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Ellipsoidal Approximation of Distributions and Its Applications

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Abstract Analytical methods of the mathematical statistics of random vectors and matrices based on the parametrization of the distributions are widely used. These methods permit to design practically simple software when it is possible to have the definite information about analytical properties of the distributions under research. The main difficulty in practical applications of the methods based on the parametrization of the distributions is the rapid increase of the number of equations for the moments, the semiinvariants or the coefficients of the truncated orthogonal expansions of the dimension r or the state vector (extended in the general case) and the maximal order N of the moments involved. The number of equations for the parameters becomes exceedingly large in such cases. For structural parametrization and/or approximation of the probability densities of the random vectors we shall apply the ellipsoidal densities, i.e. the densities whose planes of the levels of equal probability are similar concentric ellipsoids (the ellipses for two-dimensional vectors, the ellipsoids for three-dimensional vectors, the hyperellipsoids for the vectors of the dimension more than three). In particular, a normal distribution in any finite-dimensional space has an ellipsoidal structure. The distinctive characteristics of such distributions consists in the fact that their probability densities are the functions of positively determined quadratic form $u = u(y) = (y^T - m^T)C(y - m)$ where m is an expectation of the random vector Y, C is some positively determined matrix. Ellipsoidal approximation method (EAM) cardinally reduces the number of parameters till $Q^{EAM} = Q^{NAM} + n_m - 1$ ($Q^{NAM} = r(r+3)/2$) where $2n_m$ being the number of probabilistic moments. While using ellipsoidal linearization method (ELM) we get $Q^{ELM} = Q^{NAM}$. Basic EAM and ELM foundations and applications to problems of mathematical statistics and ellipsoidal distributions with invariant measure in populational Volterra differential stochastic nonlinear systems are considered.

Keywords Distributions with Invariant Measure, Ellipsoidal Approximation Method (EAM), Ellipsoidal Linearization Method (ELM), Generalized Student Distribution, Gaussian (Normal) Distribution, Wishart Distribution, Volterra Stochastic Systems, χ^2- and β -Distributions

1. Introduction

Among the various qualitative analytical methods of the mathematical statistics of random vectors and matrices based on the parametrization of the distributions are widely used. These methods permit to design practically simple software when it is possible to have the definite information about analytical properties of the distributions under research.

While deriving equations for the parameters of the one-dimensional distribution in specific problems it is useful to remember that the number N_q^r of the moments of the q^{th} order of the r-dimensional random vector and the total number P_q^N of the moments of orders not exceeding N of the r-dimensional random vector are determined by formulae

$$N_q^r = C_{q+r-1}^r = \frac{(q+r-1)!}{r!(q-1)!}, \tag{1}$$

$$P_q^N = \sum_{r=1}^{N} N_q^r = C_{N+q}^N - 1 = \frac{(N+q)!}{N!q!} - 1. \tag{2}$$

So the main difficulty in practical applications of the methods based on the parametrization of the distributions is the rapid increase of the number of equations for the moments, the semiinvariants or the coefficients of the truncated orthogonal expansions on the dimension r of the state vector (extended in the general case) and the maximal order N of the moments involved. Table 1 [1–4] illustrates the dependence of the number of equations for the parameters of the one-dimensional distribution on r and N. But in practice one often has to deal with the cases where r amounts to 100 or even to 200. The number of equations for the parameters becomes exceedingly large in such cases. For instance, at $r=100$, $N=6$ the number of equations for the parameters amounts to $1.7\cdot 10^9$.

To reduce the number of equations for the distribution parameters, one may use such an approximation of the distribution which involves the mixed moments or semiinvariants only of the second order and is independent of the mixed moments or semiinvariants of higher orders. Table 1 shows the dependence of the number of equations for the parameters of the distribution on the dimension r of the state vector and the highest order N of the moments of every component involved into such an approximation. The line corresponding to $N=2$ shows the number of equations for the first and the second order moments for the normal approximation method. The number of equations for the distribution parameters may be greatly reduced by using the approximations of distributions involving mixed moments or semiinvariants only of the second order. In particular, in the above example of $r=100$, $N=6$ the number of equations is reduced from $1.7\cdot 10^9$ to 5500, i.e. to approximately $1/(3\cdot 10^5)$ of its primary value. At $r=10$, $N=6$ the number of equations is reduced from 8007 to 105, i.e. to approximately 1/80 of its primary value.

Normal approximation method (NAM) includes

$$Q^{NAM} = r(r+3)/2 \tag{3}$$

parameters. Formulae (1), (2) and Table 1 give the number of distribution parameters for general parametrization method [1–4].

For structural parametrization and/or approximation of the probability densities of the random vectors we shall apply the ellipsoidal densities, i.e. the densities whose planes of the levels of equal probability are similar concentric ellipsoids (the ellipses for two-dimensional vectors, the ellipsoids for three-dimensional vectors, the hyperellipsoids for the vectors of the dimension more than three). In particular, a normal distribution in any finite-dimensional space has an ellipsoidal structure. The distinctive characteristics of such distributions consists in the fact that their probability densities are the functions of positively determined quadratic form $u=u(y)=(y^T-m^T)C(y-m)$ where m is an expectation of the random vector Y, C is some positively determined matrix.

Ellipsoidal approximation method (EAM) cardinally reduces the number of parameters till

$$Q^{EAM} = Q^{NAM} + n_m - 1 \tag{4}$$

where $2n_m$ being the number of probabilistic moments.

While using ellipsoidal linearization method (ELM) we get

$$Q^{ELM} = Q^{NAM}. \tag{5}$$

The developed methods cardinally differs from of known methods of parametric statistics by using structural ellipsoidal density approximation.

Let us consider EAM and ELM foundations and applications to some basic problems of mathematical statistics and populational Volterra stochastic systems.

Table 1. Number of distribution parameters

Moments Max. Order N	State Vector Dimension r							
	10	20	30	40	50	100	150	200
2	65	230	495	860	1325	5150	11475	20300
4	85	270	555	840	1425	5350	11475	20700
6	105	310	615	1020	1525	5550	12075	21100
8	125	350	675	1100	1625	5750	12375	21500
10	145	390	735	1180	1725	5950	12675	21900

2. Ellipsoidal Approximation Method

Following [1–4] let us find ellipsoidal approximation (EA) for the density of r -dimensional random vector by means of the truncated expansion based on biorthogonal polynomials $\{p_{r,\nu}(u(y)), q_{r,\nu}(u(y))\}$, depending only on the quadratic form $u=u(y)$ $u=u(y)$ for which some probability density of the ellipsoidal structure $w(u(y))$ serves as the weight:

$$\int_{-\infty}^{\infty} w(u(y))\, p_{r,\nu}(u(y)) q_{r,\mu}(u(y)) dy = \delta_{\nu\mu}. \quad (6)$$

The indexes ν and μ at the polynomials mean their degrees relatively to the variable u. The concrete form and the properties of the polynomials are determined further. But without the loss of generality we may assume that $q_{r,0}(u) = p_{r,0}(u) = 1$. Then the probability density of the vector Y may be approximately presented by the expression of the form

$$f(y) \approx f^*(u) = w(u) \sum_{\nu=0}^{N} c_{r,\nu} p_{r,\nu}(u). \quad (7)$$

For determining the coefficients $c_{r,\nu}$ in (7) we multiply term-wise equality (7) by $q_{r,\mu}(u)$ and integrate over the whole space R^r. Then by virtue of condition (6) for $\mu = 1,\dots,N$ we obtain

$$\int_{-\infty}^{\infty} f(y) q_{r,\mu}(u) dy = \int_{-\infty}^{\infty} w(u) \sum_{\nu=0}^{N} c_{r,\nu} p_{r,\nu}(u) q_{r,\mu}(u) dy = c_{r,\mu}.$$

Thus the coefficients $c_{r,\nu}$ are determined by formula

$$c_{r,\nu} = \int_{-\infty}^{\infty} f(y) q_{r,\nu}(u) dy = Eq_{r,\nu}(U), \quad (\nu = 1,\dots,N). \quad (8)$$

As $p_{r,0}(u)$ and $q_{r,0}(u)$ are reciprocal constants (the polynomials of zero degree) then always $c_{r,0} p_{r,0} = 1$ and we come to the result:

$$f(y) \approx f^*(u) = w(u)\left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u)\right]. \quad (9)$$

So formula (9) expresses the essence of the EA of the probability density of the random vector Y.

