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Equilibrium of Gravitating System of Spherical Gas-Dust Cloud

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Abstract In this paper, we take several different analytical and numerical approaches to studying the equilibrium of a gravitating system of a gas-dust cloud. We consider one-dimensional self-gravitating spherically symmetric fluid flow of a gas-dust cloud. We discuss the equilibrium of the system of gas-dust cloud by using simplified analytic stellar, polytropic models. A condition for the equilibrium of the cloud in the form of a differential equation is used. Using mass density in the cloud as a given function, we obtained the corresponding pressure analytically within the cloud and determined the central pressure. In dealing with a polytropic model, we found the analytical and numerical solution of the Lane-Emden equation for the various values of the polytropic index after that, central density and central pressure were obtained. Finally, the density and pressure of the cloud for the various values of the polytropic index are calculated. The result found by the simplified analytical method and the polytropic method is compared.

Keywords Self-gravitating, Polytropic model, Lane-Emden equation, Equilibrium

1 Introduction

A fundamental problem in the theory of star and planet formation is the physical structure of gas-dust clouds. The theory of the structure and evolution of the stars and planets have been formulated in the first half of the twentieth century; it is connected with names like Lane, Emden, Eddington, Chandrasekhar, and Fowle [1]. In the second half of the twentieth century, theories about the formation star and planet were greatly refined; this is due in part to new observation techniques and to computer simulations of its structure and evolution. It is connected with names like Penston, Larson, Hunter, Shu, and Safronov [2–6].

The equilibrium stellar models have been built by many authors e.g. [1, 6–8]. The first stellar models before the age of computers were polytropic models [9]. The polytropic models provided pretty good estimates of the density and pressure in stars. In 1966 Stein [10] discussed some way in which one could study stars using analytic models. The linear analytic model has been used in an extensive exploration of the essential features of the stellar structure.

In this work, we will study a model of the self-gravitating gas-dust cloud. We assume the model is one-dimensional of a non-rotating, non-magnetic, spherical self-gravitating gas-dust cloud. The assumption of spherical is playing a fundamental role in explaining qualitative behaviour through the numerical and analytical solving of equations of equilibrium.

The cloud changes very slowly during most of its life and so may be considered to be in hydrostatic equilibrium. In hydrostatic equilibrium, there is no motion in the fluid, so the velocity and time derivative terms in equations can be set to zero. Then the equations for hydrostatic equilibrium may be derived from the static version of the Euler equation and Poisson's equation for the gravitational field.

In this paper, we intend to investigate the equilibrium of the gravitating system of a gas-dust cloud by the simplified analytical method and to compare the findings with the polytropic method.

2 Equations of hydrostatic equilibrium

The equation of hydrostatic equilibrium is

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{1}$$

where $G = 6.672.10^{-11} m^3 s^{-2} Kg^{-1}$ is the gravitational constant, $M(r)$ is the mass inside a cloud of radius r, the minus sign takes care of the fact that the pressure decreases outward. The equation of mass conversation is

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \tag{2}$$

so that

$$M(r) = \int_0^r 4\pi\rho z^2 dz. \tag{3}$$

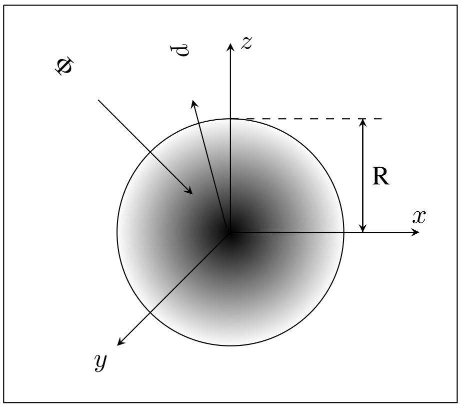

Figure 1. The plot of a spherical symmetric gas-dust cloud, the gravity force Φ is balanced by pressure force p

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3 Simplified analytical stellar models

