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Approximations for Theories of Abelian Groups

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Abstract Approximations of syntactic and semantic objects play an important role in various fields of mathematics. They can create theories and structures in one given class by means of others, usually simpler. For instance, in certain situations, infinite objects can be approximated by finite or strongly minimal ones. Thus, complicated objects can be collected using simplified ones. Among these objects, Abelian groups, their first order theories, connections and dynamics are of interests. Theories of Abelian groups are characterized by Szmielew invariants leading to the study and descriptions of approximations in terms of these invariants. In the paper we apply a general approach for approximating theories to the class of theories of Abelian groups which characterizes the approximability of a theory of Abelian groups by a given family of theories of Abelian groups in terms of Szmielew invariants and their limits. We describe some forms of approximations for theories of Abelian groups. In particular, approximations of theories of Abelian groups by theories of finite ones are characterized. In addition, we describe approximations by quasi-cyclic and torsion-free Abelian groups and their combinations with respect to given families of prime numbers. Approximations and closures of families of theories with respect to standard Abelian groups for various sets of prime numbers are also described.

Keywords Approximation, Theory, Abelian Group, Approximable Theory

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1 Introduction

The class of Abelian groups admits a good elementary classification by Szmielew invariants [1, 2, 3] reducing Abelian groups to standard ones which are represented by direct sums of a given collection. These invariants lead to closure control for families of theories of Abelian groups [4], for counting ranks for these families, for characterizing approximability over these families and, in particular, the pseudofiniteness, i.e., the approximability by theories of finite Abelian groups [5].

In this paper we apply a general approach for approximations of theories [6] to natural families of theories of Abelian groups and describe some forms of approximability.

The paper is organized as follows. In Section 2, we consider general notions of approximability, notions and notations for Abelian groups and their first-order theories, including Szmielew invariants, standard groups, and representations of theories of Abelian groups through direct sums of standard groups. In addition, we consider the criteria for the pseudofiniteness of Abelian groups and the e-minimality of families of theories of Abelian groups. We also introduce some natural families of theories of Abelian groups and study connections and closures for these families, including families of theories of finite, divisible, and torsion-free Abelian groups.

In Section 3, we characterize the approximability of a given theory of Abelian groups by a given family of theories of Abelian groups in terms of Szmielew invariants and their limits (Theorem 3.1). We describe some forms of approximations for theories of Abelian groups. In particular, approximations of theories of Abelian groups by theories of finite ones are characterized (Corollaries 3.2 and 3.4). In addition, we describe approximations by quasi-cyclic and torsion-free Abelian groups and their combinations with respect to given families of prime numbers (Corollaries 3.6 and 3.7). Approximations and closures of families of theories with respect to standard Abelian groups for various sets of prime numbers are also described (Corollary 3.8).

Illustrations of approximations are considered in Section 4.

2 Preliminaries

Throughout we consider families $\mathcal{T}$ of complete first-order theories of a language $\Sigma = \Sigma(\mathcal{T})$. For a sentence φ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$ which is called the *φ-neighbourhood* in $\mathcal{T}$.

Definition [6]. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called *$\mathcal{T}$-approximated*, or *approximated by $\mathcal{T}$*, or *$\mathcal{T}$-approximable*, or a *pseudo-$\mathcal{T}$-theory*, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [6] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Let $\mathcal{A}$ be an Abelian group in the language $\Sigma = \langle +^{(2)}, -^{(1)}, 0^{(0)} \rangle$. Then $k\mathcal{A}$ denotes its subgroup $\{ka \mid a \in A\}$ and $\mathcal{A}[k]$ denotes the subgroup $\{a \in A \mid ka = 0\}$. Let P be the set of all prime numbers. If $p \in P$ and $pA = \{0\}$ then $\mathrm{dim}\mathcal{A}$ denotes the dimension of the group $\mathcal{A}$, considered as a vector space over a field with p elements. The following numbers, for arbitrary $p \in P$ and $n \in \omega \setminus \{0\}$ are called the *Szmielew invariants* for the group A [3, 1]:

$$\alpha_{p,n}(\mathcal{A}) = \min\{\dim((p^n\mathcal{A})[p]/(p^{n+1}\mathcal{A})[p]), \omega\},$$

$$\beta_p(\mathcal{A}) = \min\{\inf\{\dim((p^n\mathcal{A})[p] \mid n \in \omega\}, \omega\},$$

$$\gamma_p(\mathcal{A}) = \min\{\inf\{\dim((\mathcal{A}/\mathcal{A}[p^n])/p(\mathcal{A}/\mathcal{A}[p^n])) \mid n \in \omega\}, \omega\},$$

$$\varepsilon(\mathcal{A}) \in \{0, 1\},$$

and $\varepsilon(\mathcal{A}) = 0 \Leftrightarrow (n\mathcal{A} = \{0\}$ for some $n \in \omega, n \neq 0)$.