For the applications the case when the normal distribution is chosen as the distribution $w(u)$ is of great importance

$$w(u) = w(x^T C x) = \frac{1}{\sqrt{(2\pi)^r |K|}} \exp(-x^T K^{-1} x/2); \quad (10)$$

accounting that $C = K^{-1}$ we reduce the condition of the biorthonormality (6) to the form

$$\frac{1}{2^{r/2}\Gamma(r/2)} \int_0^{\infty} p_{r,\nu}(u) q_{r,\mu}(u) u^{r/2-1} e^{-u/2} du = \delta_{\nu\mu}, \quad (11)$$

where $\Gamma(\cdot)$ is gamma function [5]. Thus the problem of the choosing of the polynomials system $\{p_{r,\nu}(u) q_{r,\mu}(u)\}$ which is used at the EA of the densities (9) and (8) is reduced to finding a biorthonormal system of the polynomials for which the χ^2 -distribution with r degrees of the freedom serves as the weigh.

A system of the polynomials which are orthogonal relatively to χ^2 -distribution with r degrees of the freedom is described by series:

$$S_{r,\nu}(u) = \sum_{\mu=0}^{\nu} (-1)^{\nu+\mu} C_\nu^\mu \frac{(r+2\nu-2)!!}{(r+2\mu-2)!!} u^\mu. \quad (12)$$

The main properties of polynomials $S_{r,\nu}$ are given in [2–4] between the polynomials $S_{r,\nu}(u)$ and the system of the polynomials $\{p_{r,\nu}(u), q_{r,\mu}(u)\}$ the following relations exist:

$$p_{r,\nu}(u) = S_{r,\nu}(u), \quad (13)$$

$$q_{r,\nu}(u) = \frac{(r-2)!!}{(r+2\nu-2)!!(2\nu)!!} S_{r,\nu}(u), \quad r \ge 2. \quad (14)$$

Example 1. Formulae for polynomials $p_{r,\nu}(u)$, $q_{r,\nu}(u)$ and its derivatives for some r and ν are as follows [4]:

at $r = 2$, $\nu \ge 2$,

$$p_{2,\nu}(u) = u^\nu, \quad q_{2,\nu}(u) \equiv 0, \quad q'_{2,\nu}(u) \equiv 0, \quad q''_{2,\nu}(u) \equiv 0;$$

at $r \ge 2$, $\nu = 2$

$$p_{r,2}(u) = u^2, \quad q_{r,2}(u) = \frac{1}{8}u^2, \quad q'_{r,2}(u) = \frac{1}{4}u, \quad q''_{r,2}(u) = \frac{1}{4}.$$

For $r = 2$ at $\nu = 3$ we have

$$p_{2,3}(u) = u^3, \quad q_{2,3}(u) \equiv 0, \quad q'_{2,3}(u) \equiv 0, \quad q''_{2,3}(u) \equiv 0;$$

at $r = 3$

$$\left.\begin{aligned} & p_{3,3}(u) = S_{3,3}(u), \quad q_{3,3}(u) = \frac{1}{5040} S_{3,3}(u), \\ & q'_{3,3}(u) = \frac{1}{5040} S'_{3,3}(u), \quad q''_{3,3}(u) = \frac{1}{5040} S''_{3,3}(u), \\ & S_{3,3}(u) = -105 + 105u - 21u^2 + u^3, \\ & S'_{3,3}(u) = 105 - 42u + 3u^2, \quad S''_{3,3}(u) = -42 + 6u; \end{aligned}\right\}$$

at $r = 4$:

$$\left.\begin{aligned} &p_{4,3}(u)=S_{4,3}(u), \quad q_{4,3}(u)=\frac{1}{9216}S_{4,3}(u), \\ &q'_{4,3}(u)=\frac{1}{9216}S'_{4,3}(u), \quad q''_{4,3}(u)=\frac{1}{9216}S'_{4,3}(u), \\ &S_{4,3}(u)=-197+144u-24u^2+u^3, \\ &S'_{4,3}(u)=144-48u+3u^2, \quad S''_{4,3}(u)=-48+6u. \end{aligned}\right\}$$

Now following [5] we consider the H -space $L_2(R^r)$ and the orthogonal system of the functions in them where the polynomials $S_{r,\nu}(u)$ are given by formula (12), and $w(u)$ is a normal distribution of the r -dimensional random vector (10). This system is not complete in $L_2(R^r)$. But the expansion of the probability density $f(u)=f((y^T-m^T)C(y-m))$ of the random vector Y which has an ellipsoidal structure over the polynomials $p_{r,\nu}(u)=S_{r,\nu}(u)$,

$$f(u)=w(u)\sum_{\nu=0}^{\infty}c_{r,\nu}\,p_{r,\nu}(u), \tag{15}$$

m.s. converges to the function $f(u)$ itself. The coefficients of the expansion in this case are determined by relation

$$c_{r,\nu}=\int_{-\infty}^{\infty}f(u)p_{r,\nu}(u)dy\Big/\frac{(2\nu)!!(r+2\nu-2)!!}{(r-2)!!}. \tag{16}$$

Thus the system of the functions $\{\sqrt{w(u)}S_{r,\nu}(u)\}$ forms the basis in the subspace of the space $L_2(R^r)$ generated by the functions $f(u)$ of the quadratic form $u=(y-m)^T C(y-m)$.

At the expansion over the polynomial $S_{r,\nu}(u)$ the probability densities of the random vector Y and all its possible projections are consistent. In other words, at integrating the expansions over the polynomials $S_{h+l,\nu}(u)$, $h+l=r$, of the probability densities of the r -dimensional vector Y,

$$f(y)=\frac{1}{\sqrt{(2\pi)^{h+l}\,|K|}}e^{-u/2}\left[1+\sum_{\nu=2}^{N}c_{h+l,\nu}S_{h+l,\nu}(u)\right],$$

$$u=(y-m)^T K^{-1}(y-m), \quad y=[y'^T\;y''^T]^T, \tag{17}$$

on all the components of the l -dimensional vector y'' we obtain the expansion over the polynomials $S_{h,\nu}(u_1)$ of the probability density of the h -dimensional vector Y' with the same coefficients,

$$f(y')=\frac{1}{\sqrt{(2\pi)^{h}\,|K_{11}|}}e^{-u_1/2}\left[1+\sum_{\nu=2}^{N}c_{h,\nu}S_{h,\nu}(u_1)\right],$$

$$u_1=(y'-m')^T K_{11}^{-1}(y'-m'), \quad c_{h,\nu}=c_{h+l,\nu}, \tag{18}$$

where K_{11} is a covariance matrix of the vector Y'.

