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Properties of Karcı's Fractional Order Derivative

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Abstract The derivative concept was defined by Newton and Leipzig. After these scientific, there are many approaches about the order of derivative, since derivative defined by Newton and Leipzig considered as order of 1. Many scientists such as Caputo, Riemann, etc. defined the fractional order derivative. Karcı is one of them who defined fractional order derivative. $\frac{\partial}{\alpha}Kf(t)$ was defined by Karcı, and it is not a linear derivative operator; it is a non-linear derivative operator. In this paper, we verified the most important properties of $\frac{\partial}{\alpha}Kf(t)$. $\frac{\partial}{\alpha}Kf(t)$ has got an α parameter and this parameter can be any complex number. The properties of $\frac{\partial}{\alpha}Kf(t)$, which are derivative of product, derivative of quotient, the chain rule, the relationship between $\frac{\partial}{\alpha}Kf(t)$ and complex numbers, etc., were verified in this paper. The most of these properties were not satisfied by other definitions for fractional order derivatives such as Caputo, Riemann-Lioville, Euler, etc. Khallil and his friends also defined fractional order derivative in a special case. This derivative satisfies these properties for special functions; in general, this definition also does not satisfy these properties.

Keywords Fractional Calculus, Fractional Order Derivative, Variational Calculus, Karcı Derivative

1 Introduction

The fractional calculus (variational calculus) is a three centuries old concept and one of the branch of the fractional calculus is fractional order derivatives. The fractional order derivative concept was defined by many scientists such as Euler, Caputo, Riemman-Lioville, etc. (Das, 2011). There is an idea such that the fractional calculus may depict the behaviours of nature almost in real behaviours of nature (Das, 2011).

Riemann-Liouville derivative definition for α∈[n-1,n]

$$D_a^\alpha\big(f(t)\big)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{f(x)}{(t-x)^{\alpha-n+1}}dx \qquad (1)$$

Caputo derivative definition for α∈[n-1,n]

$$D_a^\alpha\big(f(t)\big)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx \qquad (2)$$

The Fractional Order Derivative (FOD) methods in the literature can be considered as different approaches instead of classical derivatives. There are a lot of studies in this area and the most of these studies have used Euler, Riemann-Liouville and Caputo FODs. This paper was focused on the properties of fractional order derivatives defined by different scientists (Pooseh et al, 2013; Mirevski et al, 2017; Schiavone and Lamb, 1990; Li et al, 2011). Karcı defined fractional order derivative in a different manner and gave some properties of fractional order derivative (Karcı, 2013a; Karcı, 2013b; Karcı, 2015a; Karcı, 2015b; Karcı, 2015c; Karcı, 2015d; Karcı, 2015e; Karcı, 2016a; Karcı, 2016b; Karcı, 2017), since Euler, Caputo, Riemann-Liouville definitions include deficiencies (Khallil et al, 2014).

The Caputo and Riemann-Liouville derivatives do not satisfy some important properties of Newtonian derivative as follow (This information was taken from Khallil et al study).

a). Riemann-Liouville derivative does not satisfy $D_a^\alpha(c)=0$ and Caputo derivative satisfies this property.

b). Riemann-Liouville and Caputo do not satisfy the product derivative property such as

$$D_a^\alpha\big(f(t)g(t)\big)=f(t)D_a^\alpha\big(g(t)\big)+g(t)D_a^\alpha\big(f(t)\big) \qquad (3)$$

a). Riemann-Liouville and Caputo do not satisfy the quotient derivative property such as

$$D_a^\alpha\left(\frac{f(t)}{g(t)}\right)=\frac{f(t)D_a^\alpha\big(g(t)\big)-g(t)D_a^\alpha\big(f(t)\big)}{g^2(t)} \qquad (4)$$

b). Riemann-Liouville and Caputo derivatives do not satisfy the chain rule.

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^\alpha u}{\partial y^\alpha}\frac{\partial^\alpha y}{\partial t^\alpha} \qquad (5)$$

c). Riemann-Liouville and Caputo derivatives do not satisfy $D_a^\alpha D_a^\beta f(t)=D_a^{\alpha+\beta}f(t)$.

d). The Caputo definition assumes that the function f is differentiable.

