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Hyperstability and Stability of a Logarithm-type Functional Equation

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Abstract In 2001, Maksa and Páles [12] introduced a new type's stability: *hyperstability* for a class of linear functional equation $f(x) + f(y) = \frac{1}{n}\sum_{i=1}^{n} f(x\varphi_i(y))$. Riedel and Sahoo [14] have generalized a functional equation associated with the distance between the probability distributions, which is $f(pr, qs) + f(ps, qr) = 2M(rs)f(p, q) + 2M(pq)f(r, s)$. Elfen etc. [7] obtained the solution of the functional equation $f(pr, qs) + f(ps, qr) = 2f(p, q) + 2f(r, s)$ on semigroup G. The aim of this paper is to investigate the hyperstability and the Hyers-Ulam stability for the above Logarithm-type functional equation considered by Elfen, etc. Namely, if f is an approximative equation related to the above equation, then it is a solution of this equation which exists within $\varepsilon-$ bound of a given approximative function f.

Keywords Information Measure, Distance Measure, Superstability, Multiplicative Function, Stability of Functional Equation

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1 Introduction

The following stability problem is well-known as Ulam's stability problem [16]:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, Hyers [11] proved a first partial answer to Ulam's problem for an additive mapping on a Banach space. D. G. Bourgin obtained many excellent results for the stability ([3], [4]).Hyers' theorem was generalized by Aoki [1] for the case bounded by variables, and their results are improved by Rassias [13] to the case of the linear mapping and by Ger [9] . Găvruta [8] proved a further generalization of the Rassias' theorem by using a general control function.

The superstability phenomenon of the exponential equation $f(x + y) = f(x)f(y)$ was discovered by Baker, Lawrence, and Zorzitto [2] in 1979. The superstability for asymptotic phenomenon of the exponential equation was discovered by Ger [9].

In 2001, Maksa and Páles [12] proved a new type's stability for a class of linear functional equation

$$f(x) + f(y) = \frac{1}{n}\sum_{i=1}^{n} f(x\varphi_i(y)), \tag{1}$$

where f is a real-valued mapping defined on a semigroup S, and the mappings $\varphi_1, \varphi_2, \cdots, \varphi_n : S \to S$ are pairwise distinct automorphisms. That is as following:

Let $\varepsilon : S \times S \to \mathbb{R}$ be a function such that there exists a sequence u_k that satisfies

$$\lim_{k\to\infty} \varepsilon(u_k s, t) = 0 \qquad (s, t \in S).$$

Assume that $f : S \to X$ satisfies the stability inequality

$$\left|\left|f(s) + f(t) - \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t))\right|\right| \leq \varepsilon(s, t) \qquad (s, t \in S),$$

where X is a real normed space. Then, f is a solution of (1).

Such a phenomenon is called the hyperstability of the functional equation. Gselmann [10], Brazdęk and Ciepliński [5] investigated the hyperstability of functional equations. A similar concept was introduced by Sirouni and Kabbaj [15].

Riedel and Sahoo [14] solved a functional equation associated with the distance between the probability distributions. Let $M : (0.1) \to \mathbb{C}$ be a given multiplicative function. Then, if $f : (0.1)^2 \to \mathbb{C}$ satisfies the functional equation

$$f(pr, qs) + f(ps, qr) = 2M(rs)f(p, q) + 2M(pq)f(r, s)$$

if and only if

$$f(p, q) = M(p)M(q)\Big[L(p) + L(q) + l(\frac{p}{q}, \frac{p}{q})\Big],$$

where $M : (0.1) \to \mathbb{C}$ is an arbitrary logarithmic function and $l : (0.1)^2 \to \mathbb{C}$ is a bilogarithmic function. Thus, we will call it a logarithm-type functional equation

In addition, Elfen, Riedel and Sahoo [7] solved a functional equation

$$f(pr, qs) + f(sp, rq) = 2f(p, q) + 2f(r, s)$$

on semigroup G. Its solution type of f on $\bar{G}$ is given by

$$f(p, q) = A(p) + A(q) + \psi(pq^{-1}, pq^{-1}),$$

where $A : G \to \mathbb{C}$ is a homomorphism and $\psi : G \to \mathbb{C}$ is a symmetric bi-homomorphism.