But approximation (18) the probability density of h -dimensional random vector Y' obtained by the integration of expansion (17) the density of $(h+l)$ -dimensional vector is not optimal EA of the density.

For the random r -dimensional vector with an arbitrary distribution the EA (9) of its distribution determines exactly the moments till the N^{th} order inclusively of the quadratic form $U=(Y-m)^T K^{-1}(Y-m)$, i.e.

$$EU^{\mu}=E^{EA}U^{\mu}, \quad \mu\le N. \tag{19}$$

(E^{EA} stands for expectation relative to EA distribution).

The EA of any distribution determines exactly the expectation of the polynomials $p_{r,0}(u), q_{r,0}(u), \dots, p_{r,N}(u)$, $q_{r,N}(u)$. In this case the initial moments of the order s, $s=s_1+\cdots+s_r$ of the random vector Y at the approximation (9) determined by formula

$$\alpha_{s_1,\dots,s_r}=\alpha_s=EY_1^{s_1}\dots Y_r^{s_r}\approx$$

$$\approx\int_{-\infty}^{\infty}y_1^{s_1}\dots y_r^{s_r}w(u)dy+\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}y_1^{s_1}\dots y_r^{s_r}p_{r,\nu}(u)w(u)dy. \tag{20}$$

Thus at the EA of the distribution of the random vector its moments are combined as the sums of the correspondent moments of the normal distribution and the expectations of the products of the polynomials $p_{r,\nu}(u)$ by the degrees of the components of the vector Y at the normal density $w(u)$.

3. EAM Accuracy

In practice the weak convergence of the probability measures generated by the segments of the density expansion to the probability measure generated by the density itself is more important than m.s. convergence of the segments of the density expansion over the polynomials $S_{r,\nu}(u)$ to the density namely,

$$\int_A w(u)\sum_{\nu=0}^{N}c_{r,\nu}\,p_{r,\nu}(u)dy\to\int_A f(u)dy$$

uniformly relative to A at $N\to\infty$ on the σ -algebra of Borel sets of the space R^r. Thus the partial sums of series (15) give the approximation of the distribution, i.e. the probability of any event A determined by the density $f(u)$ with any degree of the accuracy. The finite segment of this expansion may be practically used for an approximate presentation of $f(u)$ with any degree of the accuracy even in those cases when $f(u)/\sqrt{w(u)}$ does not belong to $L_2(R^r)$.

In this case it is sufficient to substitute $f(u)$ by the truncated density. Expansion (15) is valid only for the densities which have the ellipsoidal structure. It is impossible in principal to approximate with any degree of the accuracy by means of the EA (9) the densities which arbitrarily depend on the vector y.

The most natural one is the way of the estimate of the accuracy of the distribution approximation in the comparison of the probability characteristics calculated by means of the known density and its approximate expression. The most complete estimate of the accuracy of the approximation may be obtained by the comparison of the probability occurrence on the sets of some given class. Besides that taking into consideration that the probability density is usually approximated by a finite segment of its orthogonal expansion for instance, over Hermite polynomials or by a finite segment of the Edgeworth series [1–5] which contain the moments till the fourth order the accuracy may be characterized by the accuracy of the definition of the moments of the random vector or its separate components, in particular, of the fourth order moments. Let compare the accuracy of EAM on the probabilities of the occurrence on the sets. Let $\mathscr{A}$ be class of the sets $\mathscr{A}$ for which it is required to estimate the accuracy of the definition of the occurrence probability. The error of the definition of the occurrence probability on Borel set $\mathscr{A}$ at the approximation of the density of the r-dimensional random vector (9) is equal to

$$\varepsilon = \int_A f^*(y)dy - \int_A f(y)dy. \qquad (21)$$

The measure of the accuracy of the approximation we assume the variable

$$\tilde{\varepsilon} = \sup_{A \subset A} \left| \int_A f^*(y)dy - \int_A f(y)dy \right|. \qquad (22)$$

Practically we may evaluate the values $\tilde{\varepsilon}$ only in the case when the class of the sets $\mathscr{A}$ contains a small number of the sets. So, for example, if as a class of the sets $\mathscr{A}$ the σ-algebra of Borel sets is chosen then the finding of ε is practically impossible. Therefore it is expedient to assume the finite systems of the typical regions as a class of the sets $\mathscr{A}$. It is natural to choose as such regions the ellipsoids in the space R^r, the ellipsoidal cylinders with the bases in the subspace of the space R^r, the layers formed by the hyperplanes of the dimension $r-1$ for the EAM distributions. These layers may be considered as the ellipsoidal cylinders with the one-dimensional bases. For calculating the probabilities of the occurrences into the ellipsoidal cylinders with the bases in the subspaces R^h of the space R^r we have to use the EA of the distribution of the correspondent h components of the vector Y.

Let E_0 be the ellipsoid $(y-m)^T C(y-m) < u_0$, $C = K^{-1}$. The probability of the occurrence in the ellipsoid E_0 at the ellipsoidal approximation (9) is equal to

$$p_0 = P(E_0) = \int_{E_0} f^{\text{EAM}}(y)dy = \int_{E_0} w(u)\left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u)\right]dy =$$

$$= \frac{|K|^{1/2}\pi^{1/2}}{\Gamma(r/2)} \int_0^{u_0} u^{r/2-1} w(u)\left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u)\right]du =$$

$$= \int_0^{u_0} \frac{u^{r/2-1}e^{-u/2}}{\Gamma(r/2)2^{r/2}}du + \sum_{\nu=2}^{N} c_{r,\nu} \int_0^{u_0} \frac{u^{r/2-1}e^{-u/2} p_{r,\nu}(u)}{\Gamma(r/2)2^{r/2}}du =$$

$$= \frac{1}{\Gamma(r/2)}\left[\gamma(r/2, u_0/2) - \frac{u_0^{r/2}e^{-u_0/2}}{2^{r/2-1}} \sum_{\nu=2}^{N} c_{r,\nu} p_{r+1,\nu-1}(u_0)\right], \qquad (23)$$

where $\gamma(x, y)$ is not complete γ function [5].

Thus for finding the ellipsoid E_0 whose probability of the occurrence $A = P(E_0)$ is given it is necessary to solve (23) relatively to u_0. The root of this equation we shall find by the Newton method known which gives the following recurrent formula for the sequential approximations for the equation $A(E_0) = \varphi(u_0)$, the root u_0:

$$u_0^{n+1} = u_0^n + \varphi'(u_0^n)^{-1}[A(E_0) - \varphi(u_0^n)] =$$

$$= u_0^n + 2^{r/2}\left(\frac{\Gamma(r/2)A(E_0) - \gamma(r/2, u_0^n/2)}{(u_0^n)^{r/2-1} e^{-u_0^n/2}\left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u_0^n)\right]} \right) +$$

$$+ 2u_0^n \frac{\sum_{\nu=2}^{N} c_{r,p} p_{r+1,\nu-1}(u_0^n)}{1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u_0^n)}. \qquad (24)$$

It is expedient to choose the value found from the equation $\gamma(r/2, u_0^0/2) = A(E_0)\Gamma(r/2)$ as zero approximation of the variable u_0.

