Mathematics and Statistics 7(4): 146-149, 2019
DOI: 10.13189/ms.2019.070408 http://www.hrpub.org

A Note on Locally Metric Connections

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Received June 17, 2019; Revised August 30, 2019; Accepted September 15, 2019

Abstract The Fundamental Theorem of Riemannian geometry states that on a Riemannian manifold there exist a unique symmetric connection compatible with the metric tensor. There are numerous examples of connections that even locally do not admit any compatible metrics. A very important class of symmetric connections in the tangent bundle of a certain manifolds (afinnely flat) are the ones for which the curvature tensor vanishes. Those connections are locally metric. S.S. Chern conjectured that the Euler characteristic of an affinely flat manifold is zero. A possible proof of this long outstanding conjecture of S.S. Chern would be by verifying that the space of locally metric connections is path connected. In order to do so one needs to have practical criteria for the metrizability of a connection. In this paper we give necessary and sufficient conditions for a connection in a plane bundle above a surface to be locally metric. These conditions are easy to be verified using any local frame. Also, as a global result we give a necessary condition for two connections to be metric equivalent in terms of their Euler class.

Keywords Affine Connections, Locally Metric Connections, Affinely Flat Manifolds, Euler Class of a Connection

1 Introduction

Throughout this paper E will denote a real vector bundle of rank m over a manifold M of dimension n.

Definition 1.1. *A connection D in E is called locally metric if and only if there is a bundle metric that is parallel with respect to D in the neighborhood of any point $p \in M$.*

Equivalently one can prove that a connection is locally metric if its restricted holonomy group $Hol_0(D)$ is compact(see [3]). However, since the explicit calculation of holonomy is very difficult, we will not make use of this variant of the definition. The literature for locally metric connections is scarce. There are not very many practical criteria to verify weather a connection is locally metric. Some promising results can be found in ([1, 2, 9, 10, 8, 4, 6]) From a pure mathematical point of view the investigation of the set of symmetric and locally metric connections in the tangent bundle of a manifold is related to a long outstanding conjecture of Chern for affinely flat manifolds. Chern's statement conjectures that the Euler characteristic of a compact afinely flat manifold is zero. The author showed in ([5]) that for a locally metric connection there is a natural cohomology class of M that coincides with the Euler class of the bundle E in the case when the connection is globally metric. He also proved that if the set of locally metric connections in the tangent bundle of an affinely flat manifold is path connected, then Chern's statement is true(see [5]).
If two symmetric and locally metric connections in the tangent bundle of a manifold M share a parallel metric in the neighborhood of any point then, by the fundamental theorem of Riemannain geometry, they are equal. If the symmetry is dropped one can no longer conclude that they are equal and the relationship becomes a weaker equivalence relation. The global result of this paper proves that two locally metric equivalent connections have the same Euler class.

2 On the local metrizability of connections

We will now establish some conditions for a connection to be metric in terms of local frames.

Lemma 2.1. *Let D be connection in E. Then D is locally metric if and only if in the neighborhood of any point there exist a local frame of the bundle*

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_m)$$

such that the connection matrix θ with respect to σ is skew symmetric.

Proof. The only if part is obvious. For the if part let

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_m)$$

as in hypothesis. Take the metric g that makes the frame σ orthonormal, that is

$$g(\sigma_i, \sigma_j) = \delta_{ij}.$$

Differentiating g in the direction $X \in TM$, we get

$$(D_X g)(\sigma_i, \sigma_j) = 0 - (\theta_{ij}(X) + \theta_{ji}(X)) = 0,$$

hence g is parallel with respect to D.