Khallil and his friends gave a new definition for fractional order derivative for sake of satisfying the mentioned problems of fractional order derivatives (Khallil et al, 2014). In order to get rid of these deficiencies Khallil and his friends defined fractional order derivative as follow.

$$T_\alpha(f(t)) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{6}$$

This definition satisfy the mentioned deficiencies for some specific functions, in general, this definition does not satisfy all mentioned deficiencies in general. For examples:

a). $T_\alpha(f(t))$ satisfies derivative property for constant functions.

b). $T_\alpha(f(t))$ does not satisfy all remaining functions/relations types in general.

In this study, it was focused on the shortcomings and wrong points involved in the methods of Euler, Riemann-Liouville and Caputo for FODs. Especially, the FODs of constant and identity functions will be obtained for Euler, Riemann-Liouville and Caputo methods. Euler and Riemann-Liouville methods were yielded shortcomings and errors in results for constant functions, and on contrary, Caputo method was yielded in correct result for constant function. All methods have not provided accurate results for identity function. We studied on fractional order derivatives which defined by Karcı, since that definition satisfies all conditions for general functions. Assume that f(t) is a differentiable function and fractional order derivative (defined by Karcı) is denoted by $\overset{\partial}{_\alpha}K$.

2. Karcı's Fractional Order Derivatives and Its Properties

Euler and Riemann-Liouville derivatives of $f(x)=cx^0$: (summarised from Karcı, 2013b)

These three methods for fractional order derivatives are mostly used methods. First of all, the results of Euler and Riemann-Liouville methods for $f(x)=cx^0$ are illustrated in Eq.7 and Eq.8 where c is a constant. For n=1 and α=1/4

$$\text{Euler: } \frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} = \frac{d^{\frac{1}{4}} x^0}{dx^{\frac{1}{4}}} = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} c x^{-\frac{1}{4}} \tag{7}$$

Riemann-Liouville method:
n=1 and α=2/3

$$\frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}} = \frac{1}{\Gamma(\frac{1}{3})} \left(-\frac{1}{(t-a)^{2/3}}\right) \neq 0 \tag{8}$$

The obtained result is inconsistent, since the result is a function of x. However, initial function is a constant function and its derivative is zero, since there is no change in the dependent variable.

The Euler and Riemann-Liouville methods do not work for constant functions as seen in Eq.7 and Eq.8.

Caputo derivative of f(x)=x:

$$\frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} = \frac{1}{\Gamma(-1/3)} \left(3(t-a)^{1/3}\right) \neq 1 \tag{9}$$

It can be seen from definition, the ratio of the change in dependent variable over to the change in independent is always 1 (one) for identity function. In this case, the derivative must be 1 in any fractional order derivative. However, fractional order derivatives of identity function with respect to Caputo method is different from 1. This means that all methods yielded inconsistent results. Due to this case, there is a need to redefine the fractional order derivative as in Karcı definition (Definition 1).

Definition1: Assume that f(t):R→R is a function, α∈R and L(.) be a L'Hospital process. The $\overset{\partial}{_\alpha}K$ of f(t) is (taken from Karcı, 2013a; Karcı, 2013b)

$$\overset{\partial}{_\alpha}K = \lim_{\Delta t \to 0} L\left(\frac{f^\alpha(t+\Delta t) - f^\alpha(t)}{(t+\Delta t)^\alpha - t^\alpha}\right) = \lim_{\Delta t \to 0} \frac{\frac{d\left(f^\alpha(t+\Delta t) - f^\alpha(t)\right)}{\Delta t}}{\frac{d\left((t+\Delta t)^\alpha - t^\alpha\right)}{\Delta t}} = \left(\frac{f(t)}{t}\right)^{\alpha-1} \frac{df(t)}{dt} \tag{10}$$

The fractional order derivative was derived due to the deficiencies of definitions for fractional order derivatives. The obtained derivative is equal to Newtonian derivative in case of α=1, and in other cases, non-linear operator exists. The properties of ${}_{\alpha}^{\partial}K$ can be listed as follow.

a). Assume that f(t)=c and c∈R, c is a constant (summarised from Karcı, 2013a; Karcı, 2013b).

