Mathematics and Statistics 7(4): 116-123, 2019 http://www.hrpub.org
DOI: 10.13189/ms.2019.070404

Implicit Two-point Block Method with Third and Fourth Derivatives for Solving General Second Order ODEs

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Abstract In this paper we present an implicit two-point block method for solving directly the general second order ordinary differential equations (ODEs). The method incorporates the first and second derivatives of $f(x, y, y')$, which are the third and fourth derivatives of the solution. The method is derived using Hermite Interpolating Polynomial as the basic function. A performance comparison of the two-point block method is compared in term of accuracy to several existing methods, which have order almost equal or higher than that of the new method. Numerical results interpret the accuracy and efficacy of the new method. Application of the new method is discussed.

Keywords Implicit, Block Method, Second Order ODEs, Hermite Interpolation

1 Introduction

The second order ordinary differential equations arise in many areas of applied science and engineering represented by mathematical models. Commonly, second order ODEs can be solved by reducing the equation into system of first order ODEs, but this could be computationally costlier than the direct methods. Some attention by some eminent scholars has been given to solve the problem directly, such as [1]- [4]. However, Awoyemi [5], Akinfenwa and Jator [6]developed methods of higher order ODEs directly. The techniques for solving the problem directly are (i) using interpolation and collocation technique, which is the most common, see [7]- [10], and (ii) using interpolation and integration technique. Several studies have been conducted to implement derivative methods in solving ODEs, see [11]- [16].

In this paper, we develop a two-point block implicit method by using interpolation and integration technique, which involved the third and fourth derivatives of the solution in (1) with a constant step-size, which aim at obtaining more accurate numerical results. We are concerned with development of the numerical solution of initial value problems (IVPs) for second-order ODEs of the form:

$$y'' = f(x, y, y') \qquad y(a) = y_0 \qquad y'(a) = y'_0 \qquad x \in [a, b] \tag{1}$$

where the third and fourth derivatives of the solution to (1) are

$$y''' = f'(x, y, y') = f_x + f_y y' + f_{y'} f = g(x, y, y'),$$

and

$$\begin{aligned} y^{(4)} &= f''(x, y, y') = f_{xx} + f_{xy} y' + f_{xy'} f + (f_{xy} \\ &+ f_{yy} y' + f_{yy'} f) y' + f_y f = q(x, y, y'), \end{aligned}$$

2 Derivation of the Method

The derivation of the proposed method is based on Hermite Interpolating Polynomial $P_2(x)$,which interpolates $f(x, y, y')$ at two-points. The polynomial has the form

$$P(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i - 1} f_i^{(k)} L_{i,k}(x) \tag{2}$$

where,

$$f_i = f(x_i, y_i, y'_i), \quad x_i = a + ih, \quad i = 0, 1, ..., n$$

$$h = \frac{b-a}{n},$$

$L_{(i,k)}(x)$is the generalized Lagrange polynomial $i = 0, 1, ..., n, \quad k = 0, 1, ..., m_i$ and n is a positive integer.

In order to compute the two approximation values of y_{n+1} and y_{n+2} simultaneously at x_{n+1} and x_{n+2}, where x_n becomes the starting point and x_{n+2} is the last point in the block with step size $2h$. The evaluation solution of y_{n+2} at the point x_{n+2} will be restored as the initial value for the next iteration. The interval $[a, b]$ is divided into a series of blocks that contained

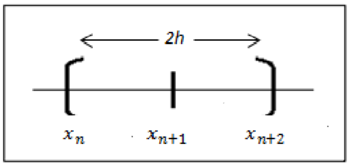

Figure 1. Two point block method.

two points at each block as shown in the Figure 1. The method will compute two points concurrently using one earlier block.

$$\begin{aligned} P_2(x) &= L_{0,0}(x)f_0 + L_{1,0}(x)f_1 + L_{2,0}(x)f_2 + L_{0,1}(x)g_0 \\ &+ L_{1,1}(x)g_1 + L_{2,1}(x)g_2 + L_{0,2}(x)q_0 + L_{1,2}(x)q_1 \\ &+ L_{2,2}(x)q_2 \end{aligned}$$

