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Some Properties of a Connected Topological Group

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Abstract In this paper, we study some topological properties of connected topological groups. From a logical point of view, the concept of a topological group arises as a simple combination of the concepts of a group and a topological space. In the same set G, operations multiplication and topological closure are specified simultaneously.

Keywords Subgroup, Coset, Dense Set, Connectedness, Hereditary Disconnected, Extremally Disconnected, Locally Connected

1. Introduction

Definition 1.1 [2]. A topological group is the set G on which both topological and group structures are given. This requires that the maps

$$G \times G \to G: (x;y) \mapsto x \circ y$$

and

$$G \to G: x \mapsto x^{-1} \text{ be continuous.}$$

A topological group (or shortly G - group) is a topological space that is simultaneously a group, with xy being a continuous function of x and y and x^{-1} being a continuous function of x. In this case, the following are added to the four axioms of the group and the two axioms of open sets:

TG1. For each neighborhood $U(ab)$ of the product ab there are neighborhoods $V(a)$ and $W(b)$ whose product is contained in $U(ab)$.

TG2. For each neighborhood $U(a^{-1})$, there exists a neighborhood $V(a)$ such that $V(a)^{-1}$ is contained in $U(a^{-1})$.

From TG1 and TG2, the statements readily follow:

TG1/1. For each neighborhood $U(a^{-1}b)$ there are neighborhoods $V(a)$ and $W(b)$ such that $V(a)^{-1}W(b)$ is contained in $U(a^{-1}b)$.

TG2/1. For each neighborhood $U(ab^{-1})$ there are neighborhoods $V(a)$ and $W(b)$ such that $V(a)W(b)^{-1}$ is contained in $U(ab^{-1})$ [2].

The topological group is usually denoted by $\langle G, \theta, \circ \rangle$.

Suppose F is a closed set, U is an open set, P is an arbitrary set, and a is some element of the topological group G. Then Fa, aF, F^{-1} are closed sets; UP, PU, U^{-1} are open sets [6].

If a basis at the unit e is given, then all neighborhoods of this element are specified: those will be the sets $U(e)$, which contain at least one of the basis neighborhoods. But then the neighborhoods and other points are known, because if $U(e)$ is a neighborhood of the unit e, then $aU(e)$ is a neighborhood of a and all neighborhoods of a can be represented in this form[6].

Indeed, if $U(a)$ is an arbitrary neighborhood of a, then $a^{-1}U(a)$ is an open set and $e \in a^{-1}U(a)$, then $U(e) = a^{-1}U(a)$, therefore, we have $U(a) = aU(e)$. One can call $aU(e)$ «neighborhood of the point e shifted by a» [4].

The sets of all neighborhoods of $e \in G$ are denoted by Ω.

The following properties are true [3]:

1. $e \in U$ for all $U \in \Omega$.
2. if $U_1, U_2 \in \Omega$, then exists $U \in \Omega$ since $U \subset U_1 \cap U_2$
3. if U is open subset and $a \in U$, then exists $V \in \Omega$ since $aV \subset U$
4. if $U \in \Omega$ and $a \in G$, then exists $V \in \Omega$ since $aVa^{-1} \subset U$
5. if $U \in \Omega$, then exists $V \in \Omega$ since $VV^{-1} \subset U$

The closure of A is denoted by $\overline{A}$.

Proposition 1.1 [5]

Let G be a topological group and $E \subset G$ be an arbitrary subset of G, then $\overline{E} = \bigcap_{U\in\Omega} UE = \bigcap_{U\in\Omega} EU$.

Proposition 1.2 [4]

Let G be a group and $g \in G$, M_α a family a subset of G for each $\alpha \in I$. Then $g(\bigcap_{\alpha\in I} M_\alpha) = \bigcap_{\alpha\in I} gM_\alpha$, where I is the set of indices.

We introduce the following definition and the corresponding theorems.

Definition 1.2 [4]

Let $(G,*)$ be a group and H be a non-empty subset of G. Then $(H,*)$ is called a subgroup of $(G,*)$, if $(H,*)$ is a group.