In the study of stars, Stein [10] used the linear stellar model, showing that this model is useful for the determination of some representative values of the stellar characteristics. The linear stellar model of a cloud represents one of a class of models for which the equation of hydrostatics equilibrium may be integrated in a closed form. In this model the density is given by the linear relation

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right), \tag{4}$$

where ρ_c is central density, R is the radius of the cloud. The linear model has been used in an extensive, although qualitative, exploration of the basic features of stellar and evolution. We use the equation (3) to derive

$$\begin{aligned} M(r) &= \int_0^r 4\pi\rho_c \left(1 - \frac{z}{R}\right) z^2 dz = \\ &= 4\pi\rho_c r^3 \left(\frac{1}{3} - \frac{r}{4R}\right). \end{aligned} \tag{5}$$

Hence, we can find the total mass M

$$M = M(R) = \frac{\pi R^3 \rho_c}{3}. \tag{6}$$

For the determination of $p(r)$ of equation (1), we have the differential equation

$$\frac{dp}{dr} = -\frac{4\pi G\rho_c^2 r}{3}\left(1 - \frac{7r}{4R} + \frac{3}{4}\frac{r^2}{R^2}\right). \tag{7}$$

Integrating the foregoing equation with respect to r, we obtain

$$p(r) - p_c = -\frac{2\pi G\rho_c^2 r^2}{3}\left[1 - \frac{7r}{6R} + \frac{3r^2}{8R^2}\right], \tag{8}$$

where p_c is the central pressure of the cloud.
According to this model, the pressure at the surface is minimum. For simplicity, let us assume pressure at the surface is zero. i.e. at $r = R$, $p(r) = 0$. Applying this boundary condition $p(R) = 0$, we get

$$p_c = \frac{5}{36}\pi G\rho_c^2 R^2, \tag{9}$$

putting the value of the central pressure from equation (9) into equation (8), we get

$$p(r) = \frac{5\pi G}{36}\rho_c^2 R^2 \left(1 - \frac{24r^2}{5R^2} + \frac{28r^3}{5R^3} - \frac{9r^4}{5R^4}\right). \tag{10}$$

Now, we will extend the linear stellar model to the general case, and this treatment will give us a better result for the investigation of the equilibrium gravitating system of the gas-dust cloud. The linear stellar model can be generalised so assuming that the density in the cloud takes an arbitrary form

$$\rho(r) = \rho_c \left(1 - \frac{r^n}{R^n}\right). \tag{11}$$

The linear stellar model can be considered as a special case of the generalised stellar model when $n = 1$ in the equation (11). We use the equation (3) to derive

$$M(r) = \int_0^r 4\pi\rho_c \left(1 - \frac{z^n}{R^n}\right) z^2 dz, \tag{12}$$

or

$$M(r) = \frac{4\pi\rho_c r^3}{3}\left[1 - \frac{3}{(n+3)}\frac{r^n}{R^n}\right]. \tag{13}$$

We can normalise the equation to the total mass, M, by noting that at $r = R$,

$$M = M(R) = \frac{4\pi\rho_c R^3}{3}\frac{n}{(n+3)}. \tag{14}$$

By substituting the value of the central density ρ_c from equation (14) into equation (11), we can calculate the density at any point of the cloud as the following:

$$\rho = \frac{3M(n+3)}{4\pi R^3}\left(1 - \frac{r^n}{R^n}\right). \tag{15}$$

For the determination of $p(r)$ of equation (1), we have the differential equation

$$\begin{aligned} \frac{dp}{dr} &= -4\pi G\rho_c^2 r \times \\ &\times \left(\frac{1}{3} - \frac{(n+6)r^n}{3(n+3)R^n} + \frac{1}{(n+3)}\frac{r^{2n}}{R^{2n}}\right). \end{aligned} \tag{16}$$

