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Non-existence of Solutions of Diophantine Equations of the Form $p^x + q^y = z^{2n}$

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Abstract Numerous researches have been devoted in finding the solutions (x, y, z), in the set of non-negative integers, of Diophantine equations of type $p^x + q^y = z^2$ (1), where the values p and q are fixed. In this paper, we also deal with a more generalized form, that is, equations of type $p^x + q^y = z^{2n}$ (2), where n is a positive integer. We will present results that will guarantee the non-existence of solutions of such Diophantine equations in the set of positive integers. We will use the concepts of the Legendre symbol and Jacobi symbol, which were also used in the study of other types of Diophantine equations. Here, we assume that one of the exponents is odd. With these results, the problem of solving Diophantine equations of this type will become relatively easier as compared to the previous works of several authors. Moreover, we can extend the results by considering the Diophantine equations $p^x + q_1^{y_1} q_2^{y_2} \dots q_k^{y_k} = z^{2n}$ (3) in the set of positive integers.

Keywords Exponential Diophantine Equations, Jacobi Symbol, Legendre Symbol

1. Introduction

For the past decade, Diophantine equations of type $p^x + q^y = z^2$ have been widely studied for various values of p and q. Commencing in 2007, Acu [1] showed that in the set $\mathbb{N}_0$ of nonnegative integers, the Diophantine equation $2^x + 5^y = z^2$ has only solutions $(x, y, z) = (3,0,3)$ and $(x, y, z) = (2,1,3)$. In 2011, Suvarnami, et. al. [2] showed that the Diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solutions in the $\mathbb{N}_0$. In 2012, Sroysang [3] showed that the Diophantine equation $8^x + 19^y = z^2$ has the only solution $(x, y, z) = (1,0,3)$ in $\mathbb{N}_0$. Sroysang [4] also showed that the Diophantine equation $3^x + 5^y = z^2$ has the only solution $(x, y, z) = (1,0,2)$ in $\mathbb{N}_0$. Then, recently, Rabago [5-6] determined all the solutions of the Diophantine equations $2^x + 31^y = z^2$, $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$. Other works related Diophantine equations of type $p^x + q^y = z^2$ can be seen in the references, such as [7-8].

The goal of this paper is to present an easier way of showing that certain Diophantine equations of type $p^x + q^y = z^2$, where p and q are fixed positive integers, may fail to have solutions in the set $\mathbb{N}$ of positive integers. This is done by using the concepts of Legendre symbol and the Jacobi symbol.

2. Preliminaries

The following definitions and well known results in Number Theory are essential in our study.

Definition 2. 1. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. If the congruence $z^2 \equiv a \ (mod\ p)$ has a solution in z, then a is said to be a quadratic residue of p. Otherwise, a is called a quadratic nonresidue of p.

Definition 2. 2. (Legendre Symbol) Let p be an odd prime and a be an integer with $\gcd(a, p) = 1$. Then the Legendre symbol $\left(\frac{a}{p}\right)$ is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{if } a \text{ is a quadratic residue of } p \\ -1, \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases} \quad (4)$$

Definition 2. 3. (Jacobi symbol) Let $\gcd(a, b) = 1$, where $b > 1$ is odd. If $b = p_1 p_2 \dots p_k$ is the prime factorization of b, where the p_i's are not necessarily distinct, then the Jacobi symbol $\left(\frac{a}{b}\right)$ is defined to be

$$\left(\frac{a}{p}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right), \quad (5)$$

where the symbols $\left(\frac{a}{p_i}\right)$ are Legendre symbols.

Lemma 2. 4. Let $\gcd(a, b) = 1$, where a is an integer and $b > 1$ is odd. If the congruence $z^2 \equiv a \ (mod\ b)$ has a solution in z, then the Jacobi symbol $\left(\frac{a}{b}\right) = 1$.

Lemma 2. 5. Consider any integer a, and an integer $n > 1$ with $\gcd(a, n) = 1$. Suppose the prime factorization of n is $n = 2^{k_0} {p_1}^{k_1} \dots {p_r}^{k_r}$. Then $z^2 \equiv a (mod\ n)$ has a solution z if and only if

- $\left(\frac{a}{p_i}\right) = 1$ for $i = 1, 2, \ldots, r$;
- $a \equiv 1\ (mod\ 4)$ if $4 \mid n$, but $8 \nmid n$; $a \equiv 1\ (mod\ 8)$ if $8 \mid n$.

The proofs of the above lemmas can be seen in [9].