It is known [3, Theorem 8.4.10] that two Abelian groups are elementary equivalent if and only if they have same Szmielew invariants. In addition, the following proposition holds.

Proposition 2.1 [3, Proposition 8.4.12]. *Let for any p and n the cardinals $\alpha_{p,n}$, β_p, $\gamma_p \leq \omega$, and $\varepsilon \in \{0, 1\}$ be given. Then there is an Abelian group $\mathcal{A}$ such that the Szmielew invariants $\alpha_{p,n}(\mathcal{A})$, $\beta_p(\mathcal{A})$, $\gamma_p(\mathcal{A})$, and $\varepsilon(\mathcal{A})$ are equal to $\alpha_{p,n}$, β_p, γ_p, and ε, respectively, if and only if the following conditions hold:*

(1) *if for prime p the set $\{n \mid \alpha_{p,n} \neq 0\}$ is infinite then $\beta_p = \gamma_p = \omega$;*

(2) *if $\varepsilon = 0$ then for any prime p, $\beta_p = \gamma_p = 0$ and the set $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ is finite.*

We denote by $\mathbf{Q}$ the additive group of rational numbers, $\mathbf{Z}_{p^n}$ — the cyclic group of the order p^n, $\mathbf{Z}_{p^\infty}$ — the quasi-cyclic group of all complex roots of 1 of degrees p^n for all $n \geq 1$, R_p — the group of irreducible fractions with denominators which are mutually prime with p. The groups $\mathbf{Q}$, $\mathbf{Z}_{p^n}$, R_p, $\mathbf{Z}_{p^\infty}$ are called *basic*. Below the notations of these groups will be identified with their universes.

Since Abelian groups with same Szmielew invariants have same theories, any Abelian group $\mathcal{A}$ is elementary equivalent to a group

$$\oplus_{p,n}\mathbf{Z}_{p^n}^{(\alpha_{p,n})} \oplus \oplus_p \mathbf{Z}_{p^\infty}^{(\beta_p)} \oplus \oplus_p R_p^{(\gamma_p)} \oplus \mathbf{Q}^{(\varepsilon)}, \qquad (1)$$

where $\mathcal{B}^{(k)}$ denotes the direct sum of k subgroups isomorphic to a group $\mathcal{B}$. Thus, any theory of an Abelian group has a model represented by a direct sum of based groups. The groups of form (1) are called *standard*.

Recall that any complete theory of an Abelian group is based by the set of positive primitive formulas [3, Lemma 8.4.5], reduced to the set of the following formulas:

$$\exists y(m_1x_1 + \ldots + m_nx_n \approx p^k y), \qquad (2)$$

$$m_1x_1 + \ldots + m_nx_n \approx 0, \qquad (3)$$

where $m_i \in \mathbf{Z}$, $k \in \omega$, p is a prime number [2], [3, Lemma 8.4.7]. Formulas (2) and (3) witness that Szmielew invariants define theories of Abelian groups modulo Proposition 2.1.

In view of Proposition 2.1 and equations (2) and (3) we have the following:

Remark 2.2. Theories of Abelian groups are forced by sentences implied by formulas of form (2) and (3) and describing dimensions with respect to $\alpha_{p,n}$, β_p, γ_p, ε as well as bounds for orders p^k of elements and possibilities for divisions of elements by p^k. Moreover, various values of Szmielew invariants are separated by some sentences modulo Proposition 2.1.

Recall [8, 7] that an infinite structure $\mathcal{M}$ is *pseudofinite* if every sentence true in $\mathcal{M}$ has a finite model. Here the theory $\mathrm{Th}(\mathcal{M})$ is also called *pseudofinite*.