Integrating the foregoing equation with respect to r, we obtain

$$\begin{aligned} p(r) &= p_c - \frac{2\pi G\rho_c^2 R^2}{3} \times \\ &\times \left[\frac{r^2}{R^2} - \frac{2(n+6)}{(n+3)(n+2)}\frac{r^{n+2}}{R^{n+2}} + \frac{3}{(n+3)(n+1)}\frac{r^{2n+2}}{R^{2n+2}}\right]. \end{aligned} \tag{17}$$

Applying the boundary condition $p(R) = 0$, we get

$$p_c = \frac{2\pi G\rho_c^2}{3}R^2\left[\frac{n^2(n+4)}{(n+3)(n+2)(n+1)}\right], \tag{18}$$

putting the value of equation (18) in equation (17), we can calculate the pressure p at any point of the cloud as the following

$$\begin{aligned} p(r) &= \frac{2\pi G\rho_c^2 R^2}{3(n+3)}\left(\frac{n^2(n+4)}{(n+2)(n+1)} - \left(\frac{r}{R}\right)^2 + \right. \\ &\left. + \frac{2(n+6)}{(n+2)}\left(\frac{r}{R}\right)^{n+2} - \frac{3}{(n+1)}\left(\frac{r}{R}\right)^{2n+2}\right). \end{aligned} \tag{19}$$

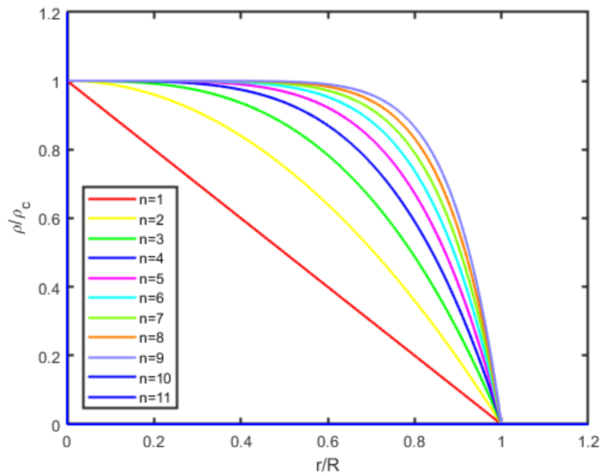

Figure 2. Graph of density ρ as function of radius r, when $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$

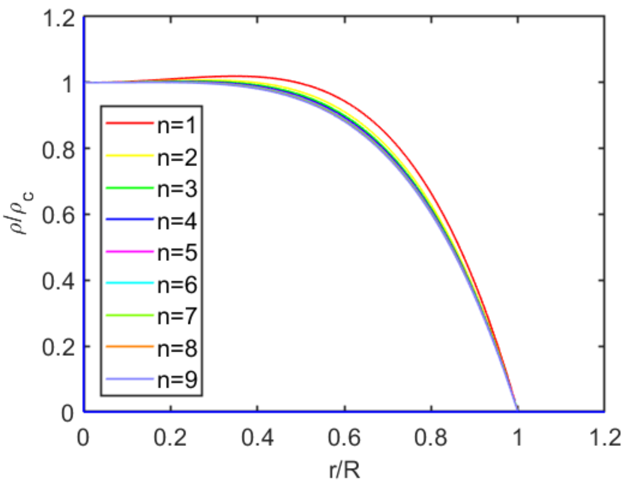

Figure 3. Graph of pressure p as function of radius r, when $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$

From equation (11), we get

$$\frac{r}{R} = \left(1 - \frac{\rho(r)}{\rho_c}\right)^{\frac{1}{n}}. \tag{20}$$

Substituting equation (20) into equation (19), we get

$$p(r) = \frac{2\pi G\rho_c^2 R^2}{3(n+3)}\left(\frac{n^2(n+4)}{(n+2)(n+1)} - \right. \\ - \left(1 - \frac{\rho(r)}{\rho_c}\right)^{\frac{2}{n}} + \frac{2(n+6)}{(n+2)} \\ \left. \left(1 - \frac{\rho(r)}{\rho_c}\right)^{\frac{n+2}{n}} - \frac{3}{(n+1)}\left(1 - \frac{\rho(r)}{\rho_c}\right)^{\frac{2n+2}{n}}\right). \tag{21}$$

The last equation can be the equation of state in the simplified analytical stellar model.
In this section, we have discussed simplified analytical stellar models that include two basic equations: the hydrostatic equilibrium equation and the mass conservation equation. Using mass density in the cloud as a given function we obtain the corresponding pressure analytically within the cloud and determine the central pressure.