The following values of the Legendre symbols $\left(\frac{2}{p}\right), \left(\frac{3}{p}\right)$ and $\left(\frac{5}{p}\right)$ are as follows: If p is an odd prime, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1, if\ p \equiv 1 \text{ (mod 8) } or\ p \equiv 7 \text{ (mod 8)} \\ -1, if\ p \equiv 3 \text{ (mod 8) } or\ p \equiv 5 \text{ (mod 8)} \end{cases}, \quad (6)$$

and if $p \neq 3$ we have

$$\left(\frac{3}{p}\right) = \begin{cases} 1, if\ p \equiv 1 \text{ (mod 12) } or\ p \equiv 11 \text{ (mod 12)} \\ -1, if\ p \equiv 5 \text{ (mod 12) } or\ p \equiv 7 \text{ (mod 12)} \end{cases}. \quad (7)$$

Also, if $p \neq 5$,

$$\left(\frac{2}{p}\right) = \begin{cases} 1, if\ p \equiv 1, 9, 11, 19 \text{ (mod 20)} \\ -1, if\ p \equiv 3, 7, 13, 17 \text{ (mod 20)} \end{cases}. \quad (8)$$

From the Generalized Quadratic Reciprocity Law, if a and b are odd integers with $a, b > 1$ and $\gcd(a, b) = 1$ then

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\cdot\frac{b-1}{2}} = \begin{cases} 1, if\ a \equiv 1 \text{ (mod 4) or } b \equiv 1 \text{(mod 4)} \\ -1, if\ a \equiv b \equiv 3 \text{ (mod 4)} \end{cases} \quad (9)$$

So if p is an odd integer that is not necessarily prime and $\gcd(3, p) = 1$, we have

$$\left(\frac{3}{p}\right) = \begin{cases} 1, if\ p \equiv 1 \text{ (mod 4)} \\ -1, if\ p \equiv 3 \text{ (mod 4)} \end{cases} \Rightarrow \left(\frac{3}{p}\right) = \begin{cases} \left(\frac{p}{3}\right), if\ p \equiv 1 \text{ (mod 4)} \\ -\left(\frac{p}{3}\right), if\ p \equiv 3 \text{ (mod 4)} \end{cases}. \quad (10)$$

Moreover, we also know that

$$\left(\frac{p}{3}\right) = \begin{cases} 1, if\ p \equiv 1 \text{ (mod 3)} \\ -1, if\ p \equiv 2 \text{ (mod 3)} \end{cases}, \quad (11)$$

which tells us that the Jacobi symbol $\left(\frac{3}{p}\right)$ coincides with the Legendre symbol. The same thing goes with the other Jacobi symbols including $\left(\frac{5}{p}\right)$, where $p \neq 5$ is an odd prime.

3. Main Results

We now present the results of the study. Examples are given to illustrate the results.

Theorem 3. 1. Let $p, q > 1$ be odd integers with $\gcd(p, q) = 1$. Then the Diophantine equation $p^x + q^y = z^{2n}$, where $n \in \mathbb{N}$, has no solution in $\mathbb{N}$ if x is odd and the Jacobi symbol $\left(\frac{p}{q}\right)$ is equal to -1. It has no solution also in $\mathbb{N}$ when y is odd and $\left(\frac{q}{p}\right) = -1$.

Proof. Suppose that $\left(\frac{p}{q}\right) = -1$ and assume in contrary that the equation $p^x + q^y = z^2$ has a solution $(x, y, z) \in \mathbb{N}^3$ with x being odd. Then taking the equation modulo q, we get that $z^2 \equiv p^x\ (mod\ q)$ has a solution. This implies that $\left(\frac{p^x}{y}\right) = 1$ by Lemma 2.4. We then have

$$1 = \left(\frac{p^x}{y}\right) = \underbrace{\left(\frac{p}{q}\right)\left(\frac{p}{q}\right)\ldots\left(\frac{p}{q}\right)}_{x \text{ times}} = \left(\frac{p}{q}\right)^x = (-1)^x = -1, \quad (12)$$

which is a contradiction. Hence $p^x + q^y = z^2$ has no solution if x is odd. Using similar arguments, one can show that the Diophantine equation $p^x + q^y = z^{2n}$, for any $n \in \mathbb{N}$, has no solutions in $\mathbb{N}$. The proof is also similar for the case where $\left(\frac{q}{p}\right) = -1$ and y is odd.