Consider the family $\mathcal{T}_{A,\mathrm{fin}}$ of all theories of finite Abelian groups and Szmielew invariants of theories in the E-closure $\mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}})$ [4, 9, 10]. Since theories of finite groups can not generate new theories of finite groups and finite Abelian groups have finitely many nonzero values $\alpha_{p,n}$, with $\beta_p = \gamma_p = \varepsilon = 0$ for any prime p, it suffices to consider, inside that closure, a description of theories of pseudofinite groups, i.e., theories in $\mathcal{T}_{A,\mathrm{pf}} = \mathrm{Cl}_E(\mathcal{T}_{A,\mathrm{fin}}) \setminus \mathcal{T}_{A,\mathrm{fin}}$.

Theorem 2.3 [5]. *For any theory T of Abelian groups the following conditions are equivalent:*

(1) *$T \in \mathcal{T}_{A,\mathrm{pf}}$;*

(2) *T has some infinite $\alpha_{p,n}$, or some $\beta_p = \gamma_p = \omega$, or $\varepsilon = 1$, moreover, for all nonzero values β_p and γ_p, $\beta_p = \gamma_p = \omega$;*

(3) *T has infinite models, and all nonzero values β_p and γ_p imply $\beta_p = \gamma_p = \omega$.*

Theorem 2.4 [5]. *For any theory T of an Abelian group $\mathcal{A}$ the following conditions are equivalent:*

(1) *T is approximated by some family of theories;*

(2) *T is approximated by some e-minimal family;*

(3) *$\mathcal{A}$ is infinite.*

In view of Proposition 2.1 we notice that all dependencies between values of Szmielew invariants in a given theory of an Abelian group are exhausted by ones given by infinite $\{n \mid \alpha_{p,n} \neq 0\}$ implying $\beta_p = \gamma_p = \omega$ as well as by infinite $\{\langle p, n\rangle \mid \alpha_{p,n} \neq 0\}$ implying $\varepsilon = 1$. It means that

Szmielew invariants, for a fixed theory and for a family, can not force positive values $\alpha_{p,n}$, β_p, γ_p using positive values for different prime p' and/or ε. Besides, all values $\alpha_{p,n}$ and natural values β_p, γ_p do not forced by other Szmielew invariants. Moreover, finite values $\alpha_{p,n}$, β_p, γ_p, for theories in $\mathrm{Cl}_E(\mathcal{T})$, can not be forced by other finite or infinite values of these invariants. Thus, as noticed in [5], all dependencies between distinct Szmielew invariants $\alpha^T_{p,n}$, β^T_p, γ^T_p, ε^T, for theories $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$, are exhausted by the following ones for sequences $(T_k)_{k\in\omega}$ of theories in $\mathcal{T}$:

1) $\alpha^T_{p,n} = \lim\limits_{k\to\infty} \alpha^{T_k}_{p,n}$,
2) $\beta^T_p = \lim\limits_{k\to\infty} \beta^{T_k}_p$,
3) $\gamma^T_p = \lim\limits_{k\to\infty} \gamma^{T_k}_p$,
3) $\varepsilon^T = \lim\limits_{k\to\infty} \varepsilon^{T_k}$,
4) $\beta^T_p = \gamma^T_p = \omega = \lim\limits_{n} \alpha^{T_k}_{p,n}$,
5) $\varepsilon^T = 1 = \lim\limits_{p,n} \alpha^{T_k}_{p,n}$.

The items 1)–5) show that limit values for Szmielew invariants are independent modulo $\alpha^{T_k}_{p,n}$, i.e., the limits of $\beta^{T_k}_p$, $\gamma^{T_k}_p$, ε^{T_k} can produce only β^T_p, γ^T_p, ε^T, respectively, whereas $\alpha^{T_k}_{p,n}$ can generate both $\alpha^T_{p,n}$, $\beta^T_p = \gamma^T_p = \omega$ and $\varepsilon^T = 1$.

