$\pi(z) = \frac{\pi_0[1-\gamma(z)]\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))\right]}{\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))-z\right]} \tag{9}$$

Since $\sum_{j=0}^{\infty}\pi_j = \pi(1) = 1$, therefore equation (9) yields

$$\pi_0 = \frac{(1-\rho)}{\gamma'(1)} \tag{10}$$

Utilizing (10) in (9), we get the required result.

Remark:1
If we take $\gamma_1 = 1$ and $\gamma_n = 0$ if $n \neq 1$, then $\gamma(z) = z$ and $\gamma'(1) = 1$. Hence, from Theorem 1, we have,

$$\pi(z) = \frac{(1-\rho)[1-z]\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))\right]}{\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))-z\right]} = \pi_o(z)\ (say) \tag{11}$$

which is the PGF of the queue length distribution at a departure epoch of queueing system.
Note that the expression (11) is consistent with the result obtained by Choudhury and Kalita (2017). Hence equation (7) can also be expressed as

$$\pi(z) = \frac{[1-\gamma(z)]\pi_o(z)}{\gamma'(1)(1-z)} = \Gamma(z)\pi_o(z) \tag{12}$$

where $\Gamma(z) = \frac{[1-\gamma(z)]}{\gamma'(1)(1-z)}$ is the PGF of the additional queue size distribution due to vacation period.
Thus, expression (12) demonstrate and justify stochastic decomposition property for our model.

Remark:2
If we consider the case of non-exhaustive service discipline, where vacation may start even when some units are present in the system (tagged vacation), then corresponding PGF of the queue size distribution at a departure epoch for our model is found to be

$$\pi(z) = \zeta(z)\frac{(1-\rho)[1-\gamma(z)]\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))\right]}{\gamma'(1)\left[\sum_{i=1}^{2}\{(1-q_i)+q_iF^*_{B_i}(\beta_i(z))\}p_iF^*_{S_i}(\beta_i(z))-z\right]}$$

where $\zeta(z)$ is the PGF of an unit that arrive during the tagged vacation.
Note that the gated service, limited service and decrementing service disciplines are examples of non-exhaustive service discipline.

4 Some Applications

In this section we will discuss very briefly some well known vacation models for our type of queueing model as an application of Theorem 1.

4.1 Randomized Vacation Policy Queueing Model

In randomized vacation policy model, as soon as the system becomes empty, the server deactivates and leaves for a vacation. Upon returning from the vacation, if at least one unit is found in the queue for any one type of service, the server starts providing service to the unit. Otherwise, if no units are found waiting in the queue, the server either remains idle in the system or takes another vacation. This pattern continues until the number of vacations taken by the server reaches J(say). If the system is still empty by the end of the J^{th} vacation, the server becomes idle in the system untill at least one unit waiting in the queue (dormant period). Thus, in this system, a random vacation period, a dormant period and a busy period constitute a cycle. Moreover, the system remains idle during a random vacation period and a dormant period and these two periods together constitute an idle period. This type of model was first investigated by Takagi (1991) (see page no. 127) for M/G/1 queueing system. Choudhury (2002) generalized the result for batch arrival case.

Next, we define:
V-vacation time random variable
$V(y)$- probability distribution function of V
$V^*(\theta) = \int_0^\infty e^{-\theta y} dV(y)$ LST of V

Now, if we define a_j by the probability that j units arrive during a vacation time V, then for $j = 0, 1, 2, ...$ we have

$$a_j = \int_0^\infty \frac{e^{-\lambda y}(\lambda y)^j}{j!} dV(y);\ j \geq 0$$

So that the probability that j units arrived and accepted to start a busy period is given by,

$$\gamma_j = a_j G_J(a_0) + (a_0)^J \delta_{j,J};\ j \geq 1 \tag{13}$$

where $G_J(a_0) = \sum_{n=0}^{J-1} (a_0)^n = \frac{(1-(a_0)^J)}{(1-a_0)}$
$a_0 = V^*(\lambda)$
and

$$\delta_{i,j} = \begin{cases} 1, & \text{if i=j} \\ 0, & \text{otherwise} \end{cases}$$

is the Kronecker's delta function.
Then, from equation (13), we have

$$\gamma(z) = \sum_{j=1}^{\infty} z^j \gamma_j = G_J(a_0)[V^*(\Psi(z)) - a_0] + [a_0]^J z \tag{14}$$

and

$$\gamma^{'}(1) = \lambda G_J(a_0) E(V) + a_0^J \tag{15}$$

Utilizing (14) and (15) in (11), we have

$$\pi(z) = \frac{[G_J(a_0)(1 - V^*(\Psi(z)) + a_0^J(1-z))]\pi_o(z)}{[\lambda G_J(a_0)E(V) + a_0^J](1-z)} \tag{16}$$