Where g and q are the third and fourth derivatives of the solution respectively and

$$\begin{aligned} L_{0,0}(x) &= (\frac{x-x_{n+1}}{x_n-x_{n+1}})^3(\frac{x-x_{n+2}}{x_n-x_{n+2}})^3[1-(\frac{3}{x_n-x_{n+1}} \\ &+ \frac{3}{x_n-x_{n+2}})((x-x_n)-\frac{(x-x_n)^2}{2}(\frac{6}{x_n-x_{n+2}} \\ &+ \frac{6}{x_n-x_{n+1}}))-(\frac{6}{(x_n-x_{n+1})^2}+\frac{6}{(x_n-x_{n+2})^2} \\ &+ \frac{18}{(x_n-x_{n+1})(x_n-x_{n+2})})(\frac{(x-x_n)^2}{2})] \\ L_{1,0}(x) &= (\frac{x-x_n}{x_{n+1}-x_n})^3(\frac{x-x_{n+2}}{x_n-x_{n+2}})^3[1-(\frac{3}{x_{n+1}-x_{n+2}} \\ &+ \frac{3}{x_{n+1}-x_n})((x-x_{n+1}) \\ &- \frac{(x-x_{n+1})^2}{2}(\frac{6}{x_{n+1}-x_{n+2}}+\frac{6}{x_{n+1}-x_n})) \\ &- (\frac{6}{(x_{n+1}-x_{n+2})^2}+\frac{6}{(x_{n+1}-x_n)^2} \\ &+ \frac{18}{(x_{n+1}-x_n)(x_{n+1}-x_{n+2})})\frac{(x-x_{n+1})^2}{2}] \\ L_{2,0}(x) &= (\frac{x-x_n}{x_{n+2}-x_n})^3(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}})^3[1 \\ &- (\frac{3}{x_{n+2}-x_n}+\frac{3}{x_{n+2}-x_{n+1}})((x-x_{n+2}) \\ &- \frac{(x-x_{n+2})^2}{2}(\frac{6}{x_{n+2}x_{n+1}}+\frac{6}{x_{n+2}-x_n})) \\ &- (\frac{6}{(x_{n+2}-x_{n+1})^2}+\frac{6}{(x_{n+2}-x_n)^2} \\ &+ \frac{18}{(x_{n+2}-x_n)(x_{n+2}-x_{n+1})})(\frac{(x-x_{n+2})^2}{2})] \end{aligned}$$

$$\begin{aligned} L_{0,1}(x) &= (\frac{x-x_{n+1}}{x_n-x_{n+1}})^3(\frac{x-x_{n+2}}{x_n-x_{n+2}})^3[(x-x_n) \\ &- (\frac{6}{x_n-x_{n+2}}+\frac{6}{x_n-x_{n+1}}(\frac{(x-x_n)^2}{2}))] \\ L_{1,1}(x) &= (\frac{x-x_n}{x_{n+1}-x_n})^3(\frac{x-x_{n+2}}{x_{n+1}-x_{n+2}})^3[(x-x_{n+1}) \\ &- (\frac{6}{x_{n+1}-x_{n+2}}+\frac{6}{x_{n+1}-x_n}(\frac{(x-x_{n+1})^2}{2}))] \\ L_{2,1}(x) &= (\frac{x-x_n}{x_{n+2}-x_n})^3(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}})^3[(x-x_{n+2}) \\ &- (\frac{6}{x_{n+2}-x_{n+1}}+\frac{6}{x_{n+2}-x_n}(\frac{(x-x_{n+2})^2}{2}))] \\ L_{0,2}(x) &= \frac{(x-x_n)^2}{2}(\frac{x-x_{n+1}}{x_n-x_{n+1}})^3(\frac{x-x_{n+2}}{x_n-x_{n+2}})^3 \\ L_{1,2}(x) &= \frac{(x-x_{n+1})^2}{2}(\frac{x-x_n}{x_{n+1}-x_n})^3(\frac{x-x_{n+2}}{x_{n+1}-x_{n+2}})^3 \\ L_{2,2}(x) &= \frac{(x-x_{n+2})^2}{2}(\frac{x-x_n}{x_{n+2}-x_n})^3(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}})^3. \end{aligned}$$

