Mathematics and Statistics 7(1): 10-13, 2019 http://www.hrpub.org
DOI: 10.13189/ms.2019.070102

Exact Traveling Wave Solutions of Nonlinear Evolution Equations: Indeterminant Homogeneous Balance and Linearizability

Barbara Abraham-Shrauner

Department of Electrical and Systems Engineering, Washington University, USA

Abstract Exact traveling (solitary) wave solutions of nonlinear partial differential equations (NLPDEs) are analyzed for third-order nonlinear evolution equations. These equations have indeterminant homogenous balance and therefore cannot be solved by the Power Index Method (PIM). Some evolution equations are linearizable where solutions are transferred from those of a linear PDE. For other evolution equations transforming to a NLPDE which has a homogenous balance gives rise to possible solutions by the PIM. The solutions for evolution equations that are not linearizable are developed here.

Keywords Evolution Equations, Homogeneous Balance, Traveling Waves, Lie Symmetry, PIM

1. Introduction

Determining exact, traveling wave solutions of some nonlinear partial differential equations (NLPDEs) has been codified recently where earlier references are noted [1,2]. The approach ties together various methods with new criteria in the PIM. A fundamental requirement of the PIM is that the homogeneous balance holds [3-6]. However, homogeneous balance does not hold for linear partial differential equations (LPDEs) but also it does not hold for some NLPDEs.

Linearization of nonlinear ordinary differential equations (NLODEs) has a long history. Single NLODEs of various orders have been studied; past efforts are reviewed in a recent paper on the linearization of fifth-order NLODEs with emphasis on Lie symmetry analysis [7]. A type of hidden symmetry has been applied to Newtonian equations for a Hamiltonian system to reduce coupled ODEs to a single second-order linear ODE [8]. The linearization of NLPDEs is even more challenging.

2. Objectives

How can exact solutions be found for NLPDEs for which homogeneous balance does not hold? This question is discussed for a particular class of NLPDEs, third-order nonlinear evolution equations. Two cases have been identified: (a) third-order nonlinear evolution equations that are linearizable and (b) other third-order nonlinear evolution equations that are not obviously linearizable.

Recent studies of third-order nonlinear evolution equations classified Lie group symmetries of a subclass of third-order evolution equations and the theorems for linearization [9]. In addition earlier symmetry analysis of third-order evolution equations was reported [10-12]. We shall consider those third-order evolution equations which possess indeterminant homogenous balance.

3. Methods

The main method is to define the class of third-order nonlinear evolution equations that have indeterminant homogenous balance and demonstrate by examples.

3.1. Third-Order Nonlinear Evolution Equations

A subclass of third-order nonlinear evolution equations is given by [9],

$$u_t = u_{xxx} + F(t, x, u, u_x, u_{xx}). \qquad (1)$$

The theorems for linearizability of (1) depend on the Lie group symmetry analysis. For linearizability either (1) or a suitably transformed equation (1) should be invariant under an infinite-dimensional subalgebra. We restrict the form of (1) to those cases which have an indeterminant homogeneous balance condition. There is an example of (1) in [9] that obeys the homogeneous balance condition that is reduced to a linear PDE by a hodograph transformation. A classic paper on a third-order evolution equation found solutions by reducing the equation to a linear PDE [13]. The homogeneous balance condition holds for the original nonlinear third-order evolution equation and that example is discussed later.

The general form of (1) for our investigations is

$$u_t = u_{xxx} + C_1 \frac{u_x u_{xx}}{u} + C_2 \frac{u_x^3}{u^2} + C_3 \frac{u_{xx}^2}{u_x}, \qquad (2)$$

where all terms in (2) can be expressed in terms of differential invariants of the Lie group generator $u\frac{\partial}{\partial u}$ by dividing (2) by u. The relevant differential invariants are $\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{u_t}{u}$ and various combinations of these differential invariants. The theorems in [9] are used to find the coefficients C_1, C_2, C_3 that lead to linearizability. An alternative approach not introduced in [9] is to start with the third-order linear PDE

$$w_t = w_{xxx} \qquad (3)$$

and let $w = u^m$, for any m except $m = 0$. The result is

$$u_t = u_{xxx} + 3\frac{(m-1)u_x u_{xx}}{u} + \frac{(m-1)(m-2)u_x^3}{u^2} \qquad (4)$$