For the sake of concreteness let us give a couple of examples of connections which are locally metric but not globally metric. Let $M = \mathbb{T}^2$ be the two dimensional torus and let $f = (f_1, f_2)$ a global frame of commutative vector fields in TM. The dual frame will be denoted $f^* = (f^1, f^2)$. Consider the connection D that has its connection matrix with respect to f

$$\theta_f = \begin{Vmatrix} f^1 & 0 \\ 0 & 0 \end{Vmatrix}$$

This is a flat, symmetric connection in TM hence locally metric. However there is no global metric on M that will have D as its Levi Civita connection. Assume, by contradiction, that there is a global metric g that is preserved by D and consider $\gamma = \gamma(t), \ -\infty < t < \infty$ an integral curve of f_1. On M we have a globally defined smooth function defined as

$$h = g(f_1, f_1),$$

and if we look at its restriction $h(t)$ to γ, then $h(t)$ satisfies the differential equation

$$h'(t) = 2h(t),$$

and hence it is an exponential. Consequently h is not bounded on M which is a contradiction.

Before giving the second example we need to make a definition

Definition 2.1. *Let D_1 and D_2 be two locally metric connections in a vector bundle E. We say they are metric equivalent if and only if in the neighborhood of any point there exist a local bundle metric g such that*

$$D_1 g = D_2 g = 0.$$

Our second example is a connection $\tilde{\nabla}$ in the tangent bundle of a generic Riemannian manifold (M, g) that is metric equivalent $(\nabla g \equiv \tilde{\nabla} g \equiv 0)$ to its Levi Civita connection ∇ but not symmetric. We will base our example on Theorem 2.1 in ([7]). Take u a one form on M and let $u^{\#}$ be its metric dual with respect to g. Define

$$\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - G(X, Y)u^{\#} \tag{1}$$

It Is easy to verify that equation (1) defines a connection compatible with the metric g and that its torsion satisfies

$$T_{\tilde{\nabla}}(X, Y) = u(Y)X - u(X)Y,$$

and therefore $\tilde{\nabla}$ is non-symmetric if $u \neq 0$.

Next let us note that if a connection preserves a metric (locally) then it also preserves a volume form, hence we have the following criterion

Lemma 2.2. *Le D be a connection in a bundle E and θ be its connection matrix with respect to the local frame $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_m)$. Then D preserves a local volume form if and only if*

$$Tr\ \Omega = d(Tr\ \theta) = 0.$$

Proof. First let us note that

$$Tr\ \Omega = d(Tr\ \theta)$$

follows immediately from Cartan's equation

$$\omega = d\theta + \theta \wedge \theta$$

by taking the trace of both sides.
Next let ω be a volume form on E. Whitout loss of generality we may assume that the frame $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_m)$ is positive with respect to ω, that is:

$$\omega = f\sigma^1 \wedge \sigma^2 \wedge \ldots\ldots \wedge \sigma^m, \tag{2}$$

where $f > 0$ is a local smooth function and $\sigma^* = (\sigma^1, \sigma^2, \cdots, \sigma^m)$ is the dual frame of σ.. Taking the derivative of ω in the direction of an arbitrary tangent vector field X we get

$$D_X\omega = X(f)\sigma^1\wedge\sigma^2 \ldots\wedge\sigma^m + f\sum_{k=1}^{m} \sigma^1\wedge\sigma^2\wedge\ldots.\wedge D_X\sigma^k \ldots\wedge\sigma^m \tag{3}$$

Since

$$D_X e^k = -\theta_{sk}(X)e^s$$

and by using (3) we obtain

$$D_X\omega = (X(f) - f\sum_{k=1}^{m} \theta_{kk}(X))\sigma^1 \wedge \sigma^2 \ldots \wedge \sigma^m \tag{4}$$

which shows that

$$D\omega = 0$$

is equivalent to

$$df = fTr\ \theta$$

or

$$d(\ln f) = Tr\ \theta. \tag{5}$$

Obviously (5) is equivalent to $Tr\ \theta$ is closed.