Example 3. 2. In [4], Sroysang showed that the Diophantine equation $3^x + 5^y = z^2$ has a unique solution $(x, y, z) = (1, 0, 2)$ in $\mathbb{N}_0$ and has no solution in $\mathbb{N}$. We can verify this result when x is odd:

$$\left(\frac{3}{5}\right) = -1\ and\ \left(\frac{5}{3}\right) = -1. \quad (13)$$

Example 3. 3. The Diophantine equation $17^x + 27^y = z^2$ has no solution in $\mathbb{N}$. Note that if x and y are both even, then $17^x + 27^y \equiv 2\ (mod\ 4)$. On the other hand, since z^2 is even, we get $z^2 \equiv 0\ (mod\ 4)$. If x is odd, we get no solution because

$$\left(\frac{17}{27}\right) = \left(\frac{17}{3}\right)^3 = \left(\frac{2}{3}\right)^3 = (-1)^3 = -1. \quad (14)$$

If y is odd, we also get no solution because

$$\left(\frac{27}{17}\right) = \left(\frac{3}{17}\right)^3 = (-1)^3 = -1. \quad (15)$$

If 17 is replaced by any positive integer congruent to 2 modulo 3, the equation, still has no solution wherever x is odd.

We can generalize Theorem 2.1 by considering p and q to be any positive integers such that $\gcd(p, q) = 1$. This is stated in the next theorem.

Theorem 3. 4. Let p and q be positive integers such that $\gcd(p, q) = 1$. If $p = 2^{k_0} {p_1}^{k_1} \ldots {p_r}^{k_r}$ is a prime factorization of $p > 1$, then the Diophantine equation $p^x + q^y = z^{2n}$ where y is odd, has no solutions if at least one of the following is satisfied:

a. $\left(\frac{a}{p_i}\right) = -1$ for some $1 \leq i < r$
b. $q \not\equiv 1\ (mod\ 4)$ if $4 \mid p$ but $8 \nmid p; q \not\equiv 1\ (mod\ 8)\ if\ 8 \mid p$.

The result still holds if p and y are replaced by q and x, respectively.

Proof. First, suppose condition (a) is satisfied. Since y is odd, we have $\left(\frac{q^y}{p_i}\right) = \left(\frac{q}{p_i}\right)^y = (-1)^y = -1$ for some i. The contrapositive of Lemma 2.5 would mean that the congruence $z^2 \equiv q^y\ (mod\ p)$ will have no solution. Hence, if y is odd, then the Diophantine equation

$p^x + q^y = z^2$ has no solution, and consequently $p^x + q^y = z^2$ has no solution also. Now suppose condition (b) is satisfied. Since y is odd, we have $q^y \not\equiv 1 \ (mod\ 4)$ if $4 \mid p$ but $8 \nmid p$ and $q^y \not\equiv 1 \ (mod\ 8)$ if $8 \mid p$. Using Lemma 2.5, we see that the congruence $z^2 \equiv q^y \ (mod\ p)$ has no solution, which further indicates that $p^x + q^y = z^2$ has no solution also when y is odd. The proof is similar when p and y are replaced by q and x, respectively. Hence, it is omitted.

Example 3. 5. Consider the Diophantine equation $23^x + 81^y = z^2$. If x is odd and we use Lemma 2.5, we get

$$\left(\frac{23}{81}\right) = \left(\frac{23}{3}\right)^4 = \left(\frac{2}{3}\right)^4 = (-1)^4 = 1\,. \quad (16)$$

Hence, no conclusion can be drawn about the non-existence of solutions. However, if we use Theorem 3.4, since 3 is a factor of 81, we get that

$$\left(\frac{23}{3}\right) = \left(\frac{2}{3}\right) = -1\,. \quad (17)$$

This means that the equation has no solution when x is odd.

Example 3. 6. In [3], Sroysang showed that the Diophantine equations $8^x + 19^y = z^2$ has no positive integer solution. We can verify the result if x is odd, since

$$\left(\frac{8}{19}\right) = \left(\frac{2}{19}\right)^3 = (-1)^3 = -1. \quad (18)$$

Also, if y is odd, it has no solution since $8 \mid p$ but $q = 19 \not\equiv 1 \ (mod\ 8)$.

So far, we have been dealing with the case where p and q are relatively prime. Here is the extension for the case where $\gcd(p, q) > 1$.

Theorem 3. 7. Let $\gcd(p, q) = d > 1$. If

$$\frac{p}{d} = 2^{k_0} \cdot \prod_{i=1}^{r} {p_i}^{k_i} \ \text{d} \ \frac{q}{d} = 2^{l_0} \cdot \prod_{i=1}^{r} {q_i}^{k_i} \quad (19)$$

are the prime factorizations of p/d and q/d respectively, then the Diophantine equation $p^x + q^y = z^{2n}$ has no solution if

a. x is odd and $\left(\frac{p}{q_i}\right) = -1$ for some i such that $\gcd(q_i, d) = 1$

b. x is odd and $\left(\frac{q}{p_i}\right) = -1$ for some i such that $\gcd(p_i, d) = 1$.