The approximation values of y_{n+1} and y_{n+2} at the points x_{n+1} and x_{n+2} can be obtained by:

First point: integrating (1) once and twice with respect to x over the interval $[x_n, x_{n+1}]$. The following formulae can be obtained

$$\int_{x_n}^{x_{n+1}} y''dx = \int_{x_n}^{x_{n+1}} f(x,y,y')dx \qquad (3)$$

$$\int_{x_n}^{x_{n+1}}\int_{x_n}^{x} y''dxdx = \int_{x_n}^{x_{n+1}}\int_{x_n}^{x} f(x,y,y')dxdx \qquad (4)$$

Let $x_{n+1} = x_n + h$ which gives

$$y'(x_{n+1}) = y'(x_n) + \int_{x_n}^{x_{n+1}} f(x,y,y')dx \qquad (5)$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \int_{x_n}^{x_{n+1}} (x_{n+1}-x)f(x,y,y')dx \qquad (6)$$

Then, $f(x,y,y')$ in (5) and (6) will be replaced with Hermite Interpolating Polynomial $P_2(x)$.

By taking $x = x_{n+2} + sh$ and $s = \frac{x-x_{n+2}}{h}$ and $dx = hds$ and changing the limit of integration from -2 to -1, we have

$$\begin{aligned} y'(x_{n+1}) &= y'(x_n) + \int_{-2}^{-1}[f_0L_{0,0}(s)+f_1L_{1,0}(s)+f_2L_{2,0}(s) \\ &+ g_0L_{0,1}(s)+g_1L_{1,1}(s)+g_2L_{2,1}(s)+q_0L_{0,2}(s) \\ &+ q_1L_{1,2}(s)+q_2L_{2,2}(s)]hds \end{aligned} \qquad (7)$$

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + hy'(x_n) + \int_{-2}^{-1}[f_0L_{0,0}(s)+f_1L_{1,0}(s) \\ &+ f_2L_{2,0}(s)+g_0L_{0,1}(s)+g_1L_{1,1}(s)+g_2L_{2,1}(s) \\ &+ q_0L_{0,2}(s)+q_1L_{1,2}(s)+q_2L_{2,2}(s)]hds \end{aligned} \qquad (8)$$

where,

$$
\begin{aligned}
L_{0,0} &= \frac{1}{16}(s+1)^3 s^3 (24s^2+105s+116)\\
L_{1,0} &= -(2+s)^3 s^3 (3s^2+6s+4)\\
L_{2,0} &= \frac{1}{16}(2+s)^3 (s+1)^3 (24s^2-9s+2)\\
L_{0,1} &= \frac{h}{8}(2+s)(s+1)^3 s^3 (10+\frac{9}{2}s)\\
L_{1,1} &= -h(s+1)(2+s)^3 s^3\\
L_{2,1} &= \frac{-9}{16}hs(s+1)^3(2+s)^3(\frac{-2}{9}+s)\\
L_{0,2} &= \frac{h^2}{16}(s+1)^3(2+s)^2 s^3\\
L_{1,2} &= \frac{-h^2}{16}(s+1)^3(2+s)^2 s^3\\
L_{2,2} &= \frac{h^2}{16}(s+1)^3(2+s)^3 s^2
\end{aligned}
$$

Evaluating the integrals in (7) and (8) using **MAPLE**, gives,

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{h}{13440}(5669f_n + 8192f_{n+1} - 421f_{n+2})\\
&+ \frac{h^2}{4480}(303g_n - 560g_{n+1} + 47g_{n+2}) \qquad (9)\\
&+ \frac{h^3}{40320}(169q_n + 1024q_{n+1} - 41q_{n+2})\\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{60}(19f_n + 12f_{n+1} - f_{n+2})\\
&+ \frac{h^3}{20160}(911g_n - 1024g_{n+1} + 113g_{n+2}) \qquad (10)\\
&+ \frac{h^4}{20160}(53q_n + 252q_{n+1} - 11q_{n+2})
\end{aligned}
$$