4 polytropic model

The polytropic model has played a very important role in the understanding of a stellar interior [1]. In this paper, we consider a cloud of gas and dust. Assume gas and dust are one fluid.

4.1 Lane-Emden equation

The structure of the cloud can be given by the polytropic equation of state

$$p = K\rho^{1+\frac{1}{n}}, \tag{22}$$

where K and n are constants.
The Lane-Emden equation for self-gravitating, spherically symmetric polytropic fluid is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{23}$$

subject to initial conditions

$$\theta(0) = 1 \qquad \frac{d\theta(0)}{d\xi} = 0, \tag{24}$$

where ξ is a dimensionless variable which is related to radius r through the relation

$$r = \alpha\xi, \tag{25}$$

where α is calculated by formula

$$\alpha = \sqrt{\frac{(n+1)K}{4\pi G\lambda^{1-\frac{1}{n}}}}, \tag{26}$$

where $\lambda = \rho(0) = \rho_c$ is the central density. It is to be noted here that the solution of the Lane-Emden equation (23) gives the stellar structure and correctly represents the behaviour of stellar fluid. However, for a polytropic cloud, the polytropic index n should sufficiently be small as the cloud is expected to be less centrally condensed.
The dimensionless variable θ is related to variables ρ and p through the relation

$$p = K\lambda^{1+\frac{1}{n}}\theta^{n+1}, \qquad \rho = \lambda\theta^n. \tag{27}$$

Let's get the mass M of our spherical cloud. To do this, we use equation (3) and make substitutions $r = \alpha\xi$, $\rho = \lambda\theta^n(\xi)$, and $R = \alpha\xi_{max}$, to obtain

$$M = 4\pi\alpha^3\lambda\int_0^{\xi_{max}} \xi^2\theta^n(\xi)d\xi. \tag{28}$$

where ξ_{max} is the maximum value of ξ (Figure 4) where the density vanishes. Parameters ξ_{max} is calculated for the some values of the polytropic index n and represented in the Figure 4 . From Figure 4 one finds $\xi_{max} = \pi, 3.66, 4.36, 5.36$. It is

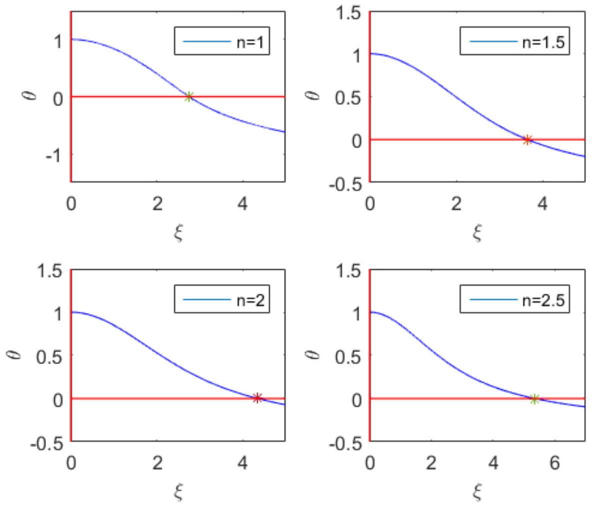

Figure 4. Solution of the Lane-Emden equation for $n = 1, 1.5, 2, 2.5$. Star show the first root of θ (ξ_{max} corresponding to the cloud radius) for each value of the polytropic index n.

