Mathematics and Statistics 7(5): 191-196, 2019 http://www.hrpub.org
DOI: 10.13189/ms.2019.070505

Backward Simulation of Correlated Negative Binomial Lévy Process

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Received August 26, 2019; Revised October 24, 2019; Accepted October 28, 2019

Abstract Recent studies on correlated Poisson processes show that the backward simulation methods are computationally efficient, and incorporate flexible and extremal correlation structures in a multivariate risk system. These methods rely on the fact that the past arrival times of a Poisson process given the number of events over a time interval, $[0, T]$, are the order statistics of uniform random variables on $[0, T]$. In this paper, we discuss an extension of the backward methods to a correlated negative binomial Lévy process which is an appealing model for over-dispersed count data such as operational losses. To obtain the conditional uniformity for the negative binomial Lévy process, we consider a particular setting in which the time interval is partitioned into equally spaced sub-intervals with unit length and the terminal time T is set to be the number of sub-intervals. Under this setting, the resulting joint probability of the increment series, conditional on the number of events over $[0, T]$, say l, is uniform for any points in the support of a $\{T, l\}$-simplex lattice. Based on this result, we establish a backward simulation method similar to that of Poisson process. Both the conditional independence and conditional dependence cases are discussed with illustrations of the corresponding time correlation patterns.

Keywords Backward Simulation, Negative Binomial Lévy Process, Correlation Structure

1 Introduction

In quantitative risk modelling, the ability for a risk model to incorporate flexible and extremal dependence structures is of main concern. Especially for the purpose of scenario analysis, the dependence in a multivariate risk system can be driven by realization of common shocks. In practice, these shocks are naturally modelled with multivariate point processes such as multivariate Poisson processes. See Kreinin [10] for details on the correlation structures of several well-known multivariate Poisson processes including the Poisson-Wiener process (Yahav and Shmueli [11]) which is a popular model for integrated scenario generations.

In order to simulate a point process in a fixed time horizon, one may consider either forward or backward method. For example, the forward simulation of a Poisson process is implemented by repeated simulations of independent and identically distributed (iid) exponential inter-arrival times until the sojourn time, the sum of the inter-arrival times, reaches the time horizon. In a bivariate (or multivariate) setting, the dependence structure is determined by that of the bivariate (or multivariate) exponential distribution for the inter-arrival times. Due to the randomness in the number of exponential inter-arrival times, this method can be computationally demanding especially when the time horizon is long and the mean Poisson rate is large. As discussed in Kreinin [10] and Bae and Kreinin [2], this method also lacks ability to realize the extremal (maximal or minimal) correlation structure which makes this forward method less appealing for quantitative risk applications. On the other hand, the backward methods which have recently been studied by Duch et al. [7] and Kreinin [10], use the fact that the past arrival times of Poisson jumps, given the total number of Poisson jumps over the time interval, have the distribution equivalent to the order statistics of uniform random variates on the time interval. Since the number of jumps are generated anterior from a Poisson distribution, the method is computational more efficient than the forward method. More importantly, under the conditional independence framework, Kreinin [10] shows that the backward method allows for extremal correlations, and thus it is appealing for scenario analysis of portfolio risks. In Bae and Kreinin [2], the conditional independence assumption has been alleviated by using several parametric copulas, which results in quite flexible time correlation patterns.

Even though the Poisson process is dominantly used in practices, it may not be suitable for data exhibiting over-dispersion. For example, Bae [1] studies an operational loss dataset in which the estimated variability of the frequency of operational losses from each risk cell (a specific business line and event type) is significantly higher than the estimated mean. Over-dispersion is also often observed in the frequencies of natural disasters such as earthquakes or tropical storms. For such cases, mixed Poisson or negative binomial models are more appealing than the Poisson. The present paper is aiming at developing a backward simulation method for correlated negative binomial processes. Amongst many possible constructions of negative binomial process, we focus on the model with independent and stationary increments, the so called negative binomial Lévy process, which appears to be a desirable model for consistent risk modelling and management over multiple time periods.