Now we consider the logarithm-type functional equation given by

$$\frac{1}{2}[f(pr, qs) + f(ps, qr)] = f(p, q) + f(r, s). \tag{2}$$

For example, if $f(x, y) = \ln xy$, then f is a solution of the equation (2). In this paper, we investigate the hyperstability and stability of the functional equation (2). Namely, we prove that if f satisfies a stability inequality for the equation (2), then it is also a solution of this equation and also we can find an another solution of it which has an $\varepsilon-$error bound for f.

2 Hyperstability of the logarithm-type functional equation

In this section, we investigate the hyperstability of the equation (2). Throughout this section, let $(G, \cdot)$ denote a noncommutative semigroup, X a real normed space, and $\mathbb{R}$ the set of real numbers. And let $\mathbb{R}_+$ denote the set of positive real numbers.

Theorem 1. *Let $\varepsilon : G^2 \times G^2 \longrightarrow \mathbb{R}$ be a function such that there exists a sequence $u_k \in G$ that satisfies*

$$\lim_{k\to\infty} \varepsilon(u_k(p, q), (r, s)) = 0$$

for all $p, q, r, s \in G$. Assume that $f : G \times G \longrightarrow X$ satisfies the stability inequality

$$\Big|\Big|\frac{1}{2}\Big[f(pr, qs) + f(ps, qr)\Big] - f(p, q) - f(r, s)\Big|\Big| \le \varepsilon((p, q), (r, s)) \tag{3}$$

for all $p, q, r, s \in G$. Then,

$$\frac{1}{2}\Big[f(pr, qs) + f(ps, qr)\Big] = f(p, q) + f(r, s).$$

Proof. Define a function $F : G^2 \times G^2 \longrightarrow X$ by

$$F((p, q), (r, s)) = f(p, q) + f(r, s) - \frac{1}{2}\Big[f(pr, qs) + f(ps, qr)\Big].$$

Then, for all $p, q, r, s, v, w \in G$, we have

$$\begin{aligned}
&F((p, q), (r, s)) + \frac{1}{2}\Big[F((pr, qs), (v, w)) + F((ps, qr), (v, w))\Big] \\
&\quad = f(p, q) + f(r, s) + f(v, w) \\
&\qquad - \frac{1}{2}\Big[f(prv, qsw) + f(prw, qsv) + f(psv, qrw) + f(psw, qrv)\Big].
\end{aligned}$$

And also, for all $p, q, r, s, v, w \in G$, we have

$$F((r,s),(v,w)) + \frac{1}{2}\Big[F((p,q),(rv,sw)) + F((p,q),(rw,sv))\Big]$$
$$= f(p,q) + f(r,s) + f(v,w)$$
$$- \frac{1}{2}\Big[f(prv,qsw) + f(psw,qrv) + f(prw,qsv) + f(psv,qrw)\Big].$$

Thus, F satisfies the following functional equation

$$F((p,q),(r,s)) + \frac{1}{2}\Big[F((pr,qs),(v,w)) + F((ps,qr),(v,w))\Big]$$
$$= F((r,s),(v,w)) + \frac{1}{2}\Big[F((p,q),(rv,sw)) + F((p,q),(rw,sv))\Big]. \tag{4}$$

By (3), we get

$$||F((p,q),(r,s))|| \le \varepsilon((p,q),(r,s)),$$

and with the assumed sequence $\{u_k\}$, we obtain

$$\lim_{k\to\infty} F(u_k(p,q),(r,s)) \le \lim_{k\to\infty} \varepsilon(u_k(p,q),(r,s)) \tag{5}$$

for all $p, q, r, s \in G$.