For the case of plane bundles we will be able to give a practical test for local metrizability. For this criteria we will need the following linear algebra lemmas

Lemma 2.3. *Let U be a nonsingular 2×2. Let A, B two matrices with determinant equal to one that satisfy*

$$A^{-1}UA = B^{-1}UB = kJ,$$

with $J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$. *Then*

$$B = AS,$$

with S orthonormal.

Proof. First, let us note the following easy to prove identity for 2×2 matrices

$$XJX^{-1} = XX^TJ, \tag{6}$$

where X is any 2×2 matrix with determinant equal to one and J is as in the hypothesis. Using the hypothesis we get

$$U = kAJA^{-1} = kBJB^{-1},$$

and by using (6) we get

$$AA^T = BB^T \tag{7}$$

and therefore

$$A^{-1}B = A^T(B^T)^{-1}. \tag{8}$$

Let's set
$$S = A^{-1}B = A^T(B^T)^{-1}.$$
We have
$$SS^T = A^T(B^T)^{-1}B^T(A^T)^{-1} = I,$$
which proves the lemma.

Lemma 2.4. *Let U be a 2×2 a nonzero real matrix. Then there exist a matrix A such that $A^{-1}UA$ is skew if and only if U has purely imaginary eigenvalues. The matrix A can be chosen such that its determinant is one ant its entries are smooth in terms of the entries of U.*

Proof. For the only if part let's assume that there exist a matrix A such that
$$A^{-1}UA = \begin{Vmatrix} 0 & a \\ -a & 0 \end{Vmatrix}$$
with $a \neq 0$. Then the characteristic equation of U is
$$\lambda^2 + a^2 = 0$$
and therefore its eigenvalues are $\pm ia$.

For the if part of the lemma let us first note that we can always find a positive defined, symmetric matrix S such that
$$SU + U^T S = 0. \tag{9}$$
Let A be the only symmetric, nonsingular matrix satisfying
$$AA^T = S. \tag{10}$$
We have
$$A(A^{-1}UA + A^T U^T (A^{-1})^T)A^T = US + SU^T = 0,$$
and therefore
$$A^{-1}UA + A^T U^T (A^{-1})^T = 0.$$

Theorem 2.1. *Let E be a plane bundle over a surface Σ and D a connection in E. Assume that the curvature of D at $p \in \Sigma$ is nonzero. Let $\sigma = (\sigma_1, \sigma_2)$ be a local frame in E around p, θ its connection matrix and Ω its curvature matrix with respect to σ. Let $e = (e^1, e^2)$ be a local frame of one forms around $p \in \Sigma$ and let U be the matrix with real entries defined by the equation*
$$\Omega = (e^1 \wedge e^2)U.$$
Then the D is locally metric if and only if the following two conditions are satisfied

(a) U has purely imaginary eigenvalues at every point in the neighborhood of $p \in \Sigma$

(b) If A is a matrix as in Lemma (2.4) then the connection matrix is skew symmetric with respect to the frame $\tilde{\sigma} = \sigma A$

Proof. For the "only if" part, since D is assumed to be locally metric, according to Lemma there exist a frame τ such that the connection matrix ψ with respect to this frame is skew and consequently its curvature matrix Ψ is also skew. Let B be the matrix defined by $\tau = \sigma B$.

We have
$$\Psi = B^{-1}\Omega B = B^{-1}(e^1 \wedge e^2)UB = (e^1 \wedge e^2)B^{-1}UB$$
and hence $B^{-1}UB$ is skew and non zero. According to Lemma (2.4) it follows that U has purely imaginary eigenvalues. Now let A be a matrix as in Lemma (2.4). Since both A and B can be chosen such that their determinants at every point are equal to one and since $B^{-1}UB$ and $A^{-1}UA$ are both skew, according to Lemma(2.3) we have
$$B = AS$$
with S orthonormal. Let $\tilde{\theta}$ be the connection matrix with respect to the frame
$$\tilde{\sigma} = \sigma A$$
Since
$$\tilde{\sigma}S = \sigma AS = \sigma B = \tau$$
it follows that
$$\tilde{\theta} = S^{-1}dS + S^{-1}\psi S.$$
But since S is orthonormal, from $S^T S = I$, by differentiating, it follows that $S^{-1}dS$ is skew as well as $S^{-1}\psi S$ and therefore $\tilde{\theta}$ is skew. The "if" part follows from Lemma (2.1).