We can also let $w = lnu$ for the $m = 0$ case. Then (3) becomes

$$u_t = u_{xxx} - \frac{3u_x u_{xx}}{u} + \frac{2u_x^3}{u^2} \qquad (5)$$

We see by comparing (4) and (5) for $m = 0$, that $3(m-1) = -3, (m-1)(m-2) = 2$ or (5) is contained in (4). Comparing (2) and (4), we find that (2) is linearizable if $C_1 = 3(m-1), C_2 = (m-1)(m-2)$ and $C_3 = 0$. We can check that (4) is linearizable by letting $u = w^{1/m}$ and find that (4) becomes (3).

The other case for (2) that is linearizable is for $C_3 \neq 0, C_1 = C_2 = 0$. Next, differentiate the altered (2) with respect to x and substitute $u_x = v$. Then we find

$$v_t = v_{xxx} + C_3\left(\frac{2v_x v_{xx}}{v} - \frac{v_x^3}{v^2}\right). \qquad (6)$$

As (6) is of the form of (4), $= \frac{1}{2}, C_3 = -\frac{3}{4}$. This case involves a clever change of variable. Another example of a clever change of variables is in the reference [13]. The nonlinear evolution equation in [13] obeys the homogeneous balance condition. However, the intermediate equation (3.4) in [13] is the same as (2) with $C_3 = -\frac{3}{4}, C_1 = C_2 = 0$ and (3.1) in [13] is the same as (3) with $C_3 = -\frac{3}{4}, C_1 = C_2 = 0$.

3.2. Homogeneous Balance Condition

We start with a single NLPDE invariant under translations in the independent variables, t and x. More general cases apply. The solution of the NLPDE $u(x,t)$ is assumed to be expressed as a finite power series in a nonlinear function $U(\beta\bar{x})$ such as a hyperbolic or a Jacobian elliptic function where $\bar{x} = x - ct$. The traveling wave may also be viewed as a nonlinear solitary wave that can be a soliton if extra conditions are met. Then

$$u = \sum_{l=0}^{p} a_l \; U(\beta\bar{x})^l, \qquad (7)$$

where the power p is an integer, a_l is an expansion coefficient, β is the wave number and c is the wave speed. The essential concept is to balance the most nonlinear terms in the NLPDE; otherwise no solution is possible.

The homogeneous balance condition is given by the power index P and the justification for P is discussed in [1,2]. Then P_j for any term in the NLPDE is

$$P_j = n_j p + d_j, \qquad (8)$$

where n_j is the number of products of u and its derivatives and d_j is the net number of derivatives, counting x and t derivatives in a term in the NLPDE. The value of power p is determined from equation (8) by equating the power index for the most nonlinear terms in the NLPDEs. If p is not an integer, change the variables in the NLPDEs so that p is an integer. For a linear PDE such as (3) all n_j are one and consequently the homogeneous balance is indeterminant as is the power.

The unique aspect of the analysis here is that the power p may be indeterminant for a NLPDE. Consider (2) where each term has its power Index indicated below.

$$u_t = u_{xxx} + C_1 \frac{u_x u_{xx}}{u} + C_2 \frac{u_x^3}{u^2} + C_3 \frac{u_{xx}^2}{u_x}, \qquad (9)$$

$$p+1 \; p+3 \;\; p+3 \;\; p+3 \;\; p+3$$

where the factors of u, u_x in the denominators subtract from the value for n_j. Since all terms have a linear power value, then the power p cannot be determined by homogenous balance. Consequently, the Power Index Method cannot be used to find exact traveling wave solutions of (3). How then can exact traveling wave solutions be found? There are three possibilities: (a) start with an evolution equation that is linearizable by the form of (4), (b) transform the nonlinear evolution equation to one that is linearizable, or (c) for evolution equations that are not obviously linearizable apply the new approach presented here. For (b) use the change of variables as was done in (6) or use a hodograph transformation that is discussed in [9] or by a clever variable transformation as discussed in [13].