Similarly,
Second point: integrating (1) once and twice with respect to x over the interval $[x_{n+1}, x_{n+2}]$. The following formulae can be obtained

$$\int_{x_{n+1}}^{x_{n+2}} y'' dx = \int_{x_{n+1}}^{x_{n+2}} f(x,y,y')dx \qquad (11)$$

$$\int_{x_{n+1}}^{x_{n+2}}\int_{x_{n+1}}^{x} y'' dxdx = \int_{x_{n+1}}^{x_{n+2}}\int_{x_{n+1}}^{x} f(x,y,y')dxdx \qquad (12)$$

Let $x_{n+2} = x_{n+1} + h$ which gives

$$y'(x_{n+2}) = y'(x_{n+1}) + \int_{x_{n+1}}^{x_{n+2}} f(x,y,y')dx \qquad (13)$$

$$y(x_{n+2}) = y(x_{n+1}) + hy'(x_{n+1}) + \int_{x_{n+1}}^{x_{n+2}} (x_{n+2}-x) f(x,y,y')dx \qquad (14)$$

Then, $f(x,y,y')$ in (13) and (14) will be replaced with Hermite Interpolating Polynomial $P_2(x)$.
By taking $x = x_{n+2} + sh$ and $s = \frac{x-x_{n+2}}{h}$ and $dx = hds$ and changing the limit of integration from -1 to 0, we have

$$
\begin{aligned}
y'(x_{n+2}) &= y'(x_{n+1}) + \int_{-1}^{0} [f_0L_{0,0}(s) + f_1L_{1,0}(s)\\
&+ f_2L_{2,0}(s) + g_0L_{0,1}(s) + g_1L_{1,1}(s) + g_2L_{2,1}(s)\\
&+ q_0L_{0,2}(s) + q_1L_{1,2}(s) + q_2L_{2,2}(s)]hds \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
y(x_{n+2}) &= y(x_{n+1}) + hy'(x_{n+1})\\
&+ \int_{-1}^{0} [f_0L_{0,0}(s) + f_1L_{1,0}(s) + f_2L_{2,0}(s)\\
&+ g_0L_{0,1}(s) + g_1L_{1,1}(s) + g_2L_{2,1}(s) \qquad (16)\\
&+ q_0L_{0,2}(s) + q_1L_{1,2}(s) + q_2L_{2,2}(s)]hds
\end{aligned}
$$

Evaluating th integrals in (15) and (16) using **MAPLE**, gives,

$$
\begin{aligned}
y'_{n+2} &= y'_{n+1} + \frac{h}{13440}(-421f_n + 8192f_{n+1} + 5669f_{n+2})\\
&+ \frac{h^2}{4480}(-47g_n + 560g_{n+1} - 303g_{n+2}) \qquad (17)\\
&+ \frac{h^3}{40320}(-41q_n + 1024q_{n+1} + 169q_{n+2})\\
y_{n+2} &= y_{n+1} + hy'_{n+1} + \frac{h^2}{13440}(-197f_n + 5504f_{n+1}\\
&+ 1413f_{n+2}) + \frac{h^3}{40320}(-197g_n + 2992g_{n+1} \qquad (18)\\
&+ 905g_{n+2}) + \frac{h^4}{40320}(-19q_n + 520q_{n+1} + 63q_{n+2})
\end{aligned}
$$

The formulae (9),(10),(17) and(18) can be written in a matrix difference equation to obtain the order of the new method as follows :

$$\alpha Y_m = h\beta Y'_m + h^2\gamma F_m + h^3\delta G_m + h^4\zeta Q_m,$$

where β, γ, δ, andζ defined as,

$$\alpha = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 0 & \frac{5669}{13440} & \frac{64}{105} & \frac{-421}{13440}\\ 0 & \frac{19}{60} & \frac{1}{5} & \frac{-1}{60}\\ 0 & \frac{-421}{13440} & \frac{64}{105} & \frac{5669}{13440}\\ 0 & \frac{-197}{13440} & \frac{43}{105} & \frac{471}{4480} \end{bmatrix}, \delta = \begin{bmatrix} 0 & \frac{303}{4480} & \frac{-1}{8} & \frac{47}{4480}\\ 0 & \frac{911}{20160} & \frac{-16}{315} & \frac{113}{20160}\\ 0 & \frac{-47}{4480} & \frac{1}{8} & \frac{-303}{4480}\\ 0 & \frac{-197}{40320} & \frac{187}{2520} & \frac{-181}{8064} \end{bmatrix},$$