Definition 1.3 [4]

Let $(G,*)$ be a group. A subset H of a group G is called a normal subgroup of G, if $aH = Ha$ for all $a \in G$.

Definition 1.4 [4]

Let H be a subgroup of G. Then the number of distinct left cosets (written $[G:H]$) of H is called the index of H in G.

A topological space X is called connected [2], if X cannot be decomposed into a union of non-empty open non-intersecting sub sets.

Proposition 1.3 [2]

A topological space X is connected if and only if X does not have its proper subset Y (i.e. $Y \neq \varnothing, X$), which is closed and open simultaneously.

A connected component [2] of a topological space X is any maximal connected subset in X.

If $x_0 \in X$ is a fixed point, then the union $K(x_0) = \cup M_{x_0}$ of all connected subsets of $M_{x_0} \subset X$ containing x_0 is connected. It is easy to show that $K(x_0)$ is not a proper subset of any other connected set, i.e. $K(x_0)$ connected component containing the point x_0.

Theorem 1.1 [7]

Let $A, B \subseteq G$ be arbitrary non-empty subsets in a topological group G and $b \in G$ an arbitrary element of G. Then the following statements are true:

1) If A is a connected set in a topological group G, then A^{-1} is a connected set in a topological group G.
2) If a A connected set in the topological group G, then bA and Ab are connected sets in the topological group G.
3) If A and B are non-empty connected sets in the topological group G, then the set $A \cdot B$ is a connected set in the topological group G.

Theorem1.2 [7]

If G is a topological group and C_e is a connected component of the unit e in the topological group G, then it is a closed normal subgroup in the topological group G, and for any element $g \in G$ the set $C_e g$ is a connected component of the point g in the topological group G and $C_e g = gC_e$.

2. Main Results

Proposition 2.1

Let G be a topological group and $E \subset G$ be an arbitrary subset, $a \in G$, then

a. $\overline{aE} = a\overline{E}$ $(\overline{Ea} = \overline{E}a)$.
b. $Int(aE) = aInt(E)$ $(Int(Ea) = Int(E)a)$.

Proof: **a)** By proposition 1.1 we have $\overline{aE} = \bigcap_{U\in\Omega}(aE)U$ $(\overline{Ea} = \bigcap_{U\in\Omega} U(Ea))$. By associatively of the group operation $(aE)U = a(EU)$ $(U(Ea) = (UE)a)$ and we have $\bigcap_{U\in\Omega}(aE)U = \bigcap_{U\in\Omega} a(EU)$ $(\bigcap_{U\in\Omega} U(Ea) = \bigcap_{U\in\Omega}(UE)a)$. Now, by virtue of proposition 1.2, we have $\bigcap_{U\in\Omega} a(EU) = a\bigcap_{U\in\Omega} EU$ $(\bigcap_{U\in\Omega}(UE)a = (\bigcap_{U\in\Omega} UE)a)$, then the equality $\overline{aE} = a\overline{E}$ $(\overline{Ea} = \overline{E}a)$ holds.

b) Firstly, we prove that a map $L_a : G \to G$ is homeomorphism, at $L_a(x) = ax$ for all $x \in G$ (where a is a fixed element of group G). The mapping L_a is one-to-one. Indeed, for every element y' there is an element x', and moreover, there is only one such that