The rest of paper is organized as follows. Section 2 introduces the negative binomial Lévy process. In Section 3, we discuss the backward simulation of a univariate negative binomial Lévy process. Sections 4 and 5 presents the backward simulations of correlated (bivariate) negative binomial Lévy process under the conditional independence and dependence, respectively, with some illustrations. Finally Section 6 concludes this paper.

2 Negative binomial Lévy process

Several different constructions of continuous time negative binomial processes are available in the literature. See Barndorff-Nielsen and Yeo [3], Brix [4] and Burnett and Wasan [5] for references. In the context of mathematical finance and quantitative risk management, stochastic processes with independent and stationary increments are of great interest due to many advantages such as mathematical tractability, easiness in estimation and simulation. Here we focus on a negative binomial process constructed naturally from the infinite divisible negative binomial distribution (Kozubowski and Podgorski [9]).

Specifically, let $\{N(t)\}_{t\geq 0}$ be a negative binomial Lévy process (NBLP) with the probability function (pf)

$$f_t(k) := \mathbb{P}(N(t) = k) = \binom{t+k-1}{k} p^t q^k,\ k = 0, 1, \dots \quad (1)$$

where $p \in (0, 1)$ and $q = 1 - p$. The probability generating function (pgf) is

$$\mathbb{E}[u^{N(t)}] = \left(\frac{p}{1-qu}\right)^t. \quad (2)$$

In particular when $t = 1$, $N(1)$ has a geometric distribution. The mean and variance of the NBLP are

$$\mathbb{E}[N(t)] = (q/p)t, \quad \mathbb{V}[N(t)] = (q/p^2)t. \quad (3)$$

Due to the fact that $\mathbb{V}[N(t)] \geq \mathbb{E}[N(t)]$, the NBLP is appropriate to account for over-dispersion in a counting data. For the purposes of multivariate extensions and simulations, the following two representations of NBLP are useful:

(i) The NBLP can be represented as a compound Poisson process. Specifically, the compound Poisson process $\sum_{n=1}^{M(t)} X_i$, where $M(t)$ is a Poisson process with intensity $\lambda = -\log p$ and $\{X_i\}$ are independent and identically distributed logarithmic random variables with the pf

$$\mathbb{P}(X = k) = -\frac{q^k}{k \log p}, k = 1, 2, \dots.$$

(ii) The NBLP admits a representation as a Poisson process subordinated by a gamma process. Specifically, the time-changed Poisson process $M(G(t))$ where $\{M(t)\}$ is a Poisson process with the rate $\lambda = \beta q/p$ and $\{G(t)\}$ is a Lévy gamma process with the pgf $\mathbb{E}[u^{G(t)}] = (1 - u/\beta)^{-t}$, has the same distribution as the model (1). Here the parameter $\beta > 0$ is arbitrary.

3 Backward simulation of NBLP

The backward simulations (BS) of a Poisson process (Bae and Kreinin [2]) rely on the fact that the past arrival times of a Poisson process $\{M(t)\}_{t\leq T}$ are the order statistics of uniform random variables on the interval $[0, T]$. Also true is that, given the number of arrivals $M(T)$ over the interval $[0, T]$, the number of arrivals in any sub-interval $[t, t+s] \in [0, T]$, i.e., the increment $M(t+s) - M(t)$, follows a binomial distribution with the number of Bernoulli trials $M(T)$ and the success probability s/T. To investigate if the negative binomial évy process possesses similar properties, we consider the conditional distribution of the increment $\Delta_t(s) = N(t+s) - N(t)$, $[t, t+s] \in [0, T]$. Specifically, as given in Kozubowski and Podgorski [9], the conditional distribution of $\Delta_t(s)$ given $N(T) = l$ is

$$\begin{aligned} &\mathbb{P}[\Delta_t(s) = k \,|\, N(T) = l] \\ = &\ \frac{\binom{s+k-1}{k} p^s q^k \binom{(T-s)+l-k-1}{l-k} p^{(T-s)} q^{l-k}}{\binom{T+l-1}{l} p^T q^l} \\ = &\ \frac{\binom{s+k-1}{k}\binom{(T-s)+l-k-1}{l-k}}{\binom{T+l-1}{l}}, \quad k = 0, 1, \dots, l. \end{aligned}$$

Therefore the conditional distribution is free of the parameter p and the starting time t. However, this distribution is less straightforward to simulate from.