$$\zeta = \begin{bmatrix} 0 & \frac{169}{40320} & \frac{8}{315} & \frac{-41}{40320}\\ 0 & \frac{53}{20160} & \frac{1}{80} & \frac{-11}{20160}\\ 0 & \frac{-41}{40320} & \frac{8}{315} & \frac{169}{40320}\\ 0 & \frac{-19}{40320} & \frac{13}{1008} & \frac{1}{640} \end{bmatrix}, , Y_m = \begin{bmatrix} y_{n-1}\\ y_n\\ y_{n+1}\\ y_{n+2} \end{bmatrix},$$

$$Y'_m = \begin{bmatrix} y'_{n-1}\\ y'_n\\ y'_{n+1}\\ y'_{n+2} \end{bmatrix}, F_m = \begin{bmatrix} f_{n-1}\\ f_n\\ f_{n+1}\\ f_{n+2} \end{bmatrix}, \begin{bmatrix} g_{n-1}\\ g_n\\ g_{n+1}\\ g_{n+2} \end{bmatrix}, Q_m = \begin{bmatrix} q_{n-1}\\ q_n\\ q_{n+1}\\ q_{n+2} \end{bmatrix}.$$

According to Henrici[16], it can be said that the new methods have**order p** if $C_0 = C_1 = C_p = 0 = C_{p+1} = 0$, $C_{p+2} \neq 0$, which is the error cosntant. Where

$$C_0 = \sum_{j=0}^{k} \alpha_j \qquad C_1 = \sum_{j=0}^{k} (j\alpha_j)$$

$$C_2=\sum_{j=0}^{k}(\frac{j^2}{2!}\alpha_j-\beta_j),$$

$$C_3=\sum_{j=0}^{k}(\frac{j^3}{3!}\alpha_j-j\beta_j-\gamma_j),$$

$$\vdots$$

$$C_q=\sum_{j=0}^{k}\frac{j^q}{q!}\alpha_j-\sum_{j=0}^{K}\frac{j^{q-1}}{(q-1)!}\beta_j-\sum_{j=0}^{K}\frac{j^{q-2}}{(q-2)!}\gamma_j,\quad q=4,5,..$$

In our method $C_0=C_1=.....C_9=0$ and $C_{10}=[-0.010194415,0,-0.048511786,0]^T$ Thus, we conclude that the method in(9),(10),(17)and(18) is of the order 8.

3 Implementation

In this section, we briefly mentioned the implementation of two-point block method. The approximation solutions in (9),(10),(17) and(18) will be estimated using the predictor-corrector schemes. The predictor equations using Taylor method:

$$\begin{aligned}
y'^p_{n+i} &= y'^p_{n+(i-1)}+hf^c_{n+(i-1)}\\
y^p_{n+i} &= y^p_{n+(i-1)}+hy'^p_{n+(i-1)}+\frac{h^2}{2!}f^c_{n+(i-1)};\ i=1,2\\
f^p_{n+i}(x,y,y') &= f(x_{n+i},y^p_{n+i},y'^p_{n+i}),\\
g^p_{n+i}(x,y,y') &= f'(x_{n+i},y^p_{n+i},y'^p_{n+i}),\\
q^p_{n+i}(x,y,y') &= f''(x_{n+i},y^p_{n+i},y'^p_{n+i})
\end{aligned}\tag{19}$$