Now we consider the second way of EAM estimation of the accuracy. It consists in the comparison of the values of the moments calculated by means of the known distribution and its ellipsoidal approximation. So, for example, the error of the definition of the initial moment of the order d at the ellipsoidal approximation (24) of the probability density of the r-dimensional random vector will be equal to

$$\delta_1(d_1, \dots, d_r) = \int_{-\infty}^{\infty} y_1^{d_1} \dots y_r^{d_r} f^{\text{EAM}}(y)dy - \int_{-\infty}^{\infty} y_1^{d_1} \dots y_r^{d_r} f(y)dy$$
$$= E_{\text{EAM}} Y_1^{d_1} \dots Y_r^{d_r} - EY_1^{d_1} \dots Y_r^{d_r}, \quad d_1 + \dots + d_r = d \qquad (25)$$

$$\delta_2(d_1,\dots,d_r)=\frac{\delta_1(d_1,\dots,d_r)}{EY_1^{d_1}\dots Y_r^{d_r}}. \tag{26}$$

The variable $\delta_1(d_1,\dots,d_r)$ represents an absolute error of the definition of the initial moment $EY_1^{d_1}\dots Y_r^{d_r}$; $\delta_2(d_1,\dots,d_r)$ represents a relative error.

Let the density $f(y)$ of the r-dimensional random vector with an arbitrary covariance matrix and the moments which exist till the order d inclusively be given. By means of the orthogonal transformation Ω we reduce the covariance matrix $K=[K_{ij}]$ to the diagonal form $\tilde{K}=\mathrm{diag}(\lambda_1,\dots,\lambda_r)=\Omega K\Omega^T$. Then $C=K^{-1}=[C_{ij}]$ will turn into the diagonal matrix $\tilde{C}=\tilde{K}^{-1}=$ diag$(1/\lambda_1,\dots,1/\lambda_r)$. Hence the quadratic form u will take the form

$$u=x^TK^{-1}x=\tilde{x}^T\tilde{K}^{-1}\tilde{x}=\sum_{i=1}^{r}\tilde{x}_i^2/\lambda_i,\quad x=y-m,\quad \tilde{x}=\Omega^Tx, \tag{27}$$

where m is an expectation of the vector Y. The probability density of the vector $\tilde{X}$ is expressed in terms of the density $f(y)$ of the vector Y by the formula

$$\tilde{f}(\tilde{x})=\frac{1}{|\Omega^T|}f(\Omega^{-1T}(\tilde{x}+\Omega^Tm)). \tag{28}$$

The estimation of the accuracy of the approximation we shall perform by the comparison of the values of the moments of the vector $\tilde{X}$. The detailed derivation and the calculations we shall perform for the moments of the fourth order $(d=4)$. While calculating the moments of an arbitrary component for instance, the first one of the r-dimensional vector $\tilde{X}$ by formula (20) and substituting density (28) by its EA we have

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{x}_1^4\tilde{f}(\tilde{x})d\tilde{x}_1\dots d\tilde{x}_r\cong\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{x}_1^4w(u)d\tilde{x}_1\dots d\tilde{x}_r+$$
$$+\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{x}_1^4p_{r,\nu}(u)w(u)d\tilde{x}_1\dots d\tilde{x}_r, \tag{29}$$

where $w(u)$ is the r-dimensional normal density. We integrate (29) over $\tilde{x}_2,\dots,\tilde{x}_r$. As a result we obtain

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{x}_1^4\tilde{f}(\tilde{x})d\tilde{x}_1\dots d\tilde{x}_r\cong\int_{-\infty}^{\infty}\tilde{x}_1^4w(u_1)d\tilde{x}_1+$$
$$+\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}\tilde{x}_1^4p_{1,\nu}(u_1)w(u_1)d\tilde{x}_1, \tag{30}$$

where $u_1=\tilde{x}_1^2/\lambda_1,\ w(u_1)$ is one-dimensional density. We express $\tilde{x}_1$ in terms of u_1 in the form $\tilde{x}_1^4=\lambda_1^2u_1^2$. Then the second item in (30) will be

$$\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}\lambda_1^2u_1^2p_{1,\nu}(u_1)w(u_1)d\tilde{x}_1. \tag{31}$$

For evaluating this integral we use property (21) of the orthogonality of the polynomial $p_{r,\nu}(u)$ to all degrees of u^λ at $\lambda<\nu$. According to (2.1), u_1^2 will be orthogonal to all polynomials $p_{1,\nu}(u_1)$ beginning with ν which is equal to 3. The sum in (31) will contain only one item correspondent to ν and equal to 2. After calculating the fourth moment of the normal distribution (29) will be written as

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{x}_1^4\tilde{f}(\tilde{x})d\tilde{x}_1\dots d\tilde{x}_r\cong 3D_1^2+c_{r,2}\int_{-\infty}^{\infty}\tilde{x}_1^4p_{1,2}(u_1)w(u_1)d\tilde{x}_1. \tag{32}$$

We shall determine the coefficients $c_{r,\nu}$ from relation (8). For further evaluations it is convenient to rewrite expression (14) for the polynomial $q_{r,\nu}(u)$ in other form:

$$q_{r,\nu}(u)=\frac{(r-2)!!}{(r+2\nu-2)!!(2\nu)!!}S_{r,\nu}(u)=$$
$$=\frac{(r-2)!!}{(r+2\nu-2)!!2^\nu\nu!}\sum_{\mu=0}^{\nu}\frac{(-1)^{\nu+\mu}\nu!(r+2\nu-2)!!}{\mu!(\nu-\mu)!(r+2\mu-2)!!}u^\mu=$$
$$=\frac{1}{2^\nu}\sum_{\mu=0}^{\nu}\frac{(-1)^{\nu+\mu}u^\mu}{\mu!(\nu-\mu)!r(r+2)\cdots(r+2\mu-2)}. \tag{33}$$

It is adopted in the latter formula that the product $r(r+2)\cdots(r+2\mu-2)$ obtained at the division of $(r-2)!!$ by $(r+2\mu-2)!!$ is equal to 1 at $\mu=0$.

Now we shall find the μ^{th} degrees of the quadratic form u written in the form of (28):

$$u^\mu=\sum_{\mu_1+\dots+\mu_r=\mu}\frac{\mu!}{\mu_1!\cdots\mu_r!}\frac{\tilde{x}_1^{2\mu_1}\cdots\tilde{x}_r^{2\mu_r}}{\lambda_1^{\mu_1}\cdots\lambda_r^{\mu_r}}. \tag{34}$$

We determine the expectation of expression (24):

$$EU^\mu=\sum_{\mu_1+\dots+\mu_r=\mu}\frac{\mu!}{\mu_1!\cdots\mu_r!}\frac{E\tilde{X}_1^{2\mu_1}\cdots\tilde{X}_r^{2\mu_r}}{\lambda_1^{\mu_1}\cdots\lambda_r^{\mu_r}}.$$

Then the expectations of the degrees U^μ and the whole polynomial $q_{r,\nu}(U)$ in new denotions will be correspondingly equal to

$$EU^\mu=\sum_{\mu_1+\dots+\mu_r=\mu}\mu!\gamma_{\mu_1\dots\mu_r},$$

$$c_{r,\nu}=Eq_{r,\nu}(U)=\frac{1}{2^\nu}\sum_{\mu=0}^{\nu}(-1)^{\mu+\nu}\frac{\sum_{\mu_1+\dots+\mu_r=\mu}\gamma_{\mu_1\dots\mu_r}}{(\nu-\mu)!r(r+2)\cdots(r+2\mu-2)}, \tag{35}$$

where

$$\gamma_{\mu_1\ldots\mu_r} = \frac{E\tilde{X}_1^{2\mu_1}\cdots\tilde{X}_r^{2\mu_r}}{\mu_1!\cdots\mu_r!\lambda_1^{\mu_1}\cdots\lambda_r^{\mu_r}}, \quad \gamma_{0\ldots 0} = 1. \quad (36)$$