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{f(t)}{t}\right)^{\alpha-1}\frac{df(t)}{dt}=\left(\frac{c}{t}\right)^{\alpha-1}\frac{c-c}{dt}=0 \quad \forall c\in R. $$

b). Assume that f(t), g(t):R→R are continuous functions, α∈R and h(t)=f(t)g(t). The fractional order derivative of h(t) is as follows (summarised from Karcı, 2015c).

$$ {}_{\alpha}^{\partial}Kh(t)=\left(\frac{f(t)g(t)}{t}\right)^{\alpha-1}\left(\frac{df(t)}{dt}g(t)+f(t)\frac{dg(t)}{dt}\right) $$

Assume that f(t) and g(t) are real functions and, so, h(t) is also a real function. The fractional order derivative of h(t) can be obtained as follow.

$$ {}_{\alpha}^{\partial}Kh(t)=\lim_{\Delta t\to 0} L\left(\frac{\left(f(t+\Delta t)g(t+\Delta t)\right)^{\alpha}-\left(f(t)g(t)\right)^{\alpha}}{(t+\Delta t)^{\alpha}-t^{\alpha}}\right) $$

$$ =\lim_{\Delta t\to 0}\left(\frac{\alpha\frac{df(t+\Delta t)}{dt}f^{\alpha-1}(t+\Delta t)g^{\alpha}(t+\Delta t)}{\alpha(t+\Delta t)^{\alpha-1}}+\frac{\alpha\frac{dg(t+\Delta t)}{dt}g^{\alpha-1}(t+\Delta t)f^{\alpha}(t+\Delta t)}{\alpha(t+\Delta t)^{\alpha-1}}\right) $$

$$ =\frac{\alpha\frac{df(t)}{dt}f^{\alpha-1}(t)g^{\alpha}(t)}{\alpha(t)^{\alpha-1}}+\frac{\alpha\frac{dg(t)}{dt}g^{\alpha-1}(t)f^{\alpha}(t)}{\alpha(t)^{\alpha-1}}=\left(\frac{f(t)g(t)}{t}\right)^{\alpha-1}\left(\frac{\alpha\frac{df(t)}{dt}g(t)}{\alpha}+\frac{\alpha\frac{dg(t)}{dt}f(t)}{\alpha}\right) $$

$$ =\left(\frac{f(t)g(t)}{t}\right)^{\alpha-1}\left(\frac{df(t)}{dt}g(t)+\frac{dg(t)}{dt}f(t)\right) $$

It can be seen easily that in case of α=1, ${}_{\alpha}^{\partial}Kh(t)={}_{1}^{\partial}Kh(t)={}^{\partial}Kh(t)$ is equal to Newtonian derivative. Not all of the other derivative definitions obey this case.

c). Assume that f(t), g(t):R→R is a continue functions, α∈R and $h(t)=\frac{f(t)}{g(t)}$. The fractional order derivative of h(t) is as follows (summarised from Karcı, 2015c).

$$ {}_{\alpha}^{\partial}Kh(t)=\left(\frac{f(t)}{tg(t)}\right)^{\alpha-1}\frac{f'(t)g(t)-f(t)g'(t)}{g^{2}(t)} $$

f(t) and g(t) are real functions, so, h(t) is also a real function. The fractional order derivative of h(t) is

$$ {}_{\alpha}^{\partial}Kh(t)=\lim_{\Delta t\to 0} L\left(\frac{\left(\frac{f(t+\Delta t)}{g(t+\Delta t)}\right)^{\alpha}-\left(\frac{f(t)}{g(t)}\right)^{\alpha}}{(t+\Delta t)^{\alpha}-t^{\alpha}}\right)=\lim_{\Delta t\to 0} L\left(\frac{f^{\alpha}(t+\Delta t)g^{\alpha}(t)-f^{\alpha}(t)g^{\alpha}(t+\Delta t)}{\left((t+\Delta t)^{\alpha}-t^{\alpha}\right)g^{\alpha}(t+\Delta t)g^{\alpha}(t)}\right) $$

$$ =\lim_{\Delta t\to 0}\left(\frac{\alpha\frac{df(t+\Delta t)}{dt}f^{\alpha-1}(x+h)g^{\alpha}(x)}{\Psi}\right)+\lim_{\Delta t\to 0}\left(\frac{-f^{\alpha}(x)\alpha\frac{dg(t+\Delta t)}{dt}g^{\alpha-1}(x+h)}{\Psi}\right) $$

$$ =\left(\frac{f(t)}{tg(t)}\right)^{\alpha-1}\frac{\frac{df(t)}{dt}g(x)-f(x)\frac{dg(t)}{dt}}{g^{2}(x)} $$

where $\Psi=\alpha(t+\Delta t)^{\alpha-1}g^{\alpha}(t+\Delta t)g^{\alpha}(t)+\left((t+\Delta t)^{\alpha}-t^{\alpha}\right)g^{\alpha}(t)\alpha g'(t+\Delta t)g^{\alpha-1}(t+\Delta t)$