Define Eq.(19) as the initial approximation and applying the derived method in (9),(10),(17) and(18) as corrector. In order to obtain the corrector iteration, the following equations will be used

$$\begin{aligned}
y'^c_{n+1} &= y'^c_n+\frac{h}{13440}(5669f^c_n+8192f^p_{n+1}-421f^p_{n+2})\\
&+ \frac{h^2}{4480}(303g^c_n-560g^p_{n+1}+47g^p_{n+2})\\
&+ \frac{h^3}{40320}(169q^c_n+1024q^p_{n+1}-41q^p_{n+2})\\
y^c_{n+1} &= y^c_n+hy'^c_n+\frac{h^2}{60}(19f^c_n+12f^p_{n+1}-f^p_{n+2})\\
&+ \frac{h^3}{20160}(911g^c_n-1024g^p_{n+1}+113g^p_{n+2})\\
&+ \frac{h^4}{20160}(53q^c_n+252q^p_{n+1}-11q^p_{n+2})\\
y'^c_{n+2} &= y'^c_{n+1}+\frac{h}{13440}(-421f^c_n+8192f^p_{n+1}+5669f^p_{n+2})\\
&+ \frac{h^2}{4480}(-47g^c_n+560g^p_{n+1}-303g^p_{n+2})\\
&+ \frac{h^3}{40320}(-41q^c_n+1024q^p_{n+1}+169q^p_{n+2})\\
y^c_{n+2} &= y^c_{n+1}+hy'^c_{n+1}+\frac{h^2}{13440}(-197f^c_n+5504f^p_{n+1}\\
&+ 1413f^p_{n+2})+\frac{h^3}{40320}(-197g^c_n+2992g^p_{n+1}\\
&+ 905g^p_{n+2})+\frac{h^4}{40320}(-19q^c_n+520q^p_{n+1}+63q^p_{n+2})
\end{aligned}\tag{20}$$

Then, the next corrector iteration will be

$$\begin{aligned}
y'^c_{n+1} &= y'^c_n+\frac{h}{13440}(5669f^c_n+8192f^p_{n+1}-421f^p_{n+2})\\
&+ \frac{h^2}{4480}(303g^c_n-560g^p_{n+1}+47g^p_{n+2})\\
&+ \frac{h^3}{40320}(169q^c_n+1024q^p_{n+1}-41q^p_{n+2})\\
y^c_{n+1} &= y^c_n+hy'^c_n+\frac{h^2}{60}(19f^c_n+12f^p_{n+1}-f^p_{n+2})\\
&+ \frac{h^3}{20160}(911g^c_n-1024g^p_{n+1}+113g^p_{n+2})\\
&+ \frac{h^4}{20160}(53q^c_n+252q^p_{n+1}-11q^p_{n+2})\\
y'^c_{n+2} &= y'^c_{n+1}+\frac{h}{13440}(-421f^c_n+8192f^c_{n+1}\\
&+ 5669f^p_{n+2})+\frac{h^2}{4480}(-47g^c_n+560g^c_{n+1}\\
&- 303g^p_{n+2})+\frac{h^3}{40320}(-41q^c_n+1024q^c_{n+1}\\
&+ 169q^p_{n+2})\\
y^c_{n+2} &= y^c_{n+1}+hy'^c_{n+1}+\frac{h^2}{13440}(-197f^c_n\\
&+ 5504f^c_{n+1}+1413f^p_{n+2})+\frac{h^3}{40320}(-197g^c_n\\
&+ 2992g^c_{n+1}+905g^p_{n+2})\\
&+ \frac{h^4}{40320}(-197q^c_n+520q^c_{n+1}+63q^p_{n+2})
\end{aligned}\tag{21}$$

4 Problem tested

We present in this section numerical results for some tested problems and application by applying the proposed method derived in this paper using **C language** code. The following notations will be used in the tables

I2PBDO8	The implicit two-point block method proposed in this paper, order 8.
Jator1	Three step hybrid linear multistep method with three off-step points[17], order 7.
Vigo-Ramos	variable-step Falkner method in the predictor-corrector mode [18] , order 8.
Jator2	The two step block hybrid third derivative method with one off-step point [19], order 8.
Akinfenwa	Four step block hybrid collocation method with four off-step points [20], order 9.
Badmus1	implicit linear multistep Hybrid Block method of Uniform order 8 at k=5 [21], order 8.
Badmus2	Uniform order eight implicit block method[22], order 8.
h	step size
NS	Total number of steps
MAXE	Maximum absolute error