The connections above imply the following:

Theorem 2.5 [5]. *For any infinite family $\mathcal{T}$ of theories of Abelian groups the following conditions are equivalent:*

(1) *$\mathcal{T}$ is e-minimal;*

(2) *for any upper bound $\xi \geq m$ or lower bound $\xi \leq m$, for $m \in \omega$, of a Szmielew invariant $\xi \in \mathbf{Szm}$, there are finitely many theories in $\mathcal{T}$ satisfying this bound; having finitely many theories with $\xi \geq m$, there are infinitely many theories in $\mathcal{T}$ with a fixed value $\alpha_{p,n} < m$, if $\xi = \alpha_{p,n}$, with a fixed value $\beta_p < m$, if $\xi = \beta_p$, with a fixed value $\gamma_p < m$, if $\xi = \gamma_p$, and with a fixed value $\varepsilon < m$, if $\xi = \varepsilon$.*

Following [4] we denote by $\overline{\mathcal{TA}}$ the family of all theories of Abelian groups. For a theory $T \in \overline{\mathcal{TA}}$ we consider the *support* $\mathrm{Supp}(T)$ of T, i.e., the set of positive Szmielew invariants for T. Now we denote by $\mathcal{FS}$ (respectively, $\mathcal{CFS}$) the set of all theories in $\overline{\mathcal{TA}}$ having (co)finite supports. By $\mathcal{ICIS}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ having infinite and co-infinite supports. By $\mathcal{F}$ we denote the set of all theories in $\overline{\mathcal{TA}}$ with finite Szmielew invariants, and by $\mathcal{INF}$ — with infinite Szmielew invariants $\alpha_{p,n} > 0$, $\beta_p > 0$, $\gamma_p > 0$.

Notice that $\mathrm{Cl}_E(\mathcal{FS}) = \overline{\mathcal{TA}}$ and $\mathrm{Cl}_E(\mathcal{CFS}) = \overline{\mathcal{TA}}$ implying $\mathrm{Cl}_E(\mathcal{ICIS}) = \overline{\mathcal{TA}}$. Note also that $\mathrm{Cl}_E(\mathcal{F}) = \overline{\mathcal{TA}}$ while $\mathcal{INF}$ is E-closed.

We denote by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$ the classes of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by $\alpha_{p,n}$, β_p, γ_p, ε, respectively. For $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U} \in \{\mathbf{A}, \mathbf{B}, \mathbf{\Gamma}, \mathbf{E}\}$ we denote by $\mathbf{XY}$, $\mathbf{XYZ}$, $\mathbf{XYZU}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}$, β_p, γ_p, ε for $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}$.

For $\mathbf{X}$ as above and for a sequence S of some Szmielew invariants, we write $\mathbf{X}_S$ for the set of of all theories T in $\mathbf{X}$ such that Szmielew invariants for T are equal to corresponding values in S. If the sequences S do not have finite positive values we denote by $\mathbf{X}^\infty$ the union of these $\mathbf{X}_S$. If for a subset P_0 of the set P of all prime numbers the sequences S do not have positive values for $p \in P \setminus P_0$ we denote by $\mathbf{X}_{P_0}$ the union of these $\mathbf{X}_S$. We write $\mathbf{X}_p$ instead $\mathbf{X}_{P_0}$ if P_0 is a singleton $\{p\}$.

As above we denote by $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}$, $\mathbf{X}_{P_0}\mathbf{Y}_{P'_0}\mathbf{Z}_{P''_0}\mathbf{U}_{P'''_0}$, respectively, the set of all theories in $\overline{\mathcal{TA}}$ whose positive Szmielew invariants are exhausted by corresponding $\alpha_{p,n}, \beta_p, \gamma_p, \varepsilon$ for $\mathbf{X}_{P_0}, \mathbf{Y}_{P'_0}, \mathbf{Z}_{P''_0}, \mathbf{U}_{P'''_0}$.

Recall [11] that a group $\mathcal{A}$ is *divisible* if for any natural $n > 0$ and any element $a \in A$ the equation $nx = a$ has a solution in $\mathcal{A}$.

Theorem 2.6 [11]. *Any divisible subgroup $\mathcal{A}$ of Abelian group $\mathcal{B}$ is a direct summand in $\mathcal{B}$.*

Theorem 2.7 [11]. *Any nonzero divisible Abelian group $\mathcal{A}$ is represented as a direct sum of groups isomorphic to $\mathbf{Q}$ or $\mathbf{Z}_{p^\infty}$.*

Recall [11, 12] that a group $\mathcal{A}$ is *bounded* if there is a positive number n such that $n\mathcal{A} = \{0\}$. Otherwise the group $\mathcal{A}$ is called *unbounded*. A group $\mathcal{A}$ is *torsion-free* if all nonunit elements have infinite order.