a general property of the solutions that ξ_{max} grows monotonically with the polytropic index n.
Using equation (23), the integrand can be written as a perfect differential, so we get

$$M = 4\pi\alpha^3\lambda(-\xi_{max}^2\theta_1^{'}). \quad (29)$$

Here we define the shorthand $\theta_1^{'} = \left[\frac{d\theta(\xi)}{d\xi}\right]_{\xi=\xi_{max}}$.
From equation (26), and equation (29), we get

$$\rho_c = \rho(0) = \lambda = \left(\frac{4\pi}{(n+1)^3\xi_{max}^4\theta_1^{'2}}\frac{G^3M^2}{K^3}\right)^{\frac{n}{3-n}}. \quad (30)$$

Table (1) shows the central values of density, and the central values of pressure of a cloud for different values of polytropic index n which are calculated by formulas (27) and (30).

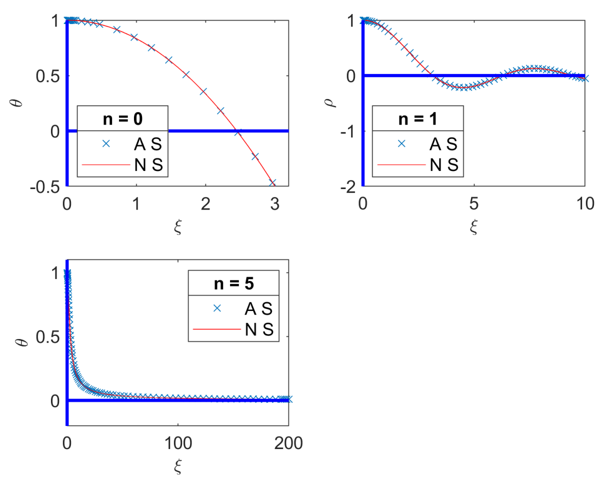

Figure 5. Exact analytical and numerical solution of the Lane-Emden equation for polytropic indexes $n = 0, 1, 5$..

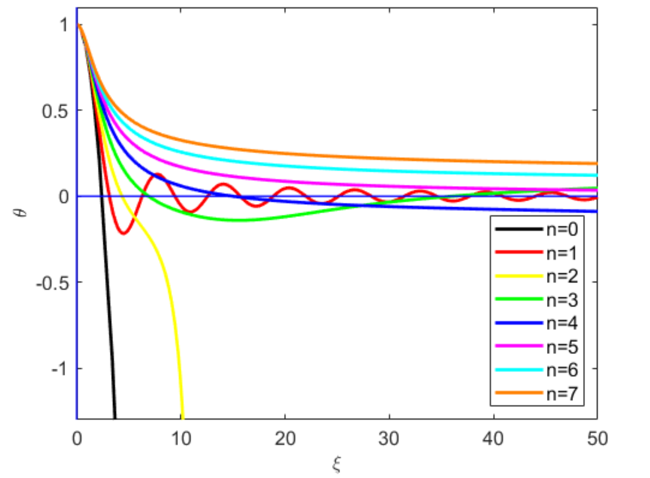

Figure 6. Numerical solution to the Lane-Emden equation for polytropic index $n \leq 7$.

4.2 Solving the Lane-Emden equation

Numerical and perturbation approaches to solve equation (23) have been considered by various authors [1, 11]. Most techniques which were used in handling the Lane-Emden-type problems are based on either series solutions or perturbation techniques. The values of n which are physically interesting, lie in the interval $[0, 5]$. The main difficulty in the analysis of this type of equation is the singularity behaviour occurring at $\xi = 0$. Unfortunately, the Lane-Emden equation (23) does not have an exact analytic solution for arbitrary values of n. In fact, there are only three exact analytic solutions [1, 12].
The first is for $n = 0$, which implies $\rho(r) = \lambda$, or a constant density sphere. For these models,

