Universal Journal of Mechanical Engineering 7(1): 11-15, 2019 http://www.hrpub.org
DOI: 10.13189/ujme.2019.070102

Pull-in Effect for Micro Beam with Tensile Force

Skubov D. Yu.[1,2,*], Privalova O. V.[2], Shtukin L. V.[1,2]

[1]Institute of Problem of Mechanical Engineering RAS & Saint-Petersburg, Russia
[2]Department of Mechanics and Processes of Control, Peter the Great St. Petersburg Polytechnic University, Russia

Abstract Recent time the development and achievement of micro- and nano-electromechanical systems (MEMS and NEMS) are appeal the great interest of physics, biologists, engineers-electricians. The designing of MEMS based on pull-in effect consists in interaction of electrostatic field with thin elastic conductive beam. This interaction leads to pull-in instability – the effect of collapse of two initially parallel conductive layers, which play the role of capacitor. The important significance of MEMS have been acquired [1, 2] such, for example, as micro-switches with forward or rotary movement. These devices may be membrane else cantilever or another type, also high speed rotational actuator – contactless micro-gyroscope.

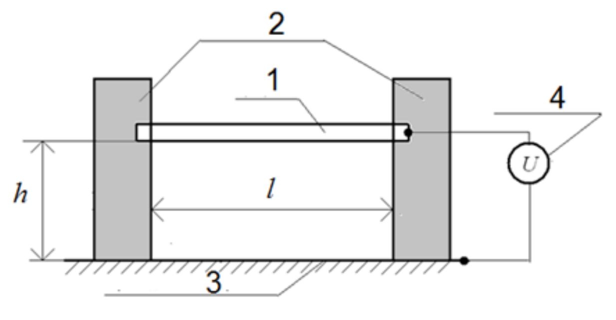
Keywords Pull-in Effect, Electrostatic Field, Micro-beam

1. Introduction

The «history» of pull-in effect is very long. The full description of this study we can find in work of J. Pelesko and T. Driscoll [3]. In this article the investigation of influence of the edge effect of distributed electric field on the equilibrium forms of conductive membrane – upper plate of drum-shaped electrostatically actuating MEMS device was fulfilled. Also we can mention work of Sl. Krylov et al. [4] in which the analysis of the influence of initially curved beam on the pull-in effect of micro beam loaded by distributed electro-static force is studied. It gives us the criteria to predict action of non-symmetric buckling in electrostatic field on constitution of the bistable forms of micro beam. The main purpose of our article is to give estimation of the relation between flexural stiffness arising due to tension of micro-rod and bending stiffness of this electro-elastic system.

Free Oscillation. Eigen Frequency and Modes

Here effect of «pull-in» instability is considered for beam with tensile force. «Pull-in» instability is an inherently nonlinear effect connected with disappearance of equilibrium forms of elastic part of sensitive or actuating mechanism, which in our task is considered as a flexible double clamped microbeam. Its parameters are: length l, bending stiffness EI, tensile force P, ρF – the density on the unit of length. Moreover we suppose that the bending stiffness is much less than the stiffness of string. This assumption gives us the possibility to build the asymptotic approximation of pull-in branching curve and estimate the influence of stiffness relation.



1 – elastic element, 2 – isolate support, 3 - motionless electrode, 4 - voltage source

Figure 1. Model of investigation

In static equation of beam taking in account the tensile force $P > 0$ has a next form

$$EIy^{IY} - Py'' + \rho F\ddot{y} = 0 \tag{1}$$

where y – deflection, $y^{IV} = y'''' = \frac{\partial^4 y}{\partial x^4}, y'' = \frac{\partial^2 y}{\partial x^2},\ \ddot{y} = \frac{\partial^2 y}{\partial t^2},\ x \in [0, l],\ P = ES\frac{\Delta l}{l}$ – longitudinal tensile load with the double-clamped boundary conditions:

$$y(0) = y(l) = 0,\ \ y'(0) = y'(l) = 0. \tag{2}$$

The eigen frequency of the flexural oscillations has a scale factor $\Omega_0 = \frac{1}{l}\sqrt{\frac{P}{\rho F}}$. Let's introduce the dimensionless variables $s = \frac{x}{l},\ \tau = \Omega_0 t$ and non-dimensional buckling $w = y/h$ (h – magnitude of gap).