Consider a plane bundle E over a surface Σ. Assume D is a connection on E with nowhere zero curvature. Our Theorem(2.1) and Lemma (2.1) provide an algorithm that allows us to determine whether a connection is locally metric. Here is a step by step description of the algorithm:

(a) Take a local frame σ and calculate the curvature matrix Ω with respect to this frame

(b) Take any local volume form on Σ and factor it out of Ω
$$\Omega = \omega U$$

(c) Calculate the eigenvalues of U. If at a point of the chosen neighborhood the eigenvalues are not purely imaginary, then the connection is not metric

(d) If the eigenvalues are purely imaginary on the entire chosen neighborhood, find a symmetric, smooth and positive solution to the system
$$US + SU^T = 0$$
and calculate $A = \sqrt{S}$

(e) Take the frame $\tilde{\sigma} = \sigma A$ and calculate the connection forms of D with respect to it.

(f) D is metric only and only if the connection matrix with respect to $\tilde{\sigma}$ is skew

3 On the metric equivalence of connections

In this section we give a necessary condition for two connections to be metric equivalent (See Definition (2.1)).

Theorem 3.1. *If D_0 and D_1 are two metric equivalent locally metric connections in E then their Euler class is the same.*

Proof. Let $\pi : M \times \mathbb{R} \to M$, denote the projection

$$\pi(p, t) = p$$

and let $\tau = \pi^*(E)$ denote the pullback of the bundle of E. Let $D_0^* = \pi^*(D_0)$ and $D_1^* = \pi^*(D_1)$ be the pullback of the two connections from E to τ. Consider the linear combination

$$D_t^* = (1 - t)D_1^* + tD_2^* \tag{11}$$

and as usual define a connection $\mathbb{D}$ in τ by

$$\mathbb{D}\sigma(p, t) = D_t^*\sigma \tag{12}$$

First we need to show that the connection $\mathbb{D}$ is locally metric. Let $(p, t) \in M \times \mathbb{R}$ be an arbitrary point. Since D_0 and D_1 are metric equivalent we can find a bundle metric g defined in a neighborhood $p \in U \in M$ such that

$$D_0 g = D_1 g = 0. \tag{13}$$

Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_m)$ be an orthonormal frame with respect to g. If we denote by $\theta_k, k = 0, 1$ the connection matrix of $D_k, k = 0, 1$ with respect to σ, then clearly θ_k's are both skew-symmetric. With respect to the pullback frame $\pi^*(\sigma)$ the connection $\mathbb{D}$ has the connection matrix θ and satisfies the equation

$$\theta = (1 - t)\theta_0 + t\theta_1, \tag{14}$$

and therefore is skew symmetric. By Lemma (2.1) it follows that $\mathbb{D}$ is locally metric. According to Lemma 3.1 in ([5]) its Euler form is well defined and closed. Let us denote the Euler class of $\mathbb{D}$ by $\mathcal{A}$ and the the Euler class of D_k by $\mathcal{A}_k$, for $k = 0, 1$.
We define a family of maps

$$i_t : M \to M \times \mathbb{R}$$

by

$$i_t(p) = (p, t).$$

We have

$$i_0^*\mathcal{A} = \mathcal{A}_0$$

and

$$i_1^*\mathcal{A} = \mathcal{A}_1.$$

Because the two maps i_0 and i_1 are homotopic and $\mathcal{A}$ is closed, they induce the same map in cohomology and it follows that

$$\mathcal{A}_0 - \mathcal{A}_1$$

is exact on M, and the conclusion of the theorem follows.

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