$y' = ax'$. Further, the mapping L_a is continuous. Indeed, if $y' = ax'$ and W are some neighborhood of the element y', then there exist neighborhoods U and V of the elements a and x' such that $UV \subset W$; but $a \in U$ and, therefore, $aV \subset W$, i.e. $L_a(V) \subset W$, which means the continuity of the mapping L_a .The continuity of the inverse mapping $L_a^{-1}(y) = a^{-1}y$ is also proved. Then by the homeomorphism property we have that $IntL_a(E) = L_a(IntE)$ .Hence, $Int(aE) = aInt(E)$ .Secondly, a map $R_a : G \to G$ (at $R_a(x) = xa$ for all $x \in G$) is a homeomorphism. Because, the R_a mapping is one-to-one. Indeed, for every element y' there is an element x', and moreover, there is only one such that $y' = x'a$. Further, the mapping R_a is continuous. Indeed, if $y' = x'a$ and W are some neighborhood of the element y' , then there exist neighborhoods U and V of the elements x' and a such that $UV \subset W$; but $a \in V$ and, therefore, $Ua \subset W$, i.e. $R_a(V) \subset W$, which means the continuity of the mapping R_a .The continuity of the inverse mapping $R_a^{-1}(y) = ya^{-1}$ is also proved. Then by the homeomorphism property we have that $IntR_a(E) = R_a(IntE)$ for an arbitrary subset of $E \subset G$. Hence, $Int(Ea) = Int(E)a$. Proposition 2.1 is completely proved.

Theorem 2.1

Let the topological group G be connected:

1). then there is no open proper subgroup of group G .

2). if the subgroup H has a finite index in the topological group G , then the subgroup H is dense in G .

3). if subgroup H has a finite index in the topological group G , then each cosets of group G over subgroup H is dense in G .

Proof. 1) Suppose the opposite. Let a topological group G be connected and there is an open proper subgroup H of the group G . We show that $H = \overline{H}$. It is known that $H \subset \overline{H}$ for arbitrary subset $H \subset G$. Choose an arbitrary point $a \in \overline{H}$. Then aH will be an open subset in the topological space G containing point a , and we have $aH \cap H \neq \varnothing$. Assume that $b \in aH$ and $b \in H$, then there exists $h \in H$ such that $b = ah \in H$. We have $a \in Hh^{-1} \subset HH^{-1} = H$. In this way, $\overline{H} = H$. This means that the subgroup H is clopen. By proposition 1.3 topological group G is disconnected. This contradicts the condition of theorem 2.1. So, there is an open proper subgroup of the group G .

2) Now suppose that H is a subgroup with a finite index of a topological group G. Then there exists a finite sequence $a_1, a_2, ..., a_n$ ($a_iH \neq a_jH$ for all $i \neq j$, $i = 1,2,....,n$ and $j = 1,2,....,n$) of G , such that $a_1H \cup a_2H \cup ... \cup a_nH = G$. It is known that, $a_iH \subset a_i\overline{H}$ for all $i = 1,2,....,n$, then $G = (a_1H \cup a_2H \cup ... \cup a_nH) \subset (a_1\overline{H} \cup a_2\overline{H} \cup ... \cup a_n\overline{H}) \subset G$. Hence have, that $a_1\overline{H} \cup a_2\overline{H} \cup ... \cup a_n\overline{H} = G$. Now we prove that $\overline{H}$ is a subgroup of a topological group G . Let $g_1, g_2 \in \overline{H}$ and U is be an arbitrary neighborhood of point g_1g_2 . Since we have continuity of the group operation with respect to both variables, there exist neighborhoods V_1 , V_2 of points g_1 and g_2 , respectively, such that $V_1V_2 \subset U$. From the definition of closure, we get $g_1 \in V_1 \cap H$ and $g_2 \in V_2 \cap H$. This means that $g_1 \in H$, $g_2 \in H$. We get that $g_1g_2 \in V_1V_2$ and since H is a group we have $g_1g_2 \in H$. Since $V_1V_2 \subset U$ we get $g_1g_2 \in U$, so $g_1g_2 \in U \cap H$. Finally, we proved that $g_1g_2 \in \overline{H}$. Let $h \in \overline{H}$ and U is be an arbitrary neighborhood of a point h^{-1} . From the continuity of the inverse map we get U^{-1} is neighborhood of a point h .From the definition of closure, we get $U^{-1} \cap H \neq \varnothing$. So, there exists point g : such that $g \in U^{-1}$ and $g \in H$,then $g^{-1} \in H$ since H is a subgroup of G . We get that $g^{-1} \in U$ and $g^{-1} \in H$, so $g^{-1} \in U \cap H$. In order to show $h^{-1} \in \overline{H}$ we need to prove $U \cap H \neq \varnothing$ for all U is an arbitrary neighborhood of a point h^{-1} . It follows that $h^{-1} \in \overline{H}$, as required. Then we have that $\overline{H}$ is a closed subgroup with a finite index in the topological group G . Then coset $a\overline{H}$ is also closed in G . The union of all finite cosets, except of $\overline{H}$, is closed again. This union is a