Proposition 2.8 [4]. 1. $\mathbf{AB} = \mathbf{A\Gamma} = \mathbf{AB\Gamma} = \mathbf{A} \subset \mathcal{FS}$, $\mathbf{A}$ *is divided into $\mathbf{A} \cap \mathcal{F}$, consisting of theories with finite models, and $\mathbf{A} \setminus \mathcal{F}$, consisting of theories with infinite bounded models.*

2. $\mathbf{B} = \mathbf{\Gamma} = \mathbf{B\Gamma} = \mathbf{O}$, *where $\mathbf{O}$ is a singleton consisting of the one-element group.*

3. $\mathbf{AE}$ *consists of theories T without β_p and γ_p in $\mathrm{Supp}(T)$ and such that sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p, i.e., with bounded quotients with respect to maximal divisible subgroups.*

4. $\mathbf{BE}$ *consists of all theories of divisible Abelian groups.*

5. $\mathbf{\Gamma E}$ *consists of all theories of torsion-free Abelian groups.*

6. $\mathbf{ABE}$ *consists of all theories of Abelian groups $\mathcal{A}$ with bounded quotients relative to maximal divisible subgroups $\mathcal{B}$, i. e., the theories $\mathrm{Th}(\mathcal{A}/\mathcal{B})$ form the set $\mathbf{A}$.*

7. $\mathbf{A\Gamma E}$ *consists of all theories of Abelian groups without β_p in $\mathrm{Supp}(T)$ and such that the sets $\{n \mid \alpha_{p,n} \neq 0\}$ are finite for each p.*

8. $\mathbf{B\Gamma E}$ *consists of all theories of Abelian groups such that quotients with respect to maximal divisible subgroups are torsion-free.*

9. $\mathbf{AB\Gamma E} = \overline{\mathcal{TA}}$.

Proposition 2.9 [4]. 1. *For any $P_0 \subseteq P$, $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0} (\mathbf{A}_p \cap \mathcal{F})\right) = \mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0} \mathbf{A}_p\right) = \bigcup\limits_{p\in P_0} \mathbf{A}_p\mathbf{B}^\infty_p\mathbf{\Gamma}^\infty_p\mathbf{E}$ with the least generating set $\bigcup\limits_{p\in P_0} (\mathbf{A}_p \cap \mathcal{F})$.*

2. *For any nonempty $P_0 \subseteq P$, $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}_{P_0}) = \mathbf{A}_{P_0}\mathbf{B}^\infty_{P_0}\mathbf{\Gamma}^\infty_{P_0}\mathbf{E}$ with the least generating set $\mathbf{A}_{P_0} \cap \mathcal{F}$; $\mathrm{Cl}_E\left(\bigcup\limits_{p\in P_0} (\mathbf{A}_p \cap \mathcal{F})\right)$ is a subset of $\mathrm{Cl}_E(\mathbf{A}_{P_0} \cap \mathcal{F})$ which is proper if and only if $|P_0| \geq 2$.*

3. $\mathrm{Cl}_E(\mathbf{A} \cap \mathcal{F}) = \mathrm{Cl}_E(\mathbf{A}) = \mathbf{AB}^\infty\mathbf{\Gamma}^\infty\mathbf{E}$ *with the least generating set $\mathbf{A} \cap \mathcal{F}$.*

3 Approximations and accumulation points

Theorem 3.1. *For any infinite family $\mathcal{T} \dot\cup \{T\}$ of theories of Abelian groups the following conditions are equivalent:*

(1) *T is $\mathcal{T}$-approximated;*

(2) *T has an infinite model and for any finite set of Szmielew invariants ξ for T there are infinitely many theories $T_k \in \mathcal{T}$, $k \in \omega$, such that each ξ either coincides for all T_k and for T, or ξ for T is a limit of corresponding Szmielew invariants for T_n (either of same name ξ or as a limit for $\alpha^{T_k}_{p,n}$).*

Proof. If T is approximable then any group $\mathcal{A} \models T$ is infinite by Theorem 2.4. Assuming that T is $\mathcal{T}$-approximated, we have a countable e-minimal subfamily $\{T_k \mid k \in \omega\} \subseteq \mathcal{T}$ such that T is unique accumulation point for $\{T_k \mid k \in \omega\}$. Since T is uniquely defined by its Szmielew invariants, Theorem 2.5 produces approximations of Szmielew invariants for T by corresponding Szmielew invariants for T_k modulo Proposition 2.1. Since each sentence contains an information about finitely many Szmielew invariants, that approximation of T by T_k is reduced to finite sets of Szmielew invariants producing the item (2).