For both practical and simplification purposes, we set $T = m\ (\geq 1)$ be a positive integer throughout this paper. Let us consider a partition of the interval $[0, T]$ into disjoint sub-intervals with the unit length $\delta = 1$. Then, the joint distribution of the series of increments $\{\Delta_{(i-1)\delta}(\delta)\}, i = 1, \dots, m$, becomes

$$\begin{aligned} &\mathbb{P}[\Delta_0(\delta) = k_1, \dots, \Delta_{(m-1)\delta}(\delta) = k_m \,|\, N(T) = l] \\ = &\ \frac{\prod_{i=1}^m \binom{\delta+k_i-1}{k_i} p^\delta q^{k_i}}{\binom{m+l-1}{l} p^m q^l} = \frac{1}{\binom{m+l-1}{l}}, \end{aligned} \quad (4)$$

where $\sum_{i=1}^m k_i = l$ and $k_i \geq 0$, $i = 1, \dots, m$. That is, the joint probability is uniform for any points in the support or a $\{m, l\}$-simplex lattice where $\binom{m+l-1}{l}$ is the number of all points in the support. Due to the conditional uniformity under this particular setting, we can proceed the backward simulations of a univariate NBLP as follows:

Step 1. Simulate $l = N(T)$ from the negative binomial distribution with the parameters m and p.

Step 2. Simulate a series of increments $(k_1, \dots, k_m)$, a m-dimensional vector of non-negative integers that sum to l. Due to the conditional uniformity, this can be done by a random selection of a point on a $\{m, l\}$-simplex lattice. For example, one may use the R-function xsimplex in combinat R-package to generate all points on the simplex lattice and choose one randomly (see Chasalow and Brand [6] for the generation of all points on a simplex lattice). However, this exhaustive search method is computationally inefficient for large m and l values. Here we use successive simulations of the increments from recursive multinomial experiments. Specifically, the first increment $\Delta_0(\delta) = k_1$ is simulated from a multinomial distribution with the probabilities

$$\mathbb{P}[\Delta_0(\delta) = k_1 \,|\, l] = \frac{\binom{l-k_1+(m-1)-1}{l-k_1}}{\binom{l+m-1}{l}}, \; k_1 = 0, \ldots, l. \quad (5)$$

For each $j = 2, \ldots, m$, the jth increment is recursively generated with the multinomial probabilities

$$\begin{aligned} &\mathbb{P}[\Delta_{(j-1)\delta}(\delta) = k_j \,|\, l, k_1, \ldots, k_{j-1}] \\ &= \frac{\binom{l-\sum_{i=1}^{j} k_i+(m-j)-1}{l-\sum_{i=1}^{j} k_i}}{\binom{l-\sum_{i=1}^{j-1} k_i+(m-j+1)-1}{l-\sum_{i=1}^{j-1} k_i}}, \; k_j = 0, \ldots, l - \sum_{i=1}^{j-1} k_i. \end{aligned} \quad (6)$$

Step 3. Repeat Steps 1 and 2 for the required number of simulations.

4 Simulation of correlated negative binomial Lévy processes

In this paper, we use Pearson's correlation as a measure of dependence, defined as

$$\rho(t) = \frac{\mathrm{Cov}[N_1(t), N_2(t)]}{\sqrt{\mathbb{V}[N_1(t)]}\sqrt{\mathbb{V}[N_2(t)]}}, \quad t \in (0, T]. \quad (7)$$

As in Bae and Kreinin [2], the goal is to simulate correlated increment processes for a bivariate NBLP $\{N_1(t), N_2(t)\}_{t \le T}$, given the correlation, $\rho(T)$, at the terminal time T. Based on (4), the backward construction relies on the conditional dependence structure between the increment processes given a realization $(N_1(T) = l_1, N_2(T) = l_2)$ at the terminal time T. As in Duch et al. [7], we first assume that the two processes are conditionally independent given the terminal values $(N_1(T), N_2(T))$. In this case, the correlation function $\rho(t)$ is linear in time.