d). The chain rule for fractional order derivative with respect to Definition 1 is (summarised from Karcı, 2015d)

$$ {}^{\partial}_{\alpha}Ky(u(t)) = \frac{\partial^\alpha y}{\partial t^\alpha} = \frac{\partial^\alpha y}{\partial u^\alpha}\frac{\partial^\alpha u}{\partial t^\alpha} = \left(\frac{f(u)}{u}\right)^{\alpha-1}\frac{df(u)}{du}\left(\frac{g(t)}{t}\right)^{\alpha-1}\frac{dg(t)}{dt} $$

where $\alpha \in R$, y=f(u), and u=g(t).

y and u are not absolutely independent variables; however, t is an absolutely independent variable. The derivative means that the derivative is the response of dependent variable to changes in independent variables. So, the change in y with respect to u is not absolutely derivative, since u is not independent variable. The change in y with respect to t through u must be taken in care. So,

$$ \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{f^\alpha(u+\Delta u) - f^\alpha(u)}{(u+\Delta u)^\alpha - u^\alpha} \text{ and } \frac{\Delta u}{\Delta t} = \lim_{\Delta t \to 0} \frac{g^\alpha(t+\Delta t) - g^\alpha(t)}{(t+\Delta t)^\alpha - t^\alpha}. $$

The definition for fractional order derivative with respect to Definition 1 is as follow:

$$ {}^{\partial}_{\alpha}Kf(t) = \lim_{\Delta t \to 0} L\left(\frac{f^\alpha(t+\Delta t) - f^\alpha(t)}{(t+\Delta t)^\alpha - t^\alpha}\right) = \lim_{\Delta t \to 0} L\left(\frac{\frac{d(f^\alpha(t+\Delta t) - f^\alpha(t))}{dt}}{\frac{d((t+\Delta t)^\alpha - t^\alpha)}{dt}}\right) $$

The chain rule result is

$$ \frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta t} = \lim_{\Delta u \to 0} L\left(\frac{f^\alpha(u+\Delta u) - f^\alpha(u)}{(u+\Delta u)^\alpha - u^\alpha}\right) \lim_{\Delta t \to 0} L\left(\frac{g^\alpha(t+\Delta t) - g^\alpha(t)}{(t+\Delta t)^\alpha - t^\alpha}\right) $$

$$ = \lim_{\Delta u \to 0} L\left(\frac{\frac{d(f^\alpha(u+\Delta u) - f^\alpha(u))}{d\Delta u}}{\frac{d((u+\Delta u)^\alpha - u^\alpha)}{d\Delta u}}\right) \lim_{\Delta t \to 0} L\left(\frac{\frac{d(g^\alpha(t+\Delta t) - g^\alpha(t))}{d\Delta t}}{\frac{d((t+\Delta t)^\alpha - t^\alpha)}{d\Delta t}}\right) $$

$$ = \left(\left(\frac{f(u)}{u}\right)^{\alpha-1}\frac{df(u)}{du}\right)\left(\left(\frac{g(t)}{t}\right)^{\alpha-1}\frac{dg(t)}{dt}\right) $$

e). Assume that $\frac{\partial^\alpha y}{\partial t^\alpha}$ and $\frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}$ derivatives, and α, α_1, $\alpha_2 \in R$. If $\frac{\partial^\alpha y}{\partial t^\alpha}$ is left-hand side of chain rule and $\frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}$ is right-hand side of chain rule, then $\alpha=\alpha_1=\alpha_2$.

In order to prove this assumption, the left-hand side and right hand side must be obtained. The left hand side is

$$ \frac{\partial^\alpha y}{\partial t^\alpha} = \lim_{\Delta t \to 0} L\left(\frac{y^\alpha(t+\Delta t) - y^\alpha(t)}{(t+\Delta t)^\alpha - t^\alpha}\right) = \lim_{\Delta t \to 0} \frac{\frac{d\left(y^\alpha(t+\Delta t) - y^\alpha(t)\right)}{d\Delta t}}{\frac{d((t+\Delta t)^\alpha - t^\alpha)}{d\Delta t}} $$