Now assuming that (2) holds, we obtain that T is an accumulation point for $\{T_k \mid k \in \omega\}$ through approximations of Szmielew invariants for T by Szmielew invariants for T_k, since again T is uniquely defined by its Szmielew invariants, written by a set of sentences in T. Thus, T is $\{T_k \mid k \in \omega\}$-approximated and therefore $\mathcal{T}$-approximated in view of $\{T_k \mid k \in \omega\} \subseteq \mathcal{T}$ and the preservation of approximation under superfamilies. Theorem is proved.

Now we consider some applications of Theorem 3.1 to the families $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{E}$ on a base of Proposition 2.8.

Corollary 3.2. *For any infinite family $\mathcal{T} \subseteq \mathbf{A}$, $\mathcal{T}$ has an accumulation point which is a pseudofinite theory either in $\mathbf{A}$, with some infinite $\alpha_{p,n}$, or in $\mathbf{AE}$, with $\varepsilon = 1$ and infinitely many $\alpha_{p,n}$ all of which are bounded, or a theory with some $\beta_p = \gamma_p = \omega$ and infinite $\{n \mid \alpha_{p,n} \neq 0\}$.*

Proof. We can assume that $\mathcal{T}$ is not E-closed. By Proposition 2.1 and Theorem 2.3, 3.1 we consider the following possibilities for approximations of $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$ by e-minimal $\{T_k \mid k \in \omega\} \subseteq \mathcal{T}$:

(1) $\alpha^T_{p,n} = \omega = \lim\limits_{k\to\infty} \alpha^{T_k}_{p,n}$,

(2) $\beta^T_p = \gamma^T_p = \omega = \lim\limits_{n} \alpha^{T_k}_{p,n}$,

(3) $\varepsilon^T = 1 = \lim\limits_{p,n} \alpha^{T_k}_{p,n}$.

If $T \in \mathbf{A}$ then it is obtained by theories T_k whose supports are exhausted by some finite set and the only possibilities for the limits are described in (1). If $T \notin \mathbf{A}$ then either we have the limit (3) with infinite $\{\langle p, n\rangle \mid \alpha^T_{p,n} \neq 0\}$ and bounded $\alpha^T_{p,n}$ producing $T \in \mathbf{AE}$, or we have the limit (2) with infinite $\{n \mid \alpha^T_{p,n} \neq 0\}$. In the latter case T has Szmielew invariants $\beta^T_p = \gamma^T_p = \omega$. Corollary is proved.

Example 3.3. Taking theories $T_k = \mathrm{Th}(\mathbf{Z}_{p_k})$, where p_k is the k-th prime number, we obtain the e-minimal family $\{T_k \mid k \in \omega\}$ approximating $T = \mathbf{Th}(\mathbf{Q})$: $\varepsilon^T = 1 = \lim\limits_{p,n} \alpha^{T_k}_{p,n}$ and all other Szmielew invariants equal 0.

Corollary 3.2 implies that infinite families $\mathcal{T} \subseteq \mathbf{AE}$ can not approximate more theories than subfamilies of $\mathbf{A}$ and thus we immediately have:

Corollary 3.4. *For any infinite family $\mathcal{T} \subseteq \mathbf{AE}$, $\mathcal{T}$ has an accumulation point which is a pseudofinite theory either in $\mathbf{A}$, with some infinite $\alpha_{p,n}$, or in $\mathbf{AE}$, with $\varepsilon = 1$ and infinitely many $\alpha_{p,n}$ all of which are bounded, or a theory with some $\beta_p = \gamma_p = \omega$ and infinite $\{n \mid \alpha_{p,n} \neq 0\}$.*

Remark 3.5. By Corollaries 3.2 and 3.4, $\mathrm{Cl}_E(\mathbf{A}) = \mathrm{Cl}_E(\mathbf{AE})$ consists of theories with arbitrary values $\alpha_{p,n}$ and such that positive β_p or γ_p imply $\beta_p = \gamma_p = \omega$ that confirms Proposition 2.9.