For finding the fourth order moment of a separate component it is necessary to know only the value $c_{r,2}$. We evaluate separately each item in (36) putting $\nu = 2$. At $\mu = 0$ the numerator is equal to 1 and the first item is equal to $1/2$. As $D\tilde{X}_k = \lambda_k$ then at $\mu = 1$ we have

$$\frac{\sum_{\mu_1+\cdots+\mu_r=1}\gamma_{\mu_1\ldots\mu_r}}{1!r} = \frac{1}{r}\sum_{k=1}^{r}\gamma_{\underset{k}{0\ldots1,\ldots0}} = \frac{1}{r}\sum_{k=1}^{r}\frac{E\tilde{X}_k^2}{\lambda_k} = 1. \quad (37)$$

Finally, at $\mu = 2$ only two types of the expressions $\gamma_{\mu_1\ldots\mu_r}$ are possible. The first one appears when only one of μ_i is equal to 2, the others are zeros, the second is μ_i and μ_j are equal to 1, the others are zeros. As a result we obtain

$$c_{r,2} = \frac{1}{4}\left[-\frac{1}{2} + \frac{\sum_{i=1}^{r}\gamma_{\underset{r}{0\ldots2\ldots0}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\gamma_{\underset{i\quad j}{0\ldots10\ldots010\ldots0}}}{r(r+2)}\right], \quad (38)$$

where

$$\gamma_{\underset{i}{0\ldots2\ldots0}} = E\tilde{X}_i^4/\lambda_i^2, \quad \gamma_{\underset{i\quad j}{0\ldots1\ldots1\ldots0}} = E\tilde{X}_i^2\tilde{X}_j^2/\lambda_i\lambda_j. \quad (39)$$

For the second item in (32) we have

$$E_{w_1}p_{1,2}(U_1)\tilde{X}_1^4 = 2^2\cdot 2!R_{4,2} = 8R_{4,2} = 24D_1^2, \quad (40)$$

as

$$R_{4,2} = \sum_{|h|=4}C_4^h\mu_h^w\alpha_{4-h}^w = C_4^4\mu_4^w\alpha_0^w = \mu_4^w = 3D_1^2,$$

where μ_h^w and α_{4-h}^w are the central and the initial moments of the normal distribution $\boldsymbol{N}(0,\tilde{K})$ respectively.

Thus the EA of the r-dimensional distribution gives for the fourth moment of the first and the i^{th} coordinates

$$E_{\text{EA}}\tilde{X}_1^4 = 3D_1^2 + 24c_{r,2}D_1^2, \quad E_{\text{EA}}\tilde{X}_i^4 = 3D_i^2 + 24c_{r,2}D_i^2. \quad (41)$$

By virtue of (25) and (36) we have

$$\delta_1(4,0,\ldots,0) = E_{\text{EA}}\tilde{X}_1^4 - E\tilde{X}_1^4, \quad (42)$$

$$\delta_2(4,0,\ldots,0) = \frac{\delta_1(4,0,\ldots,0)}{E\tilde{X}_1^4}. \quad (43)$$

The fourth order moments of the vector may be also given by the expressions of the type $E\tilde{X}_i^2\tilde{X}_j^2$, $E\tilde{X}_i\tilde{X}_j^3$, $E\tilde{X}_i\tilde{X}_j\tilde{X}_k\tilde{X}_l$. We evaluate $E\tilde{X}_i^2\tilde{X}_j^2$. The remained two types of the moments are determined analogously. After choosing, for example, $i = 1$ and $j = 2$, we write

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2\tilde{f}(\tilde{x})d\tilde{x}_1\ldots d\tilde{x}_r \cong \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2 w(u)d\tilde{x}_1\ldots d\tilde{x}_r + $$

$$+\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2 p_{r,\nu}(u)w(u)d\tilde{x}_1\ldots d\tilde{x}_r. \quad (44)$$

Let us integrate over $\tilde{x}_3,\ldots,\tilde{x}_r$ using (44). As a result we obtain

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2\tilde{f}(\tilde{x})d\tilde{x}_1 d\tilde{x}_2 \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2 w(u_2)d\tilde{x}_1 d\tilde{x}_2 + $$

$$+\sum_{\nu=2}^{N}c_{r,\nu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{x}_1^2\tilde{x}_2^2 p_{2,\nu}(u_2)w(u_2)d\tilde{x}_1 d\tilde{x}_2, \quad (45)$$

where $u_2 = \tilde{x}_1^2/\lambda_1 + \tilde{x}_2^2/\lambda_2$, $w(u_2)$ is two-dimensional normal density. We rewrite (45) in the form of the expectations and substitute the values of the normal moments

$$E_{\text{EA}}\tilde{X}_1\tilde{X}_2 = D_1D_2 + \sum_{\nu=2}^{N}c_{r,\nu}E_w\tilde{X}_1^2\tilde{X}_2^2 p_{2,\nu}(U_2). \quad (46)$$

We evaluate the expectation in (46):

$$E_{w_1}p_{2,\nu}(U_2)\tilde{X}_1^2\tilde{X}_2^2 = E_w p_{2,2}(U_2)\tilde{X}_1^2\tilde{X}_2^2 = $$

$$= 2^2\cdot 2!R_{2,2,2} = 8R_{2,2,2} = 8D_1D_2. \quad (47)$$

According to (47) the EA (46) of the mixed fourth order moment will take the form

$$E\tilde{X}_1^2\tilde{X}_2^2 = D_1D_2 + 8c_{r,2}D_1D_2. \quad (48)$$

Therefore finally we get correspond accuracy formulate:

$$\delta_1(2,2,0,\ldots,0) = E_{\text{EA}}\tilde{X}_1^2\tilde{X}_2^2 - E\tilde{X}_1^2\tilde{X}_2^2, \quad (49)$$

$$\delta_2(2,2,0,\ldots,0) = \delta_1(2,2,0,\ldots,0)/E\tilde{X}_1^2\tilde{X}_2^2.\}. \quad (50)$$

4. Ellipsoidal Linearization Method

Now we consider ellipsoidal linearization of nonlinear transforms of random vectors using mean square error (m.s.e.) criterion optimal m.s.e. regression of vector $Z = \varphi(Y)$ on vector Y is determined by formula [6]

$$m_z(Y) = h_2Y, \quad h_2 = \Gamma_{zy}\Gamma_y^{-1} \quad (51)$$

or

$$m_z(Y) = h_1Y + a, \quad h_1 = K_{zy}K_y^{-1}, \quad a = m_z - h_1m_y. \quad (52)$$

where h_1, h_2 are equivalent linearization matrices; m_y, K_y are mathematical expectation and covariance matrix $(\det|K_y|\neq 0)$. In case (52) coefficient h_1 is equal to

$$h_1 = K_{zy}K_y^{-1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(z-m_z)(y-m_y)^T K_y^{-1} f(z,y)dzdy =$$

$$= \int_{-\infty}^{\infty} [m_z(y)-m_z](y-m_y)^T K_y^{-1} f_1(y)dy \qquad (53)$$

where $f_1(y)$ being the density of random vector Y.