The equation (4) implies

$$F((r,s),(v,w)) \tag{6}$$
$$= F((p,q),(r,s)) + \frac{1}{2}\Big[F((pr,qs),(v,w)) + F((ps,qr),(v,w))\Big]$$
$$- \frac{1}{2}\Big[F((p,q),(rv,sw)) + F((p,q),(rw,sv))\Big].$$

Let r, s, v, w, p_0, q_0 be fixed. Applying the norm and substituting $p = u_k p_0, q = u_k q_0$ in (6) , and as $k \to \infty$, respectively, we obtain

$$|| \lim_{k\to\infty} F(r,s),(v,w))||$$
$$= || \lim_{k\to\infty} \Big[F(u_k(p_0,q_0),(r,s)) + \frac{1}{2}\Big[F(u_k(p_0r,q_0s),(v,w)) + F(u_k(p_0s,q_0r),(v,w))\Big]$$
$$- \frac{1}{2}\Big[F(u_k(p_0,q_0),(rv,sw)) + F(u_k(p_0,q_0),(rw,sv))\Big]\Big]||.$$

By applying of (5) and the triangle inequalities, we obtain

$$||F(r,s),(v,w))||$$
$$\le |\lim_{k\to\infty} \Big[\varepsilon(u_k(p_0,q_0),(r,s)) + \frac{1}{2}\Big[\varepsilon(u_k(p_0r,q_0s),(v,w)) + \varepsilon(u_k(p_0s,q_0r),(v,w))\Big]\Big]|$$
$$+ |\lim_{k\to\infty} \frac{1}{2}\Big[\varepsilon(u_k(p_0,q_0),(rv,sw)) + \varepsilon(u_k(p_0,q_0),(rw,sv))\Big]|.$$

Hence, we obtain from the assumed sequence $\{u_k\}$ the required result

$$F((r,s),(v,w)) = 0$$

for any $r, s, v, w \in G$. □

Corollary 2. *Assume that* $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow X$ *satisfies the stability inequality*

$$\Big|\Big|\frac{1}{2}\Big[f(pr,qs) + f(ps,qr)\Big] - f(p,q) - f(r,s)\Big|\Big| \le \frac{rs}{pq} \quad or \quad pqrs \tag{7}$$

for all $p, q, r, s \in R_+$. *Then,*

$$\frac{1}{2}\Big[f(pr,qs) + f(ps,qr)\Big] = f(p,q) + f(r,s).$$

Proof. Let $\varepsilon((p,q),(r,s)) = \frac{rs}{pq}$ and $u_k = a^k$ for $a > 1$, or $\varepsilon((p,q),(r,s)) = pqrs$ and $u_k = a^k$ for $0 < a < 1$. Then, we obtain

$$\lim_{k\to\infty} \varepsilon(u_k(p,q),(r,s)) = 0,$$

so the result holds. □

3 Stability of the logarithm-type functional equation

In this section, we investigate the stability of the equation (2). Throughout this section, let $(G,\cdot)$ denote a commutative semigroup, N the set of natural numbers, and X a Banach space.

Theorem 3. *Let $\varepsilon > 0$. Assume that $f : G\times G \longrightarrow X$ satisfies the stability inequality*

$$\Big|\Big|\frac{1}{2}\Big[f(pr,qs)+f(ps,qr)\Big]-f(p,q)-f(r,s)\Big|\Big| \le \varepsilon \tag{8}$$

for all $p,q,r,s\in G$. Then there exists a function $F : G\times G\longrightarrow X$ such that

$$\frac{1}{2}\Big[F(pr,qs)+F(ps,qr)\Big] = F(p,q)+F(r,s)$$

and $||F(p,q)-f(p,q)|| \le \frac{39\varepsilon}{40}$ for any $p,q\in G$, where F is defined by

$$\begin{aligned}F(p,q) := \lim_{n\to\infty}\Big[&\frac{1}{2^{2^n}}f(p^{2^n},q^{2^n})\\ &+\Big(\sum_{i=0}^{n-2}\frac{1}{2^i\cdot 2^{2^{n-i}}}+\frac{1}{2^{n+1}}\Big)f((pq)^{2^{n-1}},(pq)^{2^{n-1}})\Big]\end{aligned}$$

for any $p,q,r,s\in G$.