$$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2, \quad (31)$$

with $\xi_{max} = \sqrt{6}$ to satisfy the boundary condition of $\theta(\xi_{max}) = 0$. (This can be trivially checked via substitution.)
The second solution is for the $n = 1$ case, where

$$\theta_1(\xi) = \frac{\sin\xi}{\xi}. \quad (32)$$

Note here that there are an infinite number of values of ξ_{max} for which $\theta(\xi_{max})$. However, in practice, $\xi_{max} = \pi$, since all other values would give an unrealistic zero density somewhere in the middle of the cloud.
The third exact solution is for the $n = 5$ case, where

$$\theta_5(\xi) = \frac{1}{(1+\xi^{2/3})^{1/2}}. \quad (33)$$

4.3 Density and pressure as functions of radius

As is mentioned earlier, a cloud is expected to be less centrally condensed, n is likely to be small, we consider four different values of n in our investigation, namely 0.5, 1.0, 1.5,

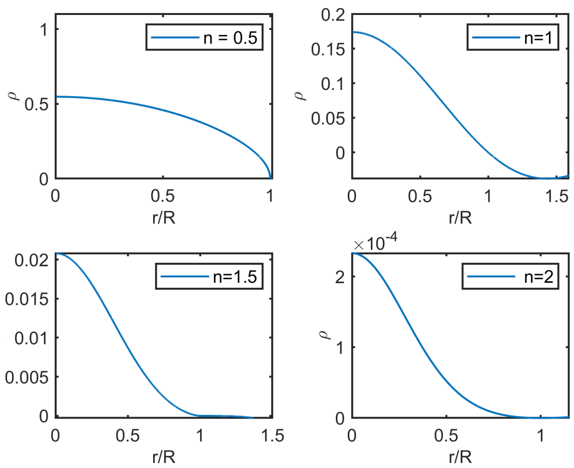

Figure 7. Density as a function of radius for polytropic index $n = 0.5, 1.0, 1.5, 2$.

n	α	$\lambda = \rho(0) = \rho_c$	$p(0) = p_c$
0.0	6.7754×10^7	1	1.6900×10^8
0.5	6.1432×10^7	0.5481	6.3381×10^5
1.0	9.5818×10^7	0.1742	1.1684×10^5
1.5	2.0424×10^8	0.0208	6.0680×10^3
2.0	9.4976×10^8	2.3309×10^{-4}	13.7008
2.5	1.0041×10^{11}	2.1745×10^{-10}	1.1423×10^{-7}
3.5	6.6738×10^2	8.5676×10^{14}	6.09387×10^{25}
4.0	6.9815×10^4	7.8931×10^8	5.0936×10^{17}
4.5	3.1064×10^5	9.2393×10^6	1.2561×10^{15}
4.9	5.5901×10^5	1.5995×10^5	1.1366×10^{14}

Table 1. Numerical central values of the density $\rho(0) = \lambda$ and pressure $p(0)$.

2. We integrated equation (23), and (24) numerically using Runge-Kutta method. Matlab is used to find central values of density λ, central values of pressure p_c, and α (see Table 1). The density ρ and pressure p inside the cloud have been calculated using relations (27), the corresponding central values of density, and pressure presented in Table 1. The results of our calculation are shown in figure 7, and figure 8.

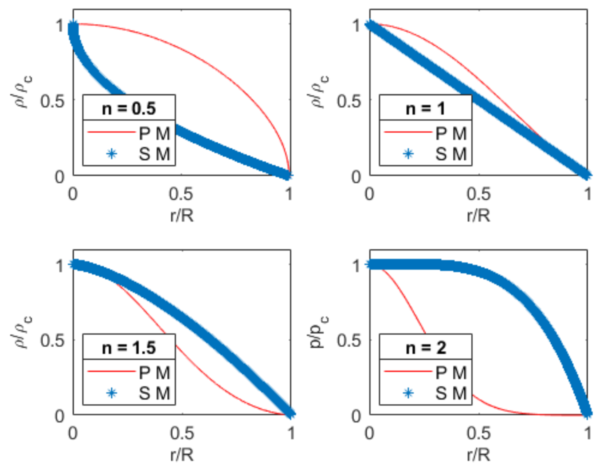

Figure 9. Graph of density for a polytropc, simplified model, when $n = 0.5, n = 1, n = 1.5, n = 2$.