It should be noted here that this type of model for batch arrival queueing system have recently been studied by Kalita and Choudhury (2019(a)).
If we take J=1 in (16), we have

$$\pi(z) = \frac{(1-\rho)[1 - V^*(\lambda - \lambda z) + V^*(\lambda)(1-z)]\left[\sum_{i=1}^{2}\{(1-q_i) + q_i F^*_{S_i}(\beta_i(z))\} p_i F^*_{B_i}(\beta_i(z))\right]}{[\lambda E(V) + V^*(\lambda)]\left[\sum_{i=1}^{2}\{(1-q_i) + q_i F^*_{S_i}(\beta_i(z))\} p_i F^*_{B_i}(\beta_i(z)) - z\right]}$$

which is the PGF of the queue size distribution at departure epoch for an $M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1$ unreliable vacation queue with delayed repair under single vacation policy, where the server takes exactly one vacation between two successive busy periods.
Similarly, if we take $J \to \infty$, then we have
$Lim_{J\to\infty}(a_0)^J \to 0$ as $\mid a_0 \mid \leq 1$
and $Lim_{J\to\infty} G_J(a_0) = \frac{1}{(1-a_0)}$
Hence, from the above equation (16), we have

$$\pi(z) = \frac{[1 - V^*(\lambda - \lambda z)]\pi_o(z)}{\lambda E(V)(1-z)}$$

which is the PFG of the queue size distribution at a departure epoch of an $M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1$ multiple vacation queue for unreliable server with delayed repair.

4.2 Queueing system with set-up time under N-Policy

Takagi (1991) and Medhi and Templeton (1992) studied such a model for M/G/1 queueing system; where the server is turned off each time as soon as the system becomes empty. The server remains idle untill the queue size builds upto to a pre-assigned threshold level N (build-up period).As soon as the queue size becomes $N(\geq 1)$ the server has to undertake a gear up time called Setup Time (SET) in order to setup the system into operative mode (set up period), on completion of which service begins (busy period).
Now, let us define the following notations,
U - SET random variable
$U(x)$ - Probability distribution function of U
$U^*(\theta) = \int_0^\infty e^{-\theta x} dU(x)$ is the LST of U
Now, if we define 'b_j'by the probability that j units arrived during a SET, then for $j \geq 0$, we have,

$$b_j = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^j}{j!} dU(x); j \geq 0$$

so that the probability that j units arrived and accepted to start a busy period is given by,

$$\gamma_j = \begin{cases} 0, if j = 1, 2, ..., N-1 \\ b_{j-N}, if j \geq N \end{cases}$$

and therefore, we have,

$$\gamma(z) = \sum_{j=1}^{\infty} z^j \gamma_j = z^N U^*(\lambda - \lambda z) \tag{17}$$

and

$$\gamma^{'}(1) = N + \lambda E(U) \tag{18}$$

Utilising (17) and (18) in equation (11) we get,

$$\pi(z) = \frac{[1 - z^N U^*(\lambda - \lambda z)]\pi_0(z)}{[N + \lambda E(U)](1 - z)} \tag{19}$$

which is the queue size distribution at departure epoch of an $M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1$ unreliable queue and delaying repair under N-policy with a random setup time.
Note that model of this nature for batch arrival queueing system have recently been studied by Kalita and Choudhury (2019(b)).
Further, if we take $N = 1$ in the above expression (19), we get

$$\pi(z) = \frac{[1 - zU^*(\lambda - \lambda z)]\pi_o(z)}{[1 + \lambda E(U)](1 - z)}$$

which is the queue size distribution at a departure epoch of an $M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1$ unreliable queue and delaying repair with a random setup time.
Note that model of this nature have been studied by Levy and Klienrock (1986) for $M/G/1$ type of vacation model.

5 Concluding Remark

This paper demonstrates decomposition result of $M/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1$ queue for unreliable server and delaying repair with generalized vacations under optional repeated service policy using the embedded Markov Chain technique. Our results cover several types of generalizations of some well known vacation models which are demonstrated in section 4.

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