4.1 problems

Proplem 1: Bessel's differential equation:

$x^2y'' + xy' + (x^2 - 0.25)y = 0 \qquad [1,8]$
$y(1) = \sqrt{\frac{2}{\pi}}sin(1),$
$y'(1) = \frac{2cos(1) - sin(1)}{\sqrt{2\pi}},$

Exact solution:
$y(x) = \sqrt{\frac{2}{\pi x}}sin(x)$

Source: Jator, S. N. (2010)

Proplem 2: A nonlinear problem

$y'' = 6y^2, \qquad [0,10]$
$y(0) = 1, \qquad y'(0) = -2$

Exact solution:
$y(x) = (1+x)^{-2}$

Source: Singh, G., & Ramos, H. (2019)

Proplem 3: A nonlinear problem

$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0 \qquad x > 0$
$y(1) = 1, \qquad y'(1) = 1$

Exact solution:
$y(x) = \frac{5}{3x} - \frac{2}{3x^4}$

Source: Badmus, A. M. (2014b)

Proplem 4: A linear problem

$y'' - 3y' = 8e^{2x},$
$y(0) = 1, \qquad y'(0) = 1$

Exact solution:
$y(x) = -4e^{2x} + 3e^{3x} + 2$

Source: Badmus, A. M. (2014b)

Table 1. Numerical results for solving Problem1.

h	NS	Method	Absolute errors
$\frac{7}{67}$	67	I2PBDO8	1.281197-13
		Jator1	6.5286-11
		Vigo-Ramos	7.1122-7
$\frac{7}{82}$	82	I2PBDO8	1.942890-14
		Jator1	1.3679-11
		Vigo-Ramos	9.2632-8
$\frac{7}{112}$	112	I2PBDO8	9.992007-16
		Jator1	81.1897-12
		Vigo-Ramos	1.2108-10
$\frac{7}{64}$	64	I2PBDO8	1.971201-013
		Jator2	1.2356-11
		Akinfenwa	9.2632-8
$\frac{7}{128}$	128	I2PBDO8	2.775558-016
		Jator2	5.9008-14
		Akinfenwa	1.2108-10

Table 2. Numerical results for solving Problem2.

h	Method	MAXE
10^{-2}	I2PBDO8	3.1326-13
	VJATOR	6.8742-12
	ode113	3.3940-11
10^{-4}	I2PBDO8	8.4359-12
	VJATOR	8.6447-12
	ode113	9.1131-12

4.2 Application: Van Der Pol oscillator(Majid et al(2012))

We consider Van Der Pol oscillator which is governed by second order differential equation

Table 3. Numerical results for solving Problem3.

Badmus1	Badmus2	I2PBDO8
$1.645-07$	8.3-08	9.977597-11
6.6035-07	1.16-06	3.886180-10
4.4141-06	6.638-06	8.716488-10
1.299366-05	9.491-06	1.537585-09
1.6377561-05	1.9535-06	2.391527-09
2.8296833-05	9.416-06	3.422115-09
5.051695-05	4.6505-05	4.634431-09
3.860932-05	4.7122-05	6.017069-09
7.490927 -05	1.86926-04	7.575093-09
1.458835-04	4.43321-04	9.297083-09

Table 4. Numerical results for solving Problem4.

Badmus1	Badmus2	I2PBDO8
$3.159-07$	1.58-07	4.440892-16
1.2709-06	3.176-06	0.000000+00
8.6554-06	1.2941-05	4.440892-16
2.59148-05	1.9323-05	0.000000+00
3.395058-05	4.0181-05	4.440892-16
5.990417-05	2.2075-05	2.220446-16
8.885833-05	8.9916-05	2.220446-16

$$y'' - 2\xi(1-y^2)y' + y = 0, \qquad [0,10]$$
$$y(0) = 0, \quad y'(0) = 0.5,$$

Where $\xi = 0.005$, y is a function of time x and ξ is a parameter that indicates the nonlinearity and the strength of damping. The problem does not have an analytical solution. The results found using I2PBDO8 are compared with the numerical solutions obtained in the *Mathematica* bulit in package *NDSolve*. Table5 shows the comparison in the numerical approximation of y at different points of x.