For ellipsoidal density $f_1(y)$ in (53)

$$f_1(y) = f_1^{\text{EL}}(y) = \tilde{f}_1^{\text{EL}}(u(y), m_y, K_y, c). \qquad (54)$$

In case (52) we get formulae of for ellipsoidal linearization method (ELM):

$$m_z(Y) \approx m_{1z}^{\text{EL}} + h_1^{\text{EL}}(m_y, K_y, c)Y^0, \qquad (55)$$

where

$$h_1^{\text{EL}} = h_1^{\text{EL}}(m_y, K_y, c) = \int_{-\infty}^{\infty} [m_z(y)-m_z](y-m_y)^T K_y^{-1} f_1^{\text{EL}}(y)dy =$$

$$= \int_{-\infty}^{\infty} [m_z(y)-m_z](y-m_y)^T K_y^{-1} \tilde{f}_1^{\text{EL}}(u(y), m_y, K_y, c)dy.$$

(56)

In case (51) we have the following formulae for ELM:

$$h_2^{\text{EL}}(\Gamma_y, c) = \int_{-\infty}^{\infty} m_z(y)y^T\Gamma_y^{-1} f_1^{\text{EL}}(y)dy =$$

$$= \int_{-\infty}^{\infty} m_z(y)y^T\Gamma_y^{-1}\tilde{f}_1(u(y), m_y, \Gamma_y, c)dy, \qquad (57)$$

$$m_z(Y) \approx h_2^{\text{EL}}(\Gamma_y, c)Y. \qquad (58)$$

At practice the following ELM generalizations are useful.

1) Let us for fixed dimension $p = \dim y$ and N in (9) with normal $w(u)$ distinguish modifications of various orders ELM $_w^{p,2}$, ELM $_w^{p,3}$,…,ELM $_w^{p,N}$. In this case $c = \{c_{p,v}\}$ characterize partial deviations from normal distributions of various orders v jointly for all p Components of vector Y (be part of quadratic form $U(Y)$.

2) At decomposition of vector Y on $l_1, l_2, ..., l_r$ random subvectors, $Y = [Y_{l_1}^T Y_{l_2}^T ... Y_{l_r}^T]^T$ we distinguish $\text{ELM}_w^{l_1,...,l_r,N}$. Coefficients $c_{l_1,v},...,c_{l_2,v}$ characterize partial deviations of subvectors from normal distribution.

3) For matrix transforms $Z = \varphi(Y) = [\varphi_1(Y) \dots \varphi_q(Y)]^T$, $\varphi_i(Y) = [\varphi_{i1}(Y) \dots \varphi_{ip}(Y)]^T \quad (i = \overline{1,q}), \quad \dim\varphi = p \times q$ we have the following formulae for ELM:

$$m_z(Y) \approx m_{1z}^{\text{EL}} + H_1^{\text{EL}}(m_y, K_y, c)(Y-m); \qquad (59)$$

$$m_z(Y) \approx H_2^{\text{EL}}Y. \qquad (60)$$

Here

$$H_1^{\text{EL}}(m_y, K_y, c) = [h_{11}^{\text{EL}}(m_y, K_y, c) \dots h_{1q}^{\text{EL}}(m_y, K_y, c)] \qquad (61)$$

$$H_2^{\text{EL}}(\Gamma_y, c) = [h_{21}^{\text{EL}}(\Gamma_y, c) \dots h_{2q}^{\text{EL}}(\Gamma_y, c)], \qquad (62)$$

Where h_{1i}^{EL} and h_{2i}^{EL} $(i = \overline{1,q})$ are determined by formulae (56) and (57).

4) For transforms depending of time process t it is useful to work with overage ELM coefficients $\langle m_{iz} \rangle$ and $\langle h_i^{\text{EL}} \rangle$.

Example 2. For nonlinear scalar function

$$Z = \varphi(Y) = Y_1Y_2,$$

(r_1, r_2 are nonnegative integers, $r_1, +r_2 = r > 2$) according (55) we have the following formula:

$$\varphi(Y) = \varphi_0 + \varphi_1 Y^0.$$

Here $Y = [Y_1\, Y_2]^T$, $Y^0 = Y - m$, $m = [m_1\, m_2]^T$ formulae (52), (56) for φ_0 , φ_{1v} and various v till probabilistic moments of the order may be presented in the following form:

- – at $v = 2$ till moments of the first order

$$\varphi_{12} = \begin{cases} 8R_{r,2} & at \quad |\, r\,|= r_1 + r_2 \geq 4; \\ 0 & at \quad |\, r\,|\, r_1 + r_2 < 4; \end{cases}$$

$$\varphi_0 = \begin{cases} \mu_{10}(m,K,t) + 8c_{2,2}R_{r,2} & at \quad |r| = r_1 + r_2 \geq 4; \\ \mu_{10}(m,K,t) & at \quad |r| = r_1 + r_2 < 4; \end{cases}$$

- at $v = 3$ till moments of the second order

$$\varphi_{13} = \begin{cases} 48R_{r,3} & at \quad |\, r\,|= r_1 + r_2 \geq 6; \\ 0 & at \quad |\, r\,|\, r_1 + r_2 < 6; \end{cases}$$

$$\varphi_0 = \begin{cases} \varphi_{10} + 48c_{2,3}R_{r,3} & at \quad |r| = r_1 + r_2 \geq 6; \\ \varphi_{10} & at \quad |r| = r_1 + r_2 < 6; \end{cases}$$

- at $v = 4$ till moments of the third order

$$\varphi_{14} = \begin{cases} 384R_{r,4} & at \quad |\, r\,|= r_1 + r_2 \geq 8; \\ 0 & at \quad |\, r\,|\, r_1 + r_2 < 8; \end{cases}$$

$$\varphi_0 = \begin{cases} \varphi_{10} + 384c_{2,4}R_{r,4} & at \quad |r| = r_1 + r_2 \geq 8; \\ \varphi_{10} & at \quad |r| = r_1 + r_2 < 8; \end{cases}$$

- at $v = 5$ till moments of the second order

$$\varphi_{15} = \begin{cases} 3840R_{r,5} & at \quad |\, r\,|= r_1 + r_2 \geq 10; \\ 0 & at \quad |\, r\,|\, r_1 + r_2 < 10; \end{cases}$$

$$\varphi_0 = \begin{cases} \varphi_{10} + 3840c_{2,5}R_{r,5} & at \quad |r| = r_1 + r_2 \geq 10; \\ \varphi_{10} & at \quad |r| = r_1 + r_2 < 10; \end{cases}$$

Here

$$R_{r_1,\dots,r_p,\nu} = R_{r,\nu} = \sum_{|h|=2\nu} C_{r_1}^{h_1} \dots C_{r_p}^{h_p} \ \alpha_{r_1-h_1, \dots, r_p-h_p}^{N}, \mu_{h_1,\dots,h_p}^{N},$$

$$R_{r,\nu} = 0 \quad \text{at} \quad \nu < 0 \quad \text{or} \quad \nu > |r|,$$

For more examples address [4].

5. Applications to Mathematical Statistics Problems

1. For the distribution of an n-dimensional random vector Y is given by the density $f^*(u) = p(u^T Cu)$ where C is a positive definite symmetrical matrix. Find the distribution of the quadratic form $U = Y^T CY$ for arbitrary ellipsoidal density defined (9). Following [6] we have:

$$f^*(u) = \frac{c}{2} u^{(n/2)-1} p(u) \mathbf{1}(u). \tag{63}$$

In the special case of a normally distributed random vector Y when

$$p(y^T Cy) = \sqrt{\frac{|c|}{(2\pi)^n}} e^{-(1/2) y^T Cy}$$

formula (63) assumes the form

$$f^*(u) = p_n(u) = \frac{\mathbf{1}(u)}{2^{n/2} \Gamma\left(\frac{n}{2}\right)} u^{(n/2)-1} e^{-(1/2)u}. \tag{64}$$

This distribution is called a χ^2-distribution since the random variable with such a distribution is usually called chi-square in statistics. The natural number n in (63) is called the number of degrees of freedom.