Proof. First, assume that condition (a) is true. Suppose in contrary, that the Diophantine equation $p^x + q^y = z^2$ has a solution when x is odd. By taking modulo q_i on the equation, we see that $z^2 \equiv p^x \ (mod\ q_i)$ has a solution. Note that $\gcd(p^x, q_i) = 1$. Hence the Legendre symbol $\left(\frac{p^x}{q_i}\right) = \left(\frac{p}{q_i}\right)^x = 1$. This implies that $\left(\frac{p}{q_i}\right) = 1$. This contradicts the assumption. Hence, the Diophantine equation $p^x + q^y = z^2$ has no solutions if x is odd. We get the same result if we consider $p^x + q^y = z^{2n}$, where $n > 1$. The proof is similar for case (b).

Example 3. 8. Consider the Diophantine equation $1155^x + 2691^y = z^2$. Here, note that $\gcd(1155, 2691) = 3$ and that $\frac{1155}{3} = 385 = 5 \cdot 7 \cdot 11$ and $\frac{2691}{3} = 897 = 3 \cdot 13 \cdot 23$. Computing the value of $\left(\frac{1155}{13}\right)$, we get

$$\left(\frac{1155}{13}\right) = \left(\frac{3}{13}\right)\left(\frac{5}{13}\right)\left(\frac{7}{13}\right)\left(\frac{11}{13}\right) = (1)(-1)\left(\frac{13}{7}\right)\left(\frac{13}{11}\right) = (-1)\left(\frac{6}{7}\right)\left(\frac{2}{11}\right) = (-1)\left(\frac{2}{7}\right)\left(\frac{3}{7}\right) = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right) = (1)(-1) = -1\,. \quad (20)$$

Thus, the equation has no solution in $\mathbb{N}$ if x is odd. Moreover,

$$\left(\frac{2691}{7}\right) = \left(\frac{3}{7}\right)\left(\frac{3}{7}\right)\left(\frac{13}{7}\right)\left(\frac{23}{7}\right) = \left(\frac{3}{7}\right)^2\left(\frac{13}{7}\right)\left(\frac{23}{7}\right) = (1)\left(\frac{2}{7}\right)^2\left(\frac{3}{7}\right) = (1)(1)(-1) = -1\,. \quad (21)$$

So, the equation has still no solution in $\mathbb{N}$ if y is odd.

We end this section by providing a result for a more general form of the Diophantine equation $p^x + q^y = z^2$. The proof is similar to the preceding theorems, hence we omit it.

Theorem 3. 9. The Diophantine equation $p^x + {q_1}^{y_1}{q_2}^{y_2}\dots{q_n}^{y_n} = z^2$ has no solutions in $\mathbb{N}$ if x is odd. Furthermore, there exists a prime r such that $\gcd(p, r) = 1, \left(\frac{p}{r}\right) = -1$ and $r \mid q_i$ for some i.

As a consequence of the Legendre symbols (or the Jacobi symbols) $\left(\frac{2}{p}\right), \left(\frac{3}{p}\right)$ and $\left(\frac{5}{p}\right)$, we have the following conclusions: If q is prime and y is odd, then

a. the Diophantine equation $2^x + q^y = z^{2n}$ has no solutions in $\mathbb{N}$ if $q \equiv 3 \ (mod\ 8)$ or $q \equiv 5 \ (mod\ 8)$.

b. the Diophantine equation $3^x + q^y = z^{2n}$ has no solutions in $\mathbb{N}$ if $q \equiv 5 \ (mod\ 12)$ or $q \equiv 7 \ (mod\ 12)$.

c. the Diophantine equation $5^x + q^y = z^{2n}$ has no solutions in $\mathbb{N}$ if $q \equiv 3, 7, 13, 17 \ (mod\ 20)$.

4. Summary

In this paper, we present criteria that guarantee non-existence of solutions of Diophantine equations (2) and other related equations. Theorems 3.1, 3.4 and 3.7 are written in a way that it can be used in a straightforward manner to show non-existence of solutions. Theorem 3.9 is an extension of the study to other related equations. We remark that these theorems only discuss the non-existence of solutions of Diophantine equations of type (2) and do not give a method of finding for solutions. Interestingly, these results reduce the cases need to be considered when finding for solutions of such equations.

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