Proof. Letting $r=p, s=q$ in (8) and dividing it by 2, we have

$$\Big|\Big|\frac{1}{2^2}\Big[f(p^2,q^2)+f(pq,pq)\Big]-f(p,q)\Big|\Big| \le \frac{\varepsilon}{2}. \tag{9}$$

And also, letting $p=q=r=s$ in (8) and dividing it by 2, we have

$$\Big|\Big|\frac{1}{2}f(p^2,p^2)-f(p,p)\Big|\Big| \le \frac{\varepsilon}{2}. \tag{10}$$

Let us show that the following inequality holds for every $n\in N$:

$$\Big|\Big|\frac{1}{2}f(p^{2^n},p^{2^n})-f(p^{2^{n-1}},p^{2^{n-1}})\Big|\Big| \le \frac{\varepsilon}{2}. \tag{11}$$

Replacing p by p^2 and q by q^2 in (9) respectively, and dividing 2^2, we have

$$\Big|\Big|\frac{1}{2^{2^2}}\Big[f(p^{2^2},q^{2^2})+f((pq)^2,(pq)^2)\Big]-\frac{f(p^2,q^2)}{2^2}\Big|\Big| \le \frac{\varepsilon}{2\cdot 2^2}. \tag{12}$$

Thus by (9),(10), and (12), we have

$$\begin{aligned}&\Big|\Big|\frac{1}{2^{2^2}}f(p^{2^2},q^{2^2})+(\frac{1}{2^{2^2}}+\frac{1}{2^2}\frac{1}{2})f((pq)^2,(pq)^2)-f(p,q)\Big|\Big|\\ &\le\Big|\Big|\frac{1}{2^{2^2}}\Big[f(p^{2^2},q^{2^2})+f((pq)^2,(pq)^2)\Big]-\frac{f(p^2,q^2)}{2^2}\Big|\Big|\\ &\quad+\Big|\Big|\frac{1}{2^2}\Big[f(p^2,q^2)+f(pq,pq)\Big]-f(p,q)\Big|\Big|\\ &\quad+\frac{1}{2^2}\Big|\Big|\frac{f((pq)^2,(pq)^2)}{2}-f(pq,pq)\Big|\Big|\\ &\le\frac{\varepsilon}{2\cdot 2^2}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2\cdot 2^2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}.\end{aligned} \tag{13}$$

In addition, by letting p by $p^{2^{n-1}}$ and q by $q^{2^{n-1}}$ in (9), and dividing $2^{2^{n-1}}$, the following inequality holds for every $n\in N$:

$$\begin{aligned}&\Big|\Big|\frac{1}{2^{2^n}}\Big[f(p^{2^n},q^{2^n})+f((pq)^{2^{n-1}},(pq)^{2^{n-1}})\Big]-\frac{f(p^{2^{n-1}},q^{2^{n-1}})}{2^{2^{n-1}}}\Big|\Big|\\ &\le\frac{\varepsilon}{2\cdot 2^{2^{n-1}}}.\end{aligned} \tag{14}$$

By (10),(12), and (14), we have

$$\begin{aligned}
&\Big|\Big|\frac{1}{2^{2^3}}f(p^{2^3},q^{2^3})+\Big(\frac{1}{2^{2^3}}+(\frac{1}{2^{2^2}}+\frac{1}{2^3})\frac{1}{2}\Big)f((pq)^{2^2},(pq)^{2^2})-f(p,q)\Big|\Big| \\
&\le \Big|\Big|\frac{1}{2^{2^3}}\Big[f(p^{2^3},q^{2^3})+f((pq)^{2^2},(pq)^{2^2})\Big]-\frac{f(p^{2^2},q^{2^2})}{2^{2^2}}\Big|\Big| \\
&\quad+\Big|\Big|\frac{1}{2^{2^2}}f(p^{2^2},q^{2^2})+(\frac{1}{2^{2^2}}+\frac{1}{2^3})f((pq)^2,(pq)^2)-f(p,q)\Big|\Big| \\
&\quad+(\frac{1}{2^{2^2}}+\frac{1}{2^3})\Big|\Big|\frac{f((pq)^{2^2},(pq)^{2^2})}{2}-f((pq)^2,(pq)^2)\Big|\Big| \\
&\le \frac{\varepsilon}{2\cdot 2^{2^2}}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\Big(\frac{\varepsilon}{2^{2^2}}+\frac{\varepsilon}{2^3}\Big)\frac{1}{2} \\
&=\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^{2^2}}+\frac{\varepsilon}{2^4} \\
&=\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^3}.
\end{aligned} \tag{15}$$