Proposition 4.1 *Suppose that two negative binomial processes $\{N_1(t)\}_{t \le T}$ and $\{N_2(t)\}_{t \le T}$ are conditionally independent given the pair $(N_1(T), N_2(T))$, then*

$$\rho(t) = \left(\frac{t}{T}\right) \rho(T), \quad 0 \le t \le T. \quad (8)$$

The proof of Proposition 4.1 relies on the following property of NBLP model (1).

Lemma 4.2 *Let $\{N(t)\}$ be an NBLP process with the pf (1). Then the conditional expectation of $N(t)$ given $N(T) = l$ is*

$$\mathbb{E}[N(t) \,\big|\, N(T) = l] = \left(\frac{t}{T}\right) l, \quad 0 \le t \le T. \quad (9)$$

PROOF. For any fixed $t \le T$, let $E_t(l) := \mathbb{E}[N(t) \,|\, N(T) = l]$ be the conditional expectation of the NBLP given its terminal value at T. Then, by the properties of independent and stationary increments,

$$\begin{aligned} E_t(l) &= \sum_{k=0}^{l} k\mathbb{P}[N(t) = k \,|\, N(T) = l] \\ &= \sum_{k=1}^{l} \frac{k\, f_t(k) f_{(T-t)}(l-k)}{f_T(l)} \\ &= \sum_{k=1}^{l} \frac{k\binom{t+k-1}{k} p^t q^k \binom{(T-t)+l-k-1}{l-k} p^{T-t} q^{l-k}}{\binom{T+l-1}{l} p^T q^l} \\ &= q \sum_{k'=0}^{l-1} \frac{(t+k')\binom{t+k'-1}{k'} p^t q^{k'} \binom{(T-t)+l-1-k'-1}{l-1-k'} p^{T-t} q^{l-1-k'}}{\binom{T+l-1}{l} p^T q^l} \\ &= q\left(\frac{f_T(l-1)}{f_T(l)}\right)[t + E_t(l-1)] \\ &= \left(\frac{l}{T+l-1}\right)[t + E_t(l-1)]. \end{aligned}$$

Thus we obtain a recursive formula for the conditional expectation. Noting that $E_t(0) = 0$, we easily obtain the solution (9). □

Now let us complete the proof of Proposition 4.1. By (9), the iterative expectation formula and the conditional independence assumption, we have

$$\begin{aligned} \mathbb{E}[N_1(t)N_2(t)] &= \mathbb{E}[\mathbb{E}[N_1(t)N_2(t) \,\big|\, (N_1(T), N_2(T))]] \\ &= \mathbb{E}\left[\mathbb{E}[N_1(t) \,\big|N_1(T)]\mathbb{E}[N_2(t) \,\big|\, N_2(T)]\right] \\ &= \left(\frac{t}{T}\right)^2 \mathbb{E}\left[N_1(T)N_2(T)\right]. \end{aligned}$$

Then,

$$\begin{aligned} &\mathrm{Cov}[N_1(t), N_2(t)] \\ = \;& \mathbb{E}[N_1(t)N_2(t)] - \mathbb{E}[N_1(t)]\mathbb{E}[N_2(t)] \\ = \;& \left(\frac{t}{T}\right)^2 \mathbb{E}\left[N_1(T)N_2(T)\right] - \left(\frac{t}{T}\right)^2 \mathbb{E}[N_1(T)]\mathbb{E}[N_2(T)] \\ = \;& \left(\frac{t}{T}\right)^2 \mathrm{Cov}[N_1(t), N_2(t)] \end{aligned}$$

and

$$\mathbb{V}[N_i(t)] = \left(\frac{t}{T}\right) \mathbb{V}[N_i(T)], \; i = 1, 2.$$

Putting these into (7) gives the desired result.