2. For the distribution of an n-dimensional random Y with spherical symmetry $f^*(y) = p(y^T Cy)$ find the distribution of the random variable

$$T = \bar{Y} \sqrt{\frac{n(n-1)}{Y^T Y - n\bar{Y}^2}}, \quad \bar{Y} = \frac{1}{n} \sum_{k=1}^{n} Y_k. \tag{65}$$

Following [6] we get

$$f^*(t) = s_{n-1}(t) = \frac{\Gamma(n/2)}{\sqrt{\pi(n-1)} \Gamma\{(n-1)/2\}} \left(1 + \frac{t^2}{n-!}\right)^{-n/2}. \tag{66}$$

This distribution is called a Student distribution, or shortly T-distribution. The quantity T determined by formula (65) is called a Student ratio. The number $k = n-1$ is called the number of degrees of freedom. The distribution of the Student ratio T is independent of the function $p(u)$. For any random vector Y whose distribution possesses a spherical symmetry the random variable T has distribution (66).

3. The joint distribution of an n-dimensional vector X and an m-dimensional vector Y possesses a spherical symmetry in the $(n+m)$-dimensional space then the random variable

$$Z = \sqrt{\frac{nm(n+m-2)}{(n+m)(X^T X - n\bar{X}^2 + Y^T Y - m\bar{Y}^2)}} (\bar{X} - \bar{Y}), \tag{67}$$

with

$$\bar{X} = \frac{1}{n} \sum_{p=1}^{n} X_p, \quad \bar{Y} = \frac{1}{m} \sum_{q=1}^{m} Y_q,$$

has a T-distribution with $n+m-2$ degrees of freedom [6]:

$$s_k(z) = \frac{\Gamma\{(k+1)/2\}}{\sqrt{\pi k} \Gamma(k/2)} \left(1 + \frac{z^2}{k}\right)^{-(k+1)/2}, \quad k = n+m-2. \tag{68}$$

4. Let X be a random $n \times m$ matrix, $m < n$, X a matrix-row whose elements are the arithmetical means of the elements of the corresponding columns of the matrix X:

$$\bar{X}_p = \frac{1}{n} \sum_{q=1}^{n} X_{qp} \quad (p = 1, \dots, m). \tag{69}$$

The random variable

$$T = \sqrt{n(n-1) \bar{X} S^{-1} \bar{X}^T}, \quad S = X^T X - n\bar{X}^T \bar{X} \tag{70}$$

is called a generalized Student ratio. For any random matrix X whose density is determined by

$$f_1(x) = p(\text{tr } xCx^T), \tag{71}$$

where $p(u)$ is any function (certainly satisfying the condition that the function $f_1(y)$ be a density, probably containing δ-functions), C is a positive definite symmetrical matrix and $\text{tr } A$ is the trace of the matrix A, the random variable T has the density [6]

$$s_{n-1,m}(t) = \frac{2\Gamma(n/2)}{(n-1)^{m/2} \Gamma(m/2) \Gamma\{(n-m)/2\}} t^{m-1} \left(1 + \frac{t^2}{n-1}\right)^{-n/2} \mathbf{1}(t). \tag{72}$$

In the special case of $m = 1$ this distribution coincides with the distribution of the absolute value of the Student ratio (66).

5. Under the conditions of subsection 5.4 the random vector (matrix-row) $\bar{X}$ and the random matrix S are uncorrelated for any function $p(u)$ and that they are independent if and only if $p(u) = \sqrt{|C|^n / (2\pi)^{nm}} e^{-u/2}$.

6. Let X_t and X_2 be the matrices $n_1 \times m$ and $n_2 \times m$ respectively, $\bar{X}_1$ and $\bar{X}_2$ the matrix-rows

whose elements are the arithmetical means of the elements of the corresponding columns of the matrices X_1 and X_2, $S_k = X_k^T X_k - n\bar{X}_k^T X_k$ $(k = 1, 2)$, $H = S_1 + S_2$. If the distribution of the block $(n_1 + n_2)\times m$ -matix $X, X^T = [X_1^T X_2^T]$ is determined by formula (71) then the random variable

$$U = \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}(\bar{X}_1 - \bar{X}_2)H^{-1}(\bar{X}_1^T - \bar{X}_2^T)} \quad (73)$$

has the density (72) where $n = n_1 + n_2 - 1$.

7. If in the problem of subsection 5.4 $p(u) = \sqrt{|C|^n/(2\pi)^{nm}}e^{-u/2}$, and $C = K^{-1}$ then the density of the matrix S is determined by

$$w_{nm}(s) = c_{nm}(2^m |K|)^{-(n-1)/2} |s|^{\{(n-m)/2\}^{-1}} e^{-(1/2)\operatorname{tr} K^{-1}}, (74)$$

in the region of positive definite matrices s of the space of $m\times m$ matrices and zero outside this region. By $|K|$ and $|s|$ in formula (74) are denoted by determinants of the matrices K and s respectively and

$$c_{nm} = \frac{1}{\pi^{\{m(m-1)\}^4}\Gamma\{(n-1)/2\}\Gamma\{(n-2)/2\}\ldots\Gamma\{(n-2)/2\}}. \quad (75)$$

The distribution determined by the density (74) is called a Wishart distribution [6].

8. If under the conditions subsection 5.4 $S' = X^T X$, then the ratio of the determinants of the matrices S and $S', V = |S|/|S'|$, has the density

$$f^*(\upsilon) = \frac{\Gamma(/2)}{\Gamma\{(n-m)/2\}\Gamma(m/2)}\upsilon^{\{(n-m)/2\}-1}(1-\upsilon)^{(m/2)-1}1(\upsilon)\mathbf{1}(1-\upsilon). \quad (76)$$

This is β -distribution with the parameters $p = (n-m)/2$ and $q = m/2$.

For more Problems address [6, 7].

6. EAM and ELM Applications for Nonlinear Stochastic Systems

Direct application of EAM to nonlinear stochastic differential equations (SDE) for one- and multidimensional densities is based on formulae (9) and set of deterministic ordinary differential Eqs for coefficients $c_{r,\mu}$ [1–3]. Number of Eqs is given by formula (4) where r being the SDE dimension. For ELM number of Eqs is given by formula (5).

7. Ellipsoidal Distributions in Volterra Stochastic Systems

Let us consider the following stochastic analogue of populational Volterra system [8] described by the set of stochastic differential equations (SDE) [9]:

$$\begin{aligned}\dot{Y}_i &= \left(\varepsilon_{0i} + \sigma_i^\varepsilon V^{(2)}\right)Y_i - \sum_{j=1}^N \left(p_{0ij} + \sigma_{ij}^p V^{(3)}\right)Y_i Y_j + \\ &+\alpha_{0i} + \sigma_i^\alpha V^{(1)}, \quad Y_i(t_0) = Y_{i0}.\end{aligned} \quad (77)$$

Here $\varepsilon_{0i}, p_{0ij}, \alpha_{0i}$ are deterministic parameters, $\sigma_i^\varepsilon, \sigma_{ij}^p, \sigma_i^\alpha$ are parameters of addive and multiplicative noises.