For evolution equations that are not obviously linearizable one may guess a solution with a nonlinear function and transform the evolution equation to another evolution equation for which homogeneous balance holds. This process is not necessarily unique and the transformation should be of a particular form as indicated in the next section with examples.

3.3. Results: Exact Nonlinear Traveling Waves or Solitary Waves

Equation (4) can be reduced to the linear PDE (3) but the coefficients in (2) are consequently constrained even though a wide range of possibilities exist for different values of m. In this section evolution equations that do not fit (4) immediately are presented. The first example is

$$u_t = u_{xxx} - \frac{3u_x u_{xx}}{u}, \tag{10}$$

$$p+1 \quad p+3 \quad p+3$$

where the power index values are noted below each term. Since p is linear in all terms, homogenous balance does not hold. We start with a guess for the solution. Let $u = \sinh(\beta\bar{x})$, $u = \cosh(\beta\bar{x})$, $u = \mathrm{sech}(\beta\bar{x})$ or $u = \mathrm{csch}(\beta\bar{x})$. Note that the amplitudes of the solutions are omitted as they cancel in each term. Then $c = 2\beta^2$. Other solutions are $u = \tanh(\beta\bar{x})$ or $u = \coth(\beta\bar{x})$ with $c = -4\beta^2$. Now transform (10) to another nonlinear evolution equation for which homogeneous balance holds. The transformation of the dependent variable must be of exponential form. Letting u be a power of w or a derivative of w leaves the homogenous balance undetermined. Let

$$u = \exp\left[\int_0^x \mathrm{w}(\beta\bar{x}')\,\partial \mathrm{x}'\right]. \tag{11}$$

Next divide (10) by u, then differentiate the result with respect to x and substitute (11) into the differentiated result. The new evolution equation is

$$w_t = w_{xxx} - 6w^2 w_x, \tag{12}$$

which is the mKdV equation; the power $p = 1$ a it obeys homogenous balance [1,2]. The solutions of (12) have been known [1,2]. Some solutions are $A\tanh(\beta\bar{x})$ or $A\coth(\beta\bar{x})$. If we invert (11), we find $= \frac{u_x}{u}$. Using the guessed solutions: $u = \sinh(\beta\bar{x}))$ or $\cosh(\beta\bar{x})$,we find $A = \beta$. For guessed solutions $u = \mathrm{sech}(\beta\bar{x})$ or $u = \mathrm{csch}(\beta\bar{x})$, $A = -\beta$. In addition $w = \frac{\beta sech^2(\beta\bar{x})}{tanh(\beta\bar{x})}$ for the guessed solution $u = \tanh(\beta\bar{x})$ and that is not the usual solution from the Power Index Method for w but is a solution found by the G'/G method [14] for a nonlinear appended ODE for $G = \tanh(\beta\bar{x})$. For the guessed solution $u = \coth(\beta\bar{x})$ $w = -\beta\tanh(\beta\bar{x})$.

The solitary waves of (12) have the amplitude proportional to β as well as the coefficient of t that is proportional to β^3. This dependence is characteristic of nonlinear solitary waves.

A second example is

$$u_t = u_{xxx} + C_2\frac{u_x^3}{u^2}. \tag{13}$$

The solutions are subtle here. We can apply (11) or let $= \frac{u_x}{u}$. Then (14) becomes

$$w_t = w_{xxx} + \frac{3}{2}(w^2)_{xx} + (C_2+1)(w^3)_x. \tag{14}$$

$$p+1 \quad p+3 \quad 2p+2 \quad 3p+1$$

The power index values below (14) can be used to compute the power $p = 1$ by equating the two most nonlinear terms. The dependent variable w may be an odd function of the argument as can be seen by evaluating all terms in (14). Then

$$w = A\tanh(\beta\bar{x}). \tag{15}$$

One might assume that $u = \cosh(\beta\bar{x})$ or $u = \sinh(\beta\bar{x})$ but this is incorrect. We let

$$u = \cosh^n(\beta\bar{x}) \tag{16}$$

and substitute (16) into (13). The general result is

$$C_2 = -\frac{(n-1)(n-2)}{n^2} \text{ and } c = \beta^2\ (2 - 3n), \tag{17}$$

where there is no solution for $n = 1{,}2$. If we substitute (17) in $w = \frac{u_x}{u}$, we find that $A = n\beta$.