Note that

$$\begin{aligned}
&\sum_{j=4}^{n}\sum_{i=2}^{j-2}\frac{\varepsilon}{2^i\cdot 2^{2^{j-i}}}+\sum_{i=1}^{n}\frac{\varepsilon}{2^{2^{i-1}}}+\sum_{i=3}^{n}\frac{\varepsilon}{2^{i+1}} \\
&\le \frac{\varepsilon}{2^2}\Big(\frac{1}{2^{2^2}}+\frac{1}{2^{2^3}}+\frac{1}{2^{2^4}}+\cdots\Big)+\frac{\varepsilon}{2^3}\Big(\frac{1}{2^{2^2}}+\frac{1}{2^{2^3}}+\frac{1}{2^{2^4}}+\cdots\Big) \\
&\quad+\cdots+\frac{\varepsilon}{2^{n-1}}\Big(\frac{1}{2^{2^2}}+\frac{1}{2^{2^3}}+\frac{1}{2^{2^4}}+\cdots\Big) \\
&\quad+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\varepsilon\Big(\frac{1}{2^{2^2}}+\frac{1}{2^{2^3}}+\frac{1}{2^{2^4}}+\cdots\Big)+\frac{\varepsilon}{2^4}\Big(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\Big) \\
&=\varepsilon\Big(\frac{1}{2^{2^2}}+\frac{1}{2^{2^3}}+\frac{1}{2^{2^4}}+\cdots\Big)\Big(1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\Big) \\
&\quad+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^4}\Big(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\Big) \\
&=\varepsilon\cdot\frac{1}{15}\cdot\frac{3}{2}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^3} \\
&=\frac{39\varepsilon}{40}.
\end{aligned} \tag{16}$$

Suppose that the following inequality holds for $n \ge 4$ and for any $p, q \in G$:

$$\begin{aligned}
&\Big|\Big|\frac{1}{2^{2^n}}f(p^{2^n},q^{2^n})+\Big(\sum_{i=0}^{n-2}\frac{1}{2^i\cdot 2^{2^{n-i}}}+\frac{1}{2^{n+1}}\Big)f((pq)^{2^{n-1}},(pq)^{2^{n-1}})-f(p,q)\Big|\Big| \\
&\quad\le\sum_{j=4}^{n}\sum_{i=2}^{j-2}\frac{\varepsilon}{2^i\cdot 2^{2^{j-i}}}+\sum_{i=1}^{n}\frac{\varepsilon}{2^{2^{i-1}}}+\sum_{i=3}^{n}\frac{\varepsilon}{2^{i+1}}.
\end{aligned} \tag{17}$$

Note that

$$\sum_{i=0}^{n-1}\frac{1}{2^i\cdot 2^{2^{n+1-i}}}=\frac{1}{2^{2^{n+1}}}+\frac{1}{2}\sum_{i=0}^{n-2}\frac{1}{2^i\cdot 2^{2^{n-i}}} \tag{18}$$