and the terms in the right-hand side are

$$ \frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}} = \lim_{\Delta u \to 0} L\left(\frac{y^{\alpha_1}(u+\Delta u) - y^{\alpha_1}(u)}{(u+\Delta u)^{\alpha_1} - u^{\alpha_1}}\right) = \lim_{\Delta u \to 0} \frac{\frac{d\left(y^{\alpha_1}(u+\Delta u) - y^{\alpha_1}(u)\right)}{d\Delta u}}{\frac{d((u+\Delta u)^{\alpha_1} - u^{\alpha_1})}{d\Delta u}} $$

and

$$ \frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}} = \lim_{\Delta t \to 0} L\left(\frac{u^{\alpha_2}(t+\Delta t) - u^{\alpha_2}(t)}{(t+\Delta t)^{\alpha_2} - t^{\alpha_2}}\right) = \lim_{\Delta t \to 0} \frac{\frac{d\left(u^{\alpha_2}(t+\Delta t) - u^{\alpha_2}(t)\right)}{d\Delta t}}{\frac{d((t+\Delta t)^{\alpha_2} - t^{\alpha_2})}{d\Delta t}} $$

In the case of chain rule, $\frac{\partial^{\alpha} y}{\partial t^{\alpha}}$ and $\frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}$ must be equal.

$$\frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}} = \left(\lim_{\Delta u \to 0} \frac{\frac{d\left(y^{\alpha_1}(u+\Delta u)-y^{\alpha_1}(u)\right)}{d\Delta u}}{\frac{d((u+\Delta u)^{\alpha_1}-u^{\alpha_1})}{d\Delta u}}\right)\left(\lim_{\Delta t \to 0} \frac{\frac{d\left(u^{\alpha_2}(t+\Delta t)-u^{\alpha_2}(t)\right)}{d\Delta t}}{\frac{d((t+\Delta t)^{\alpha_2}-t^{\alpha_2})}{d\Delta t}}\right)$$

In order to validate the chain rule,

$$\frac{d\left(u^{\alpha_2}(t+\Delta t)-u^{\alpha_2}(t)\right)}{d\Delta t} = \frac{d(u^{\alpha_1}(t+\Delta t)-u(t)^{\alpha_1})}{d\Delta t}.$$

This means that $\alpha_1=\alpha_2$. Then the right-hand side will be

$$\frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}} = \left(\lim_{\substack{\Delta u \to 0\\ \Delta t \to 0}} \frac{\frac{d\left(y^{\alpha_1}(u+\Delta u)-y^{\alpha_1}(u)\right)}{d\Delta u}}{\frac{d((t+\Delta t)^{\alpha_2}-t^{\alpha_2})}{d\Delta t}}\right).$$

If this is a chain rule, then

$$\frac{\partial^{\alpha} y}{\partial t^{\alpha}} = \frac{\partial^{\alpha_1} y}{\partial u^{\alpha_1}}\frac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}$$

$$\Rightarrow \lim_{\substack{\Delta u \to 0\\ \Delta t \to 0}} \frac{\frac{d\left(y^{\alpha}(u+\Delta u)-y^{\alpha}(u)\right)}{d\Delta u}}{\frac{d((t+\Delta t)^{\alpha}-t^{\alpha})}{d\Delta t}} = \lim_{\substack{\Delta u \to 0\\ \Delta t \to 0}} \frac{\frac{d\left(y^{\alpha_1}(u+\Delta u)-y^{\alpha_1}(u)\right)}{d\Delta u}}{\frac{d((t+\Delta t)^{\alpha_2}-t^{\alpha_2})}{d\Delta t}}$$

The last equation verifies that all orders of derivative for chain rule must be equal for the commutative property of $\frac{\partial}{\alpha}Kf(t)$

Assume that f(t):R→R, and $\alpha_1, \alpha_2 \in R$. $\frac{\partial}{\alpha}K$ is not commutative with respect to orders of derivatives, when $\alpha_1 \neq \alpha_2$. This can be verified in a simple way:

$$\frac{\partial}{\alpha_1}K\left(\frac{\partial}{\alpha_2}Kf(t)\right) = \frac{\partial}{\alpha_1}K\left(\left(\frac{f(t)}{t}\right)^{\alpha_2-1}\frac{df(t)}{dt}\right)$$

And

$$\frac{\partial}{\alpha_2}K\left(\frac{\partial}{\alpha_1}Kf(t)\right) = \frac{\partial}{\alpha_2}K\left(\left(\frac{f(t)}{t}\right)^{\alpha_1-1}\frac{df(t)}{dt}\right)$$