The backward simulation algorithm of a bivariate NBLP under the conditional independence assumption is given as follows:

Step 1. Simulate $(l_1, l_2) = (N_1(T), N_2(T))$ from a bivariate negative binomial distribution with parameters (m, p_1) and (m, p_2) for the marginal distributions, and the terminal correlation $\rho(T)$. One may use the gamma-Poisson or the compound Poisson-Logarithmic representations to construct a bivariate negative binomial distribution. For example, one can simulate $(N_1(T), N_2(T))$ from a bivariate gamma-Poisson distribution where gamma random variables are correlated in terms of a bivariate normal distribution:

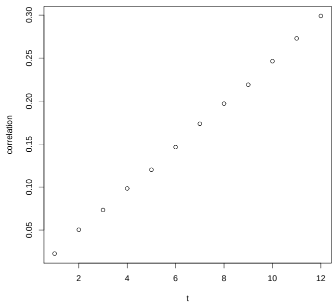
$$N_1(T) = P_1^{-1}(\Phi(W_1)),\ N_2(T) = P_2^{-1}(\Phi(W_2)),$$

where $P(\cdot)$ is the cumulative distribution functions of the gamma distribution with parameter βT, (W_1, W_2) is correlated standard normal variates with correlation ρ.

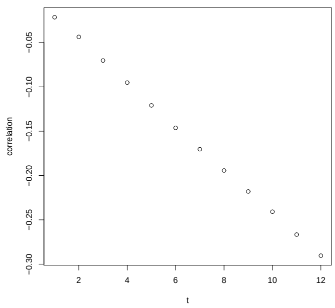
Step 2. Run two independent backward simulations of univariate NBLP to obtain the series of increments $(k_{i1}, \ldots, k_{im})$ for each $i = 1, 2$. That is, for each $i = 1, 2$, the generated series is an m-dimensional vector of non-negative integers that sums to the corresponding terminal value l_i.

Step 3. Repeat Steps 1 and 2 for the required number of simulations.

Figure 1 illustrates the correlation patterns realized based on 100,000 backward simulations using the bivariate gamma-Poisson model with $\rho \in \{0.5, -0.5\}$ and $T = 12, p_1 = p_2 = 0.4$. As expected from the theoretical result 9, the correlation function is linear in time running from zero to the terminal correlation values of about 0.3 and -0.3, respectively.



(a) $\rho = 0.5$



(b) $\rho = -0.5$

Figure 1. Correlation patterns under a conditional independence

5 Conditional dependence

Bae and Kreinin [2] construct bivariate Poisson processes with flexible time correlation structures using the Marshall–Olkin bivariate binomial distribution for the conditional law and some parametric families of bivariate copulas. A similar method can be implemented for the backward simulation of correlated NBLPs. Specifically we use uniform random variates generated from a bivariate copula for the recursive simulations of the increment series from multinomial probabilities (5) and (6). For instance, the pair of the first increments (k_{11}, k_{21}) is generated as follows. Simulate a bivariate uniform random variates (u_{11}, u_{21}) from a bivariate copula such as the Fréchet family of copulas which is defined as a linear combination of the comonotonicity copula (the Fréchet-Hoeffding upper bound), the independence copula and the countermonotonicity copula (the Fréchet-Hoeffding lower bound):

$$\begin{aligned} C(u, v) &= \alpha \min(u, v) + (1 - \alpha - \beta)uv \\ &\quad + \beta \max(u + v - 1, 0), \end{aligned} \tag{10}$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Then,

$$k_{i1} = \sum_{j=0}^{l_i} jI\left[\sum_{k=0}^{j} p(k|l_i) \leq u_{i1} < \sum_{k=0}^{j+1} p(k|l_i)\right], \quad i = 1, 2,$$