Putting

$$\begin{aligned}&Y = [Y_1 \ldots Y_N]^T; \quad V = [V^{(1)T}\ V^{(2)T}\ V^{(3)T}]^T;\\ &a = a(Y,t) = \left[\alpha_{0i} + \varepsilon_{0i}Y_i - \sum_{j=1}^N p_{0ij}Y_iY_j\right]; \quad (78)\\ &B = B(y,t) = \begin{bmatrix} b^{(1)}(Y,t) & 0 & 0\\ 0 & b^{(2)}(Y,t) & 0\\ 0 & 0 & b^{(3)}(Y,t)\end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}&b^{(1)}(Y,t) = [\sigma_\alpha^{(1)}(t)\ 0\ 0];\\ &b^{(2)}(Y,t) = [0\ \sigma_\varepsilon^{(1)}(t)\ 0]; \quad (79)\\ &b^{(3)}(Y,t) = \left[0\ 0 - \sum_{j=1}^N \sigma_{ij}^p(t)Y_iY_j\right],\end{aligned}$$

we get the following vector stochastic Ito SDE

$$\dot{Y} = a(Y,t) + B(Y,t)V, \quad Y(t_0) = Y_0, \quad (80)$$

where

$$\begin{aligned}&a = a(Y,t) = \varepsilon_0 Y - [\gamma_0 + \gamma(Y,t)]Y + \alpha_0;\\ &B = B(y,t) = \begin{bmatrix} [b^{(1)}(Y,t)] & 0 & 0\\ 0 & [b^{(2)}(Y,t)] & 0\\ 0 & 0 & [b^{(3)}(Y,t)]\end{bmatrix};\\ &b^{(1)}(Y,t) = [\sigma_\alpha^{(1)}(t)\ 0\ 0]; \quad (81)\\ &b^{(2)}(Y,t) = [0\ \sigma_\varepsilon^{(1)}(t)Y\ 0];\\ &b^{(3)}(Y,t) = \left[0\ 0 - \sigma_\gamma^{(3)} - \gamma(Y,t)Y\right],\end{aligned}$$

Nonstationary and stationary one dimensional probability density $f_1 = f_1(y;t)$ and $f_1^* = f_1^*(y)$ for Gaussian SDE (8) are described by Eqs [1–3]

$$\begin{aligned}&\frac{\partial f_1}{\partial t} = -\frac{\partial^T}{\partial y}[a(y,t)f_1(y,t)] + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial^T}{\partial y}[B(y,t)\nu(t)B(y,t)^T f_1(y;t)],\\ &f_1(y;t_0) = f_0(y);\end{aligned} \quad (82)$$

$$\frac{1}{2}\frac{\partial}{\partial y}\frac{\partial^T}{\partial y}[B(y)\nu(t)B(y)^T f_1^*(y)] - \frac{\partial^T}{\partial y}[a(y)f_1^*(y)] = 0. \quad (83)$$

Here $\nu = \nu(t)$ being the matrix intensity of compound white noise V.

Following theory of distributions with in variant measure [9–11] we get 3 basic results.

1) Ellipsoidal function $f_1 = f_1(y;\tau) = f(u)$, $u = (y-m)^T C(y-m)$ will be solution of Eq (82) where V being Gaussian white noise with intensity matrix $v = v(t)$ and diffusion matrix $\Sigma^B(y,t) = B(y,t)v(t)B(y,t)^T$ if and only if vector function $a = a(y,t)$ may be presented in the form

$$a(y,t) = a_1^1(y,t) + a_2^1(y,t) \quad (84)$$

and being the ellipsoidal invariant of ordinary differential Eq

$$\dot{Y} = a_1^1(Y,t), \quad (85)$$

i.e. it satisfy the Eq

$$\frac{\partial f_1}{\partial t} + \frac{\partial^T}{\partial y}\left(a_1^1 f_1\right) = 0, \quad (86)$$

The corresponding function $a_2^1 = a_2^1(y,t)$ is determined by formula

$$a_2^1(y,t) = \frac{1}{2}\left[\Sigma^B \frac{\partial \ln f_1}{\partial y} + \left(\frac{\partial^T}{\partial y}\Sigma^B\right)^T\right]. \quad (87)$$

2) Let at conditions (84)–(87) and addional conditions:

(1) nonnegative sealer function $\mu_1 = \mu_1(y,t)$ is ellipsoidal density of integral invariant of Eq (85);

(2) diffusion matrix Σ^B is not singular

$$\left(|\det\left(\Sigma^B\right)^{-1}| \neq 0\right);$$

(3) Vector function

$$\Gamma_1(y,t) = \left(\Sigma^B\right)^{-1}\left\{a_2^1 - \frac{1}{f_1}\left[\frac{\partial}{\partial y}\left(\Sigma^B f_1\right)\right]^T\right\} \quad (88)$$

satisfy curl absence

$$\frac{\partial \Gamma_{1i}}{\partial y_j} = \frac{\partial \Gamma_{1j}}{\partial y_i} \qquad (i,\, j = \overline{1,N}); \quad (89)$$

(4) scalar function

$$F_1(y,t) = \int \Gamma_1^T(y,t)dy \quad (90)$$

is the first integral of Eq (84).

(5) the condition of normalization is fulfilled

$$\int \mu_1(y,t)\exp F_1(y,t)dy = 1. \quad (91)$$

Then there exists the solution of Eq (82) with ellipsoidal one-dimensional density of the form

$$f_1(y,t) = \mu_1(y,t)\exp F_1(y,t). \quad (92)$$

3) The function $\mu_1 = \mu_1(y,t)$ at conditions 2) is equal to $f_1(y;t)$ if and only there exist such a matrix function $A_1 = A_1(y;t)$ that

$$a_2^1\mu_1 \equiv \left(\frac{\partial A_1}{\partial y}\right)^T, \quad A_1 + A_1^T = \Sigma^B\mu_1, \quad a_2^1 = a - a_1^1. \quad (93)$$

For more theorems concerning the ellipsoidal Gibbs distributions it is possible to get from [1–3, 9].

8. Conclusions

Methodological support for computing by EAM (ELM) and mathematical statistics for finite-dimensional random vectors problems are given. Two approaches for EAM (ELM) accuracy estimation are presented in details. Using [1–4] we get direct generalizations for random vectors and matrices defined on manifolds are possible. Special attention is paid to the one-dimensional ellipsoidal distributions of nonstationary and stationary processes with invariant measure generalizing Gibbs distributions.

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REFERENCES

[1] V. S. Pugachev, I. N. Sinitsyn. Stochastic differential systems. Analysis and filtering. John Wiley&Sons Chichester, 1987 (Russian editions 1985, 1990).

[2] V. S. Pugachev, I. N. Sinitsyn. Theory of stochastic systems. Logos, 2000 (in Russian).

[3] V. S. Pugachev, I. N. Sinitsyn. Stochastic systems. Theory and Applications. World Scientific. Singapore, 2001.

[4] I. N. Sinitsyn, V. I. Sinitsyn. Lectures of normal and ellipsoidal approximation of distributions in stochastic systems. Moscow: Torus Press, 2013 (in Russian).

[5] V. S. Pugachev, I. N. Sinitsyn. Lectures on functional analysis and applications. World Scientific. Singapore, 1999.

[6] V. S. Pugachev. Probability theory and mathematical statistics for Engineers. Pergamum Press. Oxford, 1984.

[7] Serfling R. Approximation theorems of mathematical statistics. New York, NY, USA John Wiley&Sons, 1980.

[8] Vito Volterra. Lecons sur la theorie mathematique de la lute pour la vie. Gauthiers-Villass. Paris, 1931.

[9] Sinitsyn I.N., V.I. Sinitsyn. Analytical Modeling of Distributions with invariant measure in Volterra stochastic systems // Systems and means of informatics, 2018. 28(3): 4–25.