In order to $\frac{\partial}{\alpha_1}K\left(\frac{\partial}{\alpha_2}Kf(t)\right) = \frac{\partial}{\alpha_2}K\left(\frac{\partial}{\alpha_1}Kf(t)\right)$, $\alpha_1=\alpha_2$. If $\alpha_1\neq\alpha_2$ then, $\frac{\partial}{\alpha_1}K\left(\frac{\partial}{\alpha_2}Kf(t)\right)$ and $\frac{\partial}{\alpha_2}K\left(\frac{\partial}{\alpha_1}Kf(t)\right)$ do not have to be equal.

f). Assume that f(t) is a real function, then $\frac{\partial}{\alpha}Kf(t)$ is a complex function.

In order to verify this case, assume that $\alpha = \frac{\beta}{\delta}$ and $\delta\neq 0$. There are four cases and it is known that real numbers set is a subset of complex numbers set:

1. f(t) is a non-negative function and δ is an even number: In this case,

$$\frac{\partial}{\alpha}Kf(t) = \left(\frac{f(t)}{t}\right)^{\frac{\beta}{\delta}-1}\frac{df(t)}{dt} = \left(\frac{f(t)}{t}\right)^{\frac{\beta-\delta}{\delta}}\frac{df(t)}{dt} = \sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt}$$

where $\frac{df(t)}{dt} \in R$ and δ is an even number so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in R$. Any complex number can be illustrated as a+bi where a,b∈R. In this case, b=0, so complex number is equal to a=$\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt}$.

2. f(t) is a non-negative function and δ is an odd number: In this case,

$$ {}^{\partial}_{\alpha}Kf(t) = \left(\frac{f(t)}{t}\right)^{\frac{\beta}{\delta}-1}\frac{df(t)}{dt} = \left(\frac{f(t)}{t}\right)^{\frac{\beta-\delta}{\delta}}\frac{df(t)}{dt} = \sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt} $$

where $\frac{df(t)}{dt} \in R$ and δ is an odd number so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in R$, since any power of any positive function is also positive. Any complex number can be illustrated as a+bi where a,b∈R. In this case, b=0, so complex number is equal to a=$\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt}$.

3. Assume that f(t) is a negative function and δ is an even number:

$$ {}^{\partial}_{\alpha}Kf(t) = \left(\frac{f(t)}{t}\right)^{\frac{\beta}{\delta}-1}\frac{df(t)}{dt} = \left(\frac{f(t)}{t}\right)^{\frac{\beta-\delta}{\delta}}\frac{df(t)}{dt} = \sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt} $$

It can easily be seen that $\frac{df(t)}{dt} \in R$, the Newtonian derivative of any real function is also a real function. If β-δ is an even number, then $\left(\frac{f(t)}{t}\right)^{\beta-\delta}$ is positive so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in R$. If β-δ is an odd number, then $\left(\frac{f(t)}{t}\right)^{\beta-\delta}$ is negative so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in C$, since δ is an even number.

4. Assume that f(t) is a negative function and δ is an odd number:

$$ {}^{\partial}_{\alpha}Kf(t) = \left(\frac{f(t)}{t}\right)^{\frac{\beta}{\delta}-1}\frac{df(t)}{dt} = \left(\frac{f(t)}{t}\right)^{\frac{\beta-\delta}{\delta}}\frac{df(t)}{dt} = \sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt} $$

It can easily be seen that $\frac{df(t)}{dt} \in R$, the Newtonian derivative of any real function is also a real function. If β-δ is an even number, then $\left(\frac{f(t)}{t}\right)^{\beta-\delta}$ is positive so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in R$. If β-δ is an odd number, then $\left(\frac{f(t)}{t}\right)^{\beta-\delta}$ is negative so, $\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}} \in R$. Any complex number can be illustrated as a+bi where a,b∈R. In this case, b=0, so complex number is equal to a=$\sqrt[\delta]{\left(\frac{f(t)}{t}\right)^{\beta-\delta}}\frac{df(t)}{dt}$.

The derivative of any function has got magnitude and direction, and complex numbers also have got magnitude and directions. Due to this case, there should be a relationship between derivative and complex numbers. ${}^{\partial}_{\alpha}Kf(t)$ verifies this relationship

${}^{\partial}_{\alpha}Kf(t)$ can be summarised in the following steps (in case of ${}^{\partial}_{\alpha}Kf(t)$ is complex). Any complex number z=a+bi (a, b∈R).