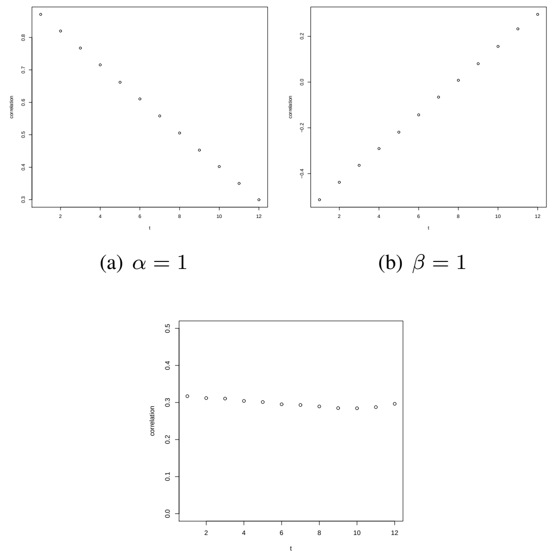
where $I[\cdot]$ is the indicator function and $p(k|l_i) := \mathbb{P}[\Delta_0(\delta) = k|l_i]$ as defined in (5). The subsequent pair of increments (k_{1j}, k_{2j}), $j = 2, \ldots, m$, can be generated recursively from the multinomial distributions (6) in the same way using the same copula.

Figure 2 illustrates the correlation patterns realized based on 100,000 backward simulations under the conditional comonotonicity ($\alpha = 1$), countermonotonicity ($\beta = 1$) and a case for $(\alpha, \beta) = (0.6, 0.4)$, with parameters $T = 12$, $p_1 = p_2 = 0.4$ and $\rho = 0.5$. Specifically, the realized correlation functions are linear for both comonotonicity and countermonotonicity. Different from the conditional independence case, however, the correlations at time zero are extremal (maximal and minimal, respectively). For the third case, the realized correlation function is (approximately) constant over time.

Note that, with various combinations of the parameters in (10) or by using other parametric copula families such as Clayton, Farlie-Gumbel-Morgenstern, Ali-Mikhail-Haq or Gumbel's survival copula (Bae and Kreinin [2]), a variety of time correlation structures may be realized. For instance, Figure 3 gives the correlation patterns realized under the Calyton copula,

$$c(u, v) = \max\{u^{-\theta} + v^{-\theta} - 1, 0\}^{-1/\theta},\ \theta \in [-1, \infty)/\{0\},$$

with a few prescribed values of parameter θ (and $T = 12$, $p_1 = p_2 = 0.4$, $\rho = 0.5$ as in Figure 1). We can see that the correlation patterns are linear toward the terminal correlation but the starting values (i.e., the correlation at $t = 1$) differ by the parameter θ in the Clayton copula. When $\theta < 0$, the negative



(a) $\alpha = 1$

(b) $\beta = 1$

(c) $\alpha = 0.6, \beta = 0.4$

Figure 2. Correlation patterns under a conditional dependence

dependence is incorporated at $t = 1$ and the opposite is true for $\theta > 0$.

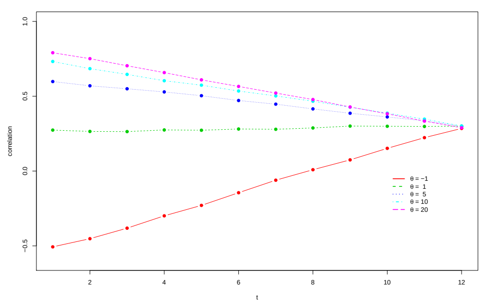


Figure 3. Correlation patterns incorporated by the Clayton copula

More general (possibly non-linear) correlation structures can be incorporated by letting the parameters in the bivariate copulas vary over time. For illustration purpose only, we consider the following time-dependent θ in the Clayton copula:

$$\theta(t) = \begin{cases} \frac{T-2t}{T-2}\theta_s + \frac{2(t-1)}{T-2}\theta_m, & 1 \le t \le T/2 \\ \frac{2(T-t)}{T}\theta_m + \frac{2t-T}{T}\theta_f, & T/2 \le t \le T, \end{cases} \tag{11}$$

where θ_s, θ_m and θ_f are parameters to be specified. Figure 4 illustrates a few non-linear correlation patterns realized based on the Clayton copula with the time-varying parameter structure (11).