for all $n \in N$. Then, for any $p, q \in G$, based on (14) and (18), we obtain

$$\begin{aligned}
&\Big|\Big|\frac{1}{2^{2^{n+1}}} f(p^{2^{n+1}}, q^{2^{n+1}}) \\
&\quad + \Big(\sum_{i=0}^{n-1} \frac{1}{2^i \cdot 2^{2^{n+1-i}}} + \frac{1}{2^{n+2}}\Big) f((pq)^{2^n}, (pq)^{2^n}) - f(p, q)\Big|\Big| \\
&\le \Big|\Big|\frac{1}{2^{2^{n+1}}}\Big[f(p^{2^{n+1}}, q^{2^{n+1}}) + f((pq)^{2^n}, (pq)^{2^n})\Big] - \frac{f(p^{2^n}, q^{2^n})}{2^{2^n}}\Big|\Big| \\
&\quad + \Big|\Big|\frac{1}{2^{2^n}} f(p^{2^n}, q^{2^n}) \\
&\qquad + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - f(p, q)\Big|\Big| \\
&\quad + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big)\Big|\Big|\frac{f((pq)^{2^n}, (pq)^{2^n})}{2} - f((pq)^{2^{n-1}}, (pq)^{2^{n-1}})\Big|\Big| \\
&= \frac{\varepsilon}{2 \cdot 2^{2^n}} + \sum_{j=4}^{n}\sum_{i=2}^{j-2} \frac{\varepsilon}{2^i \cdot 2^{2^{j-i}}} + \sum_{i=1}^{n} \frac{\varepsilon}{2^{2^{i-1}}} \\
&\quad + \sum_{i=3}^{n} \frac{\varepsilon}{2^{i+1}} + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big)\frac{\varepsilon}{2} \\
&= \Big(\sum_{j=4}^{n}\sum_{i=2}^{j-2} \frac{\varepsilon}{2^i \cdot 2^{2^{j-i}}} + \sum_{i=2}^{n-1} \frac{\varepsilon}{2^i \cdot 2^{2^{n+1-i}}}\Big) \\
&\quad + \Big(\frac{\varepsilon}{2^{2^n}} + \sum_{i=1}^{n} \frac{\varepsilon}{2^{2^{i-1}}}\Big) + \Big(\sum_{i=3}^{n} \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2^{n+2}}\Big) \\
&= \sum_{j=4}^{n+1}\sum_{i=2}^{j-2} \frac{\varepsilon}{2^i \cdot 2^{2^{j-i}}} + \sum_{i=1}^{n+1} \frac{\varepsilon}{2^{2^{i-1}}} + \sum_{i=3}^{n+1} \frac{\varepsilon}{2^{i+1}}.
\end{aligned} \tag{19}$$

Thus, by induction, inequality (17) holds for all $n \ge 4$ and for any $p, q \in G$. Now for $n \ge 4$, we have

$$\begin{aligned}
&\Big|\Big|\frac{1}{2^{2^n}} f(p^{2^n}, q^{2^n}) + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) \\
&\quad - \frac{1}{2^{2^{n-1}}} f(p^{2^{n-1}}, q^{2^{n-1}}) \\
&\quad - \Big(\sum_{i=0}^{n-3} \frac{1}{2^i \cdot 2^{2^{n-1-i}}} + \frac{1}{2^n}\Big) f((pq)^{2^{n-2}}, (pq)^{2^{n-2}})\Big|\Big| \\
&\le \Big|\Big|\frac{1}{2^{2^n}}\Big[f(p^{2^n}, q^{2^n}) + f((pq)^{2^{n-1}}, (pq)^{2^{n-1}})\Big] - \frac{1}{2^{2^{n-1}}} f(p^{2^{n-1}}, q^{2^{n-1}})\Big|\Big| \\
&\quad + \sum_{i=0}^{n-3} \frac{1}{2^i \cdot 2^{2^{n-1-i}}}\Big|\Big|\frac{f((pq)^{2^{n-1}}, (pq)^{2^{n-1}})}{2} - f((pq)^{2^{n-2}}, (pq)^{2^{n-2}})\Big|\Big| \\
&\quad + \Big|\Big|\frac{1}{2^{n+1}} f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - \frac{1}{2^n} f((pq)^{2^{n-2}}, (pq)^{2^{n-2}})\Big|\Big| \\
&\le \frac{\varepsilon}{2 \cdot 2^{2^{n-1}}} + \sum_{i=0}^{n-3} \frac{1}{2^i \cdot 2^{2^{n-1-i}}} \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2 \cdot 2^n} \\
&\le \frac{\varepsilon}{2 \cdot 2^{2^{n-1}}} + \frac{1}{2^{2^{n-1}}} \sum_{i=0}^{n-3} \frac{1}{2^{i+1}} \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2 \cdot 2^n} \quad \longrightarrow \quad 0
\end{aligned} \tag{20}$$