Corollary 3.6. *For any $P_0, P_1 \subseteq P$ each infinite family $\mathcal{T} \subseteq \mathbf{B}_{P_0}\mathbf{\Gamma}_{P_1}\mathbf{E}$, $\mathcal{T}$ has an accumulation point T which is again in $\mathbf{B}_{P_0}\mathbf{\Gamma}_{P_1}\mathbf{E}$, and each value β^T_p (respectively, γ^T_p) is either approximated by values β_p (γ_p) for theories in $\mathcal{T}$ or equals infinitely many these values.*

Proof. By Theorem 3.1 the family $\mathcal{T}$ can approximate values β^T_p and γ^T_p such that the values β_p for theories in $\mathcal{T}$ do not generate positive $\alpha_{p,n}$ and γ^T_p, and the values γ_p do not generate positive $\alpha_{p,n}$ and β^T_p. Thus for any accumulation point T, each value β^T_p (respectively, γ^T_p) is either approximated by values β_p (γ_p) for theories in $\mathcal{T}$ or equals infinitely many these values.

Corollary 3.6 immediately implies:

Corollary 3.7. *For any $P_0, P_1 \subseteq P$ the family $\mathcal{T} \subseteq \mathbf{B}_{P_0}\mathbf{\Gamma}_{P_1}\mathbf{E}$ is E-closed. In particular, the set $\mathbf{BE}$ of divisible Abelian groups is E-closed, and the set $\mathbf{\Gamma E}$ of torsion-free Abelian groups is E-closed.*

Combining Proposition 2.9 and Corollary 3.7 we obtain:

Proposition 3.8. *For any $P_0, P_1, P_2 \subseteq P$, $\mathrm{Cl}_E(\mathbf{A}_{P_0}\mathbf{B}_{P_1}\mathbf{\Gamma}_{P_2}\mathbf{E}) = \mathbf{A}_{P_0}\mathbf{B}^\infty_{P_0}\mathbf{\Gamma}^\infty_{P_0}\mathbf{B}_{P_1}\mathbf{\Gamma}_{P_2}\mathbf{E}$.*

4 Illustrations

It is known that the theory $T = \mathrm{Th}(\mathbf{Z})$ of the group $\mathbf{Z}$ of integers is not pseudofinite [8] and has Szmielew invariants $\alpha^T_{p,n} = 0$, $\beta^T_p = 0$, $\gamma^T_p = 1$, $\varepsilon^T = 1$, $n \in \omega$, $p \in P$, [13]. In view of Theorem 3.1 the theory T is approximated by theories whose Szmielew invariants converge to $\gamma_p = 1$, $p \in P$, with torsion-free parameters $\alpha_{p,n} = 0$ and $\beta_p = 0$. Thus, approximating T we have to consider a family $\mathcal{T}$ with an e-minimal subfamily $\mathcal{T}'$ such that for $\mathcal{T}'$ any γ_p equals 1 for almost all theories $\mathcal{T}'$, and $\alpha_{p,n} = 0$, $\beta_p = 0$ for almost all theories $\mathcal{T}'$. For instance, T is approximated by the e-minimal family $\{T_m \mid m \in \omega\}$ of theories of torsion-free groups, where $\gamma^{T_m}_p = 1$ for prime numbers $p = p_k$, $k \leq m$, and $\gamma^{T_m}_p = 0$ for $p = p_k$ with $k > m$.

Similarly, there are natural approximations of theories with finite β_p and γ_p. Taking, for instance, the theory T^r of the torsion-free group $\mathbf{Z}^r$, r-th direct product for copies of the group $\mathbf{Z}$, $r \in \omega \setminus \{0,1\}$, we have $\gamma_p = r$. All possibilities for approximations of T^r are reduced to the convergence step-by-step to these finite invariants γ_p. For example, we can consider the theory T^r as the limit of theories T^r_m, $m \in \omega$, where $\gamma_{p_k} = r$ for T^r_m with $k \leq m$, and $\gamma_{p_k} = 0$ for T^r_m with $k > m$.

The described approximations can be adapted for an arbitrary theory T of an Abelian group with an infinite support reducing T to a sequence of theories with finite supports.

5 Conclusion

We characterized and described the approximability and approximations of theories of Abelian group both in general and for some natural classes of Abelian groups. The machinery of that approximation in terms of Szmielew invariants as well as connections of theories by these invariants are clarified. The approximability by finite Abelian groups, by divisible and torsion-free Abelian groups with respect to various sets of prime numbers as well as closures for families of theories of these groups are described. Some illustrations for approximations of theories of Abelian groups are shown.

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