The guess in (16) is not the usual guess for $n = 1$ but demonstrates that the solution is difficult to guess from (13).

A third example is as expected from [9] and (2)

$$u_t = u_{xxx} + C_3\frac{u_{xx}^2}{u_x}. \tag{18}$$

This evolution equation is not of the correct form for the theorems on linearizability in [9]. The authors let $v = u_x$ so that any factors in the denominator in (18) are not derivatives. First, differentiate (18) by x and then replace u_x by v, etc. We find that (19) is

$$v_t = v_{xxx} + C_3\left(\frac{2v_x v_{xx}}{v} - \frac{v_x^3}{v^2}\right), \tag{19}$$

$$p+1 \quad p+3 \quad \mathrm{p}+3 \quad \mathrm{p}+3$$

where the power index values are indeterminant as is the power p.

Comparing (19) and (4), we see that for $= \frac{1}{2}$, and $C_3 = -\frac{3}{4}$ or equation (19) is linearizable. Next let

$$v = \exp\left[\int_0^x \mathrm{w}(\beta\bar{x}')\,\partial \mathrm{x}'\right] \tag{20}$$

and that results in the mKdV equation

$$w_t = w_{xxx} - \frac{3}{2}w^2 w_x, \tag{21}$$

$$p+1 \quad p+3 \quad 3p+1$$

where the power index values are below each term in (21).

A trial solution of (18) is $u = \tanh(\beta\bar{x})$. We find the trial solution is valid with $C_3 = -\frac{3}{2}$ and $c = 2\beta^2$. Next check the solution for $v = u_x = \beta\,\mathrm{sech}^2(\beta\bar{x})$ in (19) and that is valid for $C_3 = -\frac{3}{2}$ and $= 2\beta^2$. Finally, we let $w = \frac{v_x}{v} = -2\beta\tanh(\beta\bar{x})$ that is the inverse of (20). This solution is valid for $C_3 = -\frac{3}{2}$ and $c = 2\beta^2$ also. We could have started with a solution for (21) of the form $w = A\tanh(\beta\bar{x})$, determined A because the power $p = 1$ and because an odd function of the argument of w is possible.

A solution of (3) and (19) for C_3=- $\frac{3}{4}$ is $w = \cosh(\beta\bar{x})$ and $v = w^2 = \cosh^2(\beta\bar{x})$. However, the solution $u_x = v = \cosh^2(\beta\bar{x})$ is not a solution for (18) for C_3=- $\frac{3}{4}$. Actually (18) plus a constant reduces to (19) by differentiation with respect to x and $u_x = cosh^2(\beta\bar{x})$ is a solution of this other evolution equation.

The solutions have been hyperbolic functions so far but

Jacobian elliptic functions are also possible solutions. Let $u = cn(\beta\bar{x})$ in (10) where the modulus is k [15] which is suppressed in $cn(\beta\bar{x})$. The modulus is frequently defined as m with $m = k^2$. The wave speed is $c = \beta^2(4k^2 - 2)$. In the limit $k \to 1$ we find that $cn(\beta\bar{x}) \to \text{sech}(\beta\bar{x})$ and $c \to 2k^2$. For $u = sn(\beta\bar{x})$ the wave speed is $c = 2\beta^2(k^2 + 1)$. In the limit $k \to 1$ we find that $sn(\beta\bar{x}) \to tan\text{h}(\beta\bar{x})$ and $c \to 4k^2$. The limiting cases agree with those below (10). The Jacobian elliptic functions offer a wider class of exact solutions; hyperbolic solutions are a special case.