as $n \to \infty$. Thus, if we let

$$Y_n = \frac{1}{2^{2^n}} f(p^{2^n}, q^{2^n}) + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}),$$

then $\{Y_n\}$ is a Cauchy sequence due to (20), and so we can define a function $F : G \times G \longrightarrow X$ by

$$F(p,q) := \lim_{n\to\infty} \Big[\frac{1}{2^{2^n}} f(p^{2^n}, q^{2^n}) + \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}})\Big]. \tag{21}$$

Then, due to (16), (17), and (21), we have

$$||F(p,q) - f(p,q)|| \leq \frac{39\varepsilon}{40} \qquad \forall p,q \in G.$$

Finally, the function F defined in (21) holds the required equation (2) as follows:

$$\begin{aligned}
&\Big|\Big|\frac{1}{2}\Big[F(pr,qs) + F(ps,qr)\Big] - F(p,q) - F(r,s)\Big|\Big| \\
&\leq \lim_{n\to\infty} \Big|\Big|\frac{1}{2}\frac{1}{2^{2^n}} f((pr)^{2^n}, (qs)^{2^n}) \\
&\qquad + \frac{1}{2}\Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pqrs)^{2^{n-1}}, (pqrs)^{2^{n-1}}) \\
&\quad + \frac{1}{2}\frac{1}{2^{2^n}} f((ps)^{2^n}, (qr)^{2^n}) \\
&\qquad + \frac{1}{2}\Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pqrs)^{2^{n-1}}, (pqrs)^{2^{n-1}}) \\
&\quad - \frac{1}{2^{2^n}} f((p)^{2^n}, (q)^{2^n}) \\
&\qquad - \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) \\
&\quad - \frac{1}{2^{2^n}} f((r)^{2^n}, (s)^{2^n}) \\
&\qquad - \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) f((rs)^{2^{n-1}}, (rs)^{2^{n-1}})\Big|\Big| \\
&\leq \lim_{n\to\infty} \frac{1}{2^{2^n}} \Big|\Big|\frac{1}{2}\Big[f((pr)^{2^n}, (qs)^{2^n}) + f((ps)^{2^n}, (qr)^{2^n})\Big] \\
&\quad - f(p^{2^n}, q^{2^n}) - f(r^{2^n}, s^{2^n})\Big|\Big| \\
&\quad + \lim_{n\to\infty} \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big) \\
&\quad \times \Big|\Big|\frac{1}{2}\Big[f((pqrs)^{2^{n-1}}, (pqrs)^{2^{n-1}}) + f((pqrs)^{2^{n-1}}, (pqrs)^{2^{n-1}})\Big] \\
&\qquad - f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - f((rs)^{2^{n-1}}, (rs)^{2^{n-1}})\Big|\Big| \\
&\leq \lim_{n\to\infty} \frac{\varepsilon}{2^{2^n}} + \lim_{n\to\infty} \Big(\sum_{i=0}^{n-2} \frac{1}{2^i \cdot 2^{2^{n-i}}} + \frac{1}{2^{n+1}}\Big)\varepsilon \\
&= 0.
\end{aligned}$$

□

Corollary 4. *Let $\varepsilon > 0$. Assume that $f : G \times G \longrightarrow X$ satisfies the stability inequality*

$$\Big|\Big|\frac{1}{2^2}\Big[f(p^2, q^2) + f(pq, pq)\Big] - f(p,q)\Big|\Big| \leq \varepsilon \tag{22}$$

for all $p, q \in G$. Then there exists a function $F : G \times G \longrightarrow X$ such that

$$\frac{1}{2^2}\Big[F(p^2, q^2) + F(pq, pq)\Big] = F(p,q)$$

and $||F(p,q) - f(p,q)|| \leq \frac{39\varepsilon}{40}$ for any $p, q \in G$.

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