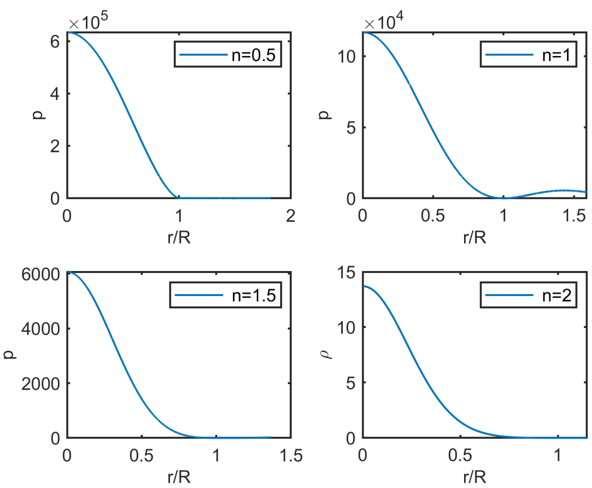

Figure 8. Pressure p as a function of radius r for polytropic index $n = 0.5, 1.0, 1.5, 2$.

5 Results and discussion

It can be observed from figure 7, and figure 8 that central density inside the cloud decreases with the increasing value of the polytropic index n, whereas for increasing the polytropic index n, the central pressure can be shown to be reduced.
The figure 9 shows the comparison between the graph of density which we found by a simplified analytical method and polytropic method for the equations of equilibrium for the four different values of n.
The figure 10 shows the comparison between the graph of pressure which we found by the simplified analytical method, and the polytropic method for the equations of equilibrium for the four different values of n.

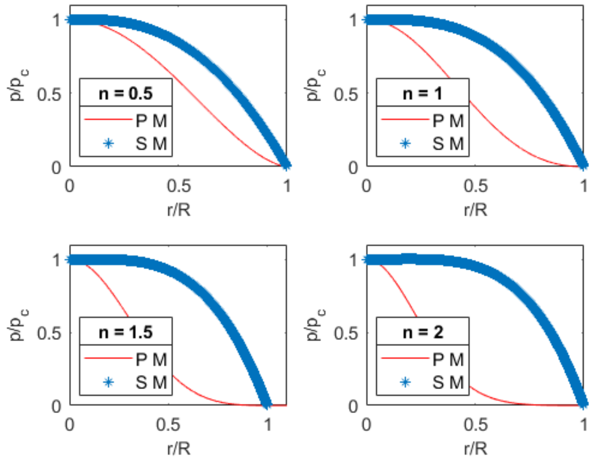

Figure 10. Graph of pressure for a polytropc, simplified model, when $n = 0.5, n = 1, n = 1.5, n = 2$.

6 Summary and conclusion

In the present paper, we have applied two methods in order to obtain solutions to equations of equilibrium gravitating system of the gas-dust cloud.

In the first part of the paper, we have discussed simplified analytical stellar models that include two basic equations: the hydrostatic equilibrium equation and the mass conservation equation. Using mass density in the cloud as a given function we obtain the corresponding pressure analytically within the cloud and determine the central pressure.

In the second part of the paper, we have investigated the problem using the polytropic method. We found the solution of the Lane-Emden equation after that, we obtained the central density and central pressure. Finally, the density and pressure of the cloud of polytropic index $n = 0.5, 1, 1.5, 2$ are calculated.

Based on the obtained results it can be pointed out here that the investigation of equilibrium of the cloud employing simplified and polytropic methods which may, therefore, give a fair representation of the density and pressure of the cloud.

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