complement of the subgroup $\overline{H}$. Consequently, $\overline{H}$ is open. Hence, $\overline{H}$ is clopen subgroup in G .Then the proposition 1.3, we have that $\overline{H}=G$. Then H is dense in G.

3) Suppose that H subgroup has finite index connected of topological group G. Then by virtue of theorem 2.1 part 2), H is dense in G, i.e. $\overline{H}=G$. By proposition 2.1 part a) we have that $\overline{xH}=x\overline{H}=xG=G$ $(\overline{Hx}=\overline{H}x=Gx=G)$ for all $x\in G$. Then xH (Hx) is everywhere dense in G. Theorem 2.1 is completely proved.

For a subset A of a group G we write $A^n=\{a^n : a\in A\}$ [4].

Definition 2.1[4]

A group G is called nilpotent if $G^n=\{e\}$ for some positive integer n.

Proposition 2.2

If G is a Hausdorff topological group and H is a nilpotent subgroup of a group G, then $\overline{H}$ also be a nilpotent subgroup of a group G.

Proof: According to theorem 2.1, $\overline{H}$ is a subgroup of a group G.

We prove that $\overline{H}$ is a nilpotent subgroup.

Assume that $H^n=\{e\}$. It means that $h^n=e$ for all $h\in H$. Suppose the opposite, and let $a\in\overline{H}$ such elements that $a^n\neq e$. Since G is a Hausdorff topological group, then in a topological group G there are such neighborhoods U and V elements a^n and e, respectively, that $U\cap V=\varnothing$.

It follows from the continuity of the multiplication operation that there exist such neighborhoods $U_1(a),U_2(a),\dots,U_n(a)$ of a such that $U_1(a)U_2(a)\dots U_n(a)\subset U$. It is known that $U(a)=\bigcap_{i=1}^{n}U_i(a)$ is a neighborhood of a. It means that for all $j=1,2,\dots,n$. In this case, we have $U(a)\cdot U(a)\cdot\ldots\cdot U(a)\subset U$ i.e. $U(a)^n\subset U$.

Since, $a\in\overline{H}$ then $U(a)\bigcap H\neq\varnothing$. If $b\in U(a)\bigcap H$, then from the nilpotent subgroup H it follows that $b^n=e$. Moreover $e=a^n\in U(a)^n\subset U$, so $U\cap V\neq\varnothing$.

We got a contradiction with the choice of neighborhoods V and U.

Denote by C_a connected component of the point a in the topological group.

The connected component of the neutral element e of group G is called the neutral component of this group [3].

Theorem 2.2

If G is a topological group and $a,b\in G$ is an arbitrary element in G. Then the following statements are true:

1. $C_a=b\cdot C_{b^{-1}a}$
2. $C_{ab}=C_a\cdot C_b$
3. $C_{a^{-1}}=C_a^{\ -1}$

Proof. 1) First we show that $C_a=bC_{b^{-1}a}$. According to theorem 1.1, part 2), $b^{-1}C_a$ will be a connected set in the topological space G containing point $b^{-1}a$. Then $b^{-1}\cdot C_a\subset C_{b^{-1}a}$ since $C_a\subset b\cdot C_{b^{-1}a}$. Similarly, $b\cdot C_{b^{-1}a}$ will be a connected set in the topological space G containing the point a .Then $b\cdot C_{b^{-1}a}\subset C_a$ .We have that $C_a=bC_{b^{-1}a}$

2) Now we show that $C_{ab}=C_a\cdot C_b$ By theorem 1.2, $C_{ab}=abC_e=ab(C_eC_e)=a(bC_e)C_e$. We have that $C_{ab}=a(C_eb)C_e=(aC_e)(bC_e)=C_aC_b$.