4. Discussion and Conclusions

Exact traveling wave (solitary) solutions of third-order nonlinear evolution equations have been solved for the nonlinear partial differential equations that have indeterminant homogenous balance. Previous results on the linearizability of the third-order evolution equations by theorems found by Lie symmetries are combined with the Power Index Method. Two main classes are found for third-order evolution equations with indeterminant homogeneous balance: those equations that are linearizable by the theorems and those equations that do not obey the theorems. The former evolution equations can be transformed to a simple linear PDE and these solutions transferred back to the nonlinear evolution equations. Solutions of linear PDEs have not been explored here but an example occurs in reference [13]. An approach for solutions of the latter class of evolution equations combines guesses of solutions of the nonlinear evolution equations and the solutions of the transferred NPDEs that are solved by the Power Index Method. Both guesses and the solutions of the transferred NLPDEs are necessary for the widest possible solutions.

Some unexpected solutions for the second class include solutions not found for hyperbolic functions and Jacobian elliptic functions suggested as nonlinear functions in the Power Index Method. In the first example the new solution could be found by a G'/G method with a nonlinear appended equation for G where the usual appended equation is linear. The second example demonstrates that the guessed function is not what is expected from the transformed NLPDE as it is a hyperbolic function raised to a power. In the third example the real surprise is that the solution for $u_x = \cosh^2(\beta\bar{x})$ with C_3=- $\frac{3}{4}$ is not of (18) but of another equation that is (18) plus a constant.

REFERENCES

[1] B. Abraham-Shrauner, Exact solutions of nonlinear partial differential equations, Discrete Contin. Dyn. l Syst. Series S, Vol. 11, 577- 582, 2018.

[2] B**.** Abraham-Shrauner, Analytic solutions of nonlinear partial differential equations by the Power Index Method, Symmetry, Vol. 10, No. 76, doc:10.3390/sym10030076, 2018.

[3] M. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A, Vol. 213, 279- 287, 1996.

[4] N. A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos, Solitons and Fractals**,** Vol. 24, 1217- 1231**,** 2005.

[5] L. Huiban, W. Kelin, Exact solutions for two nonlinear equations: I. J. Phys. A: Math. Gen., Vol. 23, 3923-3928, 1990**.**

[6] W. Malfliet, Solitary wave solution of nonlinear wave equations, Am. J. of Phys., Vol. 60, 650-654, 1992.

[7] S. Suksern, K. Naboonmee, Linearization of fifth-order Ordinary Differential Equations by Generalized Sundman Transformation; International Journal of Differential Equations; doi: org/10.1155/2018/3048423, 2018.

[8] G. Gubboitti, M. C. Nucci, Are all Classical superintegrable systems in two-dimensional space linearizable? J. Math. Phys., Vol. 58, 012902**;** doi: 10.101063/1,4974264**,** 2017**.**

[9] P. Basarab- Howath, F. Güngör, Linearization for third order evolution equations, J. Math. Phys., Vol. 58, No. 8, 081507, doi:10.1063/1.4997558, 2017.

[10] F. Güngör, V. I. Lahno, R. Z Zhdanov, Symmetry classification of KdV-type nonlinear evolution equations, J. Math. Phys., Vol. 45, 2280-2313, 2004.

[11] P. Basarab- Howath, F. Güngör and V. Lahno, V. Symmetry classification of third-order nonlinear evolution equations, Part I: Semi-simple algebras, Acta Appl. Math., Vol.124, 123- 170, 2013.

[12] P. Basarab- Howath, F. Güngö, Symmetry classification of third-order nonlinear evolution equations, Part II: Solvable algebras (unpublished).

[13] F. Calogero, The evolution partial differential equation, $u_t = u_{xxx} + 3(u_{xx}u^2 + 3u_x{}^2u) + 3u_xu^4$, J. Math. Phys., Vol. 28, 538-555, 1987.

[14] Y-M. Zhao, New Jacobi elliptic function solutions for the Zakarov equations, J. Appl. Math., Vol. 2012, doi: 10.1155/2012/854619.16, 2012.

[15] P. F. Byrd, M.D. Friedman, M. D. Handbook of Elliptic Integrals for Engineers and Scientists, 2^{nd} edition, Springer-Verlag, New York, 1971.