1. If $\alpha = \frac{\beta}{\delta}$ and δ≠0, δ is even and f(t) is a positive function then ${}^{\partial}_{\alpha}Kf(t) = \Phi$ and Φ∈R.
2. If $\alpha = \frac{\beta}{\delta}$ and δ≠0, δ is odd and f(t) is a positive function then ${}^{\partial}_{\alpha}Kf(t) = \Phi$ and Φ∈R.
3. If $\alpha = \frac{\beta}{\delta}$ and δ≠0, δ is even and f(t) is a negative function then ${}^{\partial}_{\alpha}Kf(t) = \Phi + \Theta i,\ i = \sqrt{-1}$, and Φ,Θ∈R.
4. If $\alpha = \frac{\beta}{\delta}$ and δ≠0, δ is odd and f(t) is a negative function then ${}^{\partial}_{\alpha}Kf(t) = \Phi$ and Φ∈R.

3. Conclusions

Most of the events in nature are not linear in their nature. ${}^{\partial}_{\alpha}Kf(t)$ is a non-linear operator, so this derivative can model events more realistic than Newtonian derivative. This situation easily can be observed in this paper. Another important feature of ${}^{\partial}_{\alpha}Kf(t)$ reveals the relationships between derivative and complex numbers. This case is also very important.

REFERENCES

[1] Das, S., "Functional Fractional Calculus", Springer-Verlag Berlin Heidelberg, 2011.

[2] Pooseh,S., Almeida, R., Torres, D.F.M., "Discrete direct methods in the fractional calculus of variations", Computers and Mathematics with Applications, Vol.66, Issue. 5, pp.668-676, 2013.

[3] Mirevski, S.P., Boyadjiev,L., Scherer, R.,"On the Riemann-Liouville Fractional Calculus, g-Jacobi Functions and F.Gauss Functions", Applied Mathematics and Computation, Vol.187, pp.315-325, 2007.

[4] Schiavone, S.E., Lamb, W., "A Fractional Power Approach to Fractional Calculus", Journal of Mathematical Analysis and Applications, Vol.149, pp.377-401, 1990.

[5] Li, C., Qian, D., Chen, Y.-Q., "On Riemann-Liouville and Caputo Derivatives", Discrete Dynamics in Nature and Society, Vol. 2011, Article ID: 562494, 15 pages, 2011.

[6] Karcı, A.,"Kesirli Türev için Yapılan Tanımlamaların Eksiklikleri ve Yeni Yaklaşım", TOK-2013 Turkish Automatic Control National Meeting and Exhibition, 2013a.

[7] Karcı,A., "A New Approach for Fractional Order Derivative and Its Applications", Universal Journal of Engineering Sciences, Vol:1, pp: 110-117, 2013b.

[8] Karcı, A., "Properties of Fractional Order Derivatives for Groups of Relations/Functions", Universal Journal of Engineering Sciences, vol: 3, pp: 39-45, 2015a.

[9] Karcı, A.,"The Linear, Nonlinear and Partial Differential Equations are not Fractional Order Differential Equations", Universal Journal of Engineering Sciences, vol: 3, pp: 46-51, 2015b.

[10] Karcı, A.,"Generalized Fractional Order Derivatives for Products and Quotients", Science Innovation, vol:3, pp:58-62, 2015c.

[11] Karcı, A.,"Chain Rule for Fractional Order Derivatives", Science Innovation, vol:3, 63-67, 2015d.

[12] A. Karcı, "The Properties of New Approach of Fractional Order Derivative", Journal of the Faculty of Engineering and Architecture of Gazi University, Vol.30, pp: 487-501, 2015e.

[13] Karcı, A.,"Fractional order entropy New perspectives", Optik - International Journal for Light and Electron Optics, vol:127, no:20, pp:9172-9177, 2016a.

[14] Karcı, A.,"New Kinds of Entropy Fractional Entropy", International Conference on Natural Science and Engineering, 1-4, 2016b.

[15] Karcı, A.,"Malatya Functions: Symmetric Functions Obtained by Applying Fractional Order Derivative to Karcı Entropy", Anatolian Science-Journal of Computer Sciences, Vol:2, Issue: 2, pp:1-8, 2017.

[16] Khallil, R., Al Horani, M., Yousef, A., Sababheh, M., "A New Definition of Fractional Derivative", Journal of Computational and Applied Mathematics, Vol.264, pp: 65-70, 2014.