3) By virtue of theorem 1.2, $C_{a^{-1}}=a^{-1}C_e$. It is known that C_e is a normal subgroup of the topological group G. Then $C_e=C_e^{\ -1}$ So $C_{a^{-1}}=a^{-1}C_e=a^{-1}C_e^{\ -1}=(C_ea)^{-1}=C_a^{-1}$. Theorem 2.2 is completely proved.

A topological space X is called hereditarily disconnected [1] if X does not contain any connected subsets of cardinality larger than one. Hence, a space X is hereditarily disconnected if and only if the component of any point $x\in X$ consists of the point x alone. Since the components of a space are closed, every hereditarily disconnected space is a T_1-space.

A topological space X is called extremally disconnected [1], if X is a Hausdorff space and for every open set $U\subset X$ the closure $\overline{U}$ is open in X.

We say that a topological space X is locally connected [1], if for every point $x \in X$ and any neighborhood U of the point x there exists a connected set $C \subset U$ such that $x \in IntC$

Theorem 2.3

Let G be a topological group. Then

a. G is hereditary disconnected if and only if there is a connected component of a point $x \in G$ consisting only of x.

b. G is extremally disconnected if and only if there exists an open set $U \subset G$ whose closure $[U]$ is open in G.

c. G is locally connected, if and only if for unit point $e \in G$ and any neighborhood U of e there is a connected set $C \subset U$ such that $e \in IntC$.

Proof: ***a)*** 1) It is clear that a topological group G is hereditarily disconnected, then the component of every point $x \in G$ consists only of x.

2) Let $C_x = \{x\}$,where is C_x is be a connected component of the point x of the topological group G. We show that $C_a = \{a\}$ for all $a \in G$, where C_a is connected component of a point a. Then, by theorem 1.2, $xC_e = C_x$ since, $xC_e = \{x\}$. This equality of the product of both parts $ax^{-1} \in G$, we get $aC_e = \{a\}$. Now, still using theorems 1.2, it follows that $aC_e = C_a$. From where we have that $C_a = \{a\}$.

b) 1) It is known that if a topological group G is extremally disconnected, then for every open set $U \subset X$ the closure $\overline{U}$ is open at G.

2) Suppose that there is an open set $U \subset G$, such that the closure $\overline{U}$ is open in G. We choose point $x \in U$, then there exists a neighborhood U_e of a point of a neutral element $e \in G$ such that $U = xU_e$. Then, by part a) of proposition 2.1the closure $[U] = x[U_e]$ is open in G. Consequently, the closure $[U_e]$ is open at G. Choose arbitrary point $y \in G$. Then the set $y[U_e] = [yU_e]$ is open at G. Then for every open set $U_y \subset G$ the closure $[U_y]$ is open in G, where U_y is a neighborhood of an arbitrary point $y \in G$. Therefore, the topological group G is extremally disconnected.

c) 1) It is known that if a topological group G is locally connected, then for each point $x \in G$ in particular $e \in G$ and any its neighborhood U there exists a connected set $C \subset U$ such that $e \in IntC$.

2) Choose an arbitrary point $x \in G$ topological group G. Let for a single point $e \in G$ and any its neighborhood U there exists a connected set $C \subset U$, such that $e \in IntC$. Then, any of its neighborhoods U' of a point x, there exists a neighborhood U of a point $e \in G$ such that $U' = xU$ and $xC \subset xU = U'$. It is easy to see that the set $C' = xC$ are connected. By virtue we get $xe \in xIntC$. By proposition 2.1 part b) implies that $x \in Int(xC)$. Then for each point $x \in G$ and any its neighborhood U' there exists a connected set $C' \subset U'$, such that $x \in IntC'$. Therefore, a topological group G is locally connected. Theorem 2.3 is completely proved.

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