Table 5. Numerical results for solving Van Der Pol.

x	NDSolve	I2PBDO8
0.0	0.0000000	0.0000000
0.1	0.0500417	0.0500419
0.5	0.242704	0.242703
1.3	0.496936	0.496911
2.1	0.453212	0.453153
3.1	0.0227741	0.0227339
4.1	-0.45029	-0.450239
5.1	-0.521174	-0.521028
6.1	-0.105769	-0.105662
7.1	0.429448	0.429381
8.1	0.584845	0.584604
9.1	-0.286323	-0.286329
9.9	-0.286329	-0.286323

In addition, Figure 2, depicts the numerical approximations for the the equation in $[0, 10]$. It is obtained by I2PBDO8 with $h = 0.1$ which agree very well with the solution obtained by*Mathematica* bulit in package *NDSolve*

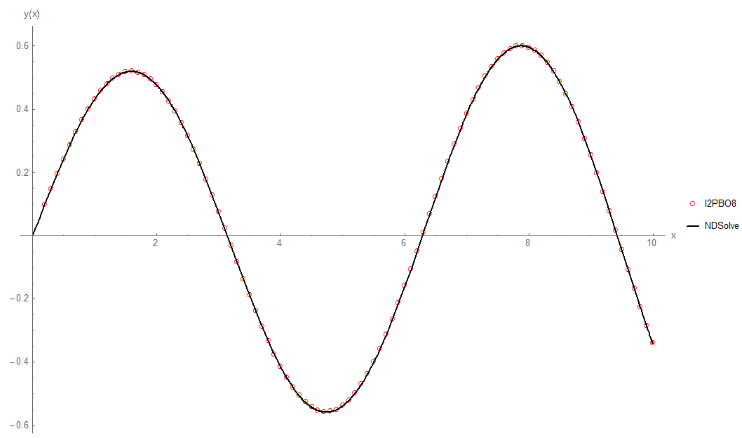

Figure 2. Plot of solution for Van Der Pol.

5 Discussion

We presented the numerical results of new method compared to the existing methods in tables. The basis for comparison with these methods is chosen because the order of the numerical methods presented in the separate works is either equal or higher to that of the new method. The data given in tables, show the superiority of I2PBDO8 in a term of accuracy.

For the Bessel's IVP, we have considered the seventh order hybrid block method in [17], the variable-step Falkner method of eighth order implemented in a predictor-corrector mode in [18], the eighth order two-step block hybrid third derivative method with one off-step point in [19] and the ninth order four step block hybrid collection method in [20]. In Table 1, comparison in term of absolute errors for different number of steps have been considered at x=8 which made by using our new method with previously literature methods.

For problem 2, the numerical results have been obtained by considering $h = 10^j, j = -2, -4$ which show maximum absolute errors compared to the variable step-size Jator's block method (VJATOR). Table 2 shows evidence for the good performance of the proposed method.

For problem 3 and 4, we have considered the eight order block methods, implicit linear multistep Hybrid Block method of Uniform order 8 in [21] and uniform order eight implicit block method in [22]. It is observed from Table 3 and 4 that the new method has better accuracy compared with previously existing methods [21] and [22].

In Figure 2 and Table 5, for the Van Der Pol problem, it is obvious that the solutions obtained by I2PBDO8 with step size $h = 0.1$ agree very well with the observation of *Mathematica* built in package *NDSolve*.

6 Conclusions

In this paper, we have developed the two-point block method with the third and fourth derivatives of the solution for directly solving general second order ODEs. The method has shown a significant improvement in accuracy. Therefore, it can be concluded that the method constructed is suitable for solving general second order ODEs.

Acknowledgements

The authors would like to thank the Universiti Putra Malaysia for the financial support of Putra grant (project No GPB/2017/9543500).

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