In practice the three copula parameters, θ_s, θ_m and θ_f, can be chosen to match the empirical correlation estimates at $t = 1$, $t = T/2$ and $t = T$.

To illustrate the relevance and practical significance of the proposed BS method, we further consider the time correlation structure of the numbers of category 4 and 5 hurricanes in Atlantic and Pacific oceans. The annual data spanning from 1950

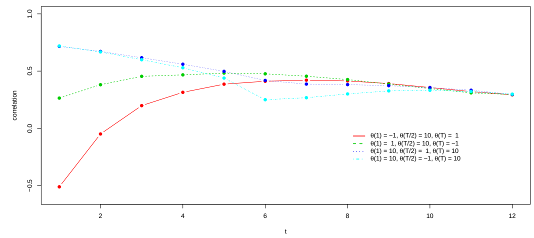


Figure 4. Correlation patterns incorporated by the Clayton copula with the time-varying parameters

to 2018 is complied from the National Hurricane Center (NHC) Data Archive. Table 1 gives a summary of the historical data.

	# of hurricanes	Mean	Variance	$\hat{p}$
Atlantic	98	1.44	1.77	0.81
Pacific	144	2.11	5.74	0.37

Table 1. Frequencies of category 4 and 5 hurricanes

For both Atlantic and Pacific oceans, the sample variance of annual frequency is larger than the sample mean, and thus, the data exhibits over-dispersion. By matching the mean and variance in (3) with the sample quantities, we estimate the parameter p in the NBLP, i.e., $\hat{p} = \bar{N}/S^2$ where $\bar{N}$ and S^2 are the sample mean and variance of annual number of hurricanes, respectively. To see the empirical correlation pattern (as a function of the length of time interval), for each $t = 1, \ldots, 5$ (years), the correlation estimate of two regions is computed by splitting the time horizon into non-overlapping intervals of size t. To reduce the impact of a specific starting point, the correlation estimates are computed by moving the starting year from 1950 to 1964 are averaged.

Figure 5 shows the empirical correlation pattern and the one obtained from the backward simulated NBLPs. We set the parameters for the BS as $p_1 = 0.81$, $p_2 = 0.37$ for the NBLPs, and $\rho = -0.31$ for a bivariate normal distribution in simulating correlated gamma-Poisson random numbers. For the the time-varying parameter structure (11) in the Clayton copula, we use $\theta_s = -0.62$, $\theta_m = 0.17$ and $\theta_f = -0.8$, which provides a correlation pattern close to the empirical one.

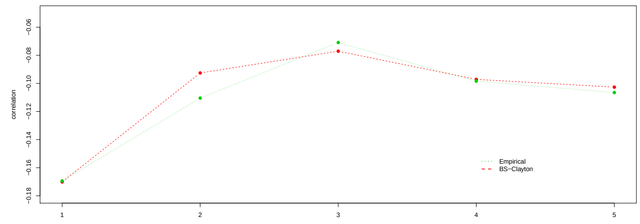


Figure 5. Empirical and simulated correlation patterns for the frequencies of hurricanes

6 Summary

In the context of quantitative risk modelling, the backward simulation methods have recently been developed to incorporate extremal and flexible correlation structures. In this paper, we have extended the method to the negative binomial Lévy process which possesses several desirable features for financial modelling of over-dispersed data. The simplistic conditional independence assumption has been generalized by using a parametric family of copulas for the conditional law to provide enough flexibility in correlation patterns. Even though we focused on bivariate case, with a specification of multivariate copula for the simulation of dependent increment series, our approach is readily extended to multivariate cases. We will pursue this study in a sequel.

Acknowledgements

T. Bae is supported by the Discovery Grant program of the Natural Science and Engineering Research Council of Canada (NSERC).

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