The equation (1) can be written in the form:

$$\varepsilon\alpha w^{IY} - w'' + \ddot{w} = 0, \tag{3}$$

where the dimensionless parameter $\varepsilon\alpha = \frac{EI}{Pl^2}$ and $\varepsilon = \frac{h}{l}$ are introduced (further believed small).

At first let's consider free harmonic oscillation, assuming that $w = W(s)e^{i\Omega\tau}$, here Ω –eigen frequency. We obtain the homogeneous differential equation:

$$\varepsilon\alpha W^{IY} - W'' - \Omega^2 W = 0 \qquad (4)$$

Believing that $W = Ce^{\lambda s}$ we obtain the dimensionless characteristic biquadratic equation

$$\varepsilon\alpha\lambda^4 - \lambda^2 - \Omega^2 = 0, \qquad (5)$$

which has two roots: $\lambda_{1,2}^2 = \frac{1}{2\varepsilon\alpha} \pm \frac{1}{2\varepsilon\alpha}\sqrt{1 + 4\varepsilon\alpha\Omega^2}$. These roots can be separate in two groups: positive and negative. Let's input designations

$$\nu = \sqrt{\frac{1}{2\varepsilon\alpha}\left(\sqrt{1 + 4\varepsilon\alpha\Omega^2} - 1\right)},$$

$$\kappa = \sqrt{\frac{1}{2\varepsilon\alpha}\left(\sqrt{1 + 4\varepsilon\alpha\Omega^2} + 1\right)}\ . \qquad (6)$$

The solution of equation (4) can be written in the form:

$$W_i = Asin\,\nu s + Bcos\,\nu s + Csh\,\kappa s + Dch\,\kappa s \qquad (7)$$

Based on the boundary condition the eigen modes and frequency are determined from homogenous algebraic system:

$$B + D = 0, \qquad \nu A + \kappa C = 0,$$
$$A\sin\nu + B\cos\nu + Csh\,\kappa + Dch\,\kappa = 0, \qquad (8)$$
$$\nu A\cos\nu - \nu B\sin\nu + \kappa Cch\,\kappa + \kappa Dsh\,\kappa = 0.$$

Having used system of equation (8) we obtain the characteristic equation

$$2\nu\kappa(\cos\nu\,\mathrm{ch}\,\kappa - 1) + (\nu^2 - \kappa^2)\sin\nu\,\mathrm{sh}\,\kappa = 0, \qquad (9)$$

where $\nu\kappa = \Omega\sqrt{\frac{1}{\varepsilon\alpha}}$, $\nu^2 - \kappa^2 = -\frac{1}{\varepsilon\alpha}$. From (9) we find the eigen frequencies Ω_i , $i = 1, 2, \ldots$ The modes of free oscillations W_i corresponded to i eigen frequency has a form

$$W_i(s) = A_i(\sin\nu_i s - G_i\cos\nu_i s - \frac{\nu_i}{\kappa_i}\,\mathrm{sh}\,\kappa_i s + G_i\,\mathrm{ch}\,\kappa_i s), \qquad (10)$$

Here $G_i = \frac{\sin\nu_i - \frac{\nu_i}{\kappa_i}\mathrm{sh}\,\kappa_i}{\cos\nu_i - \mathrm{ch}\,\kappa_i}$. The coefficients A_i is determined from condition of orthonormality

$$\int_0^1 W_i(s)W_k(s)\,ds = \delta_{ik}.$$

Table1. The some results of calculation are shown

$\varepsilon\alpha$	Ω_1	Ω_2	Ω_3
0.01	4.10	9.12	15.60
0.1	7.78	20.65	39.50
0.2	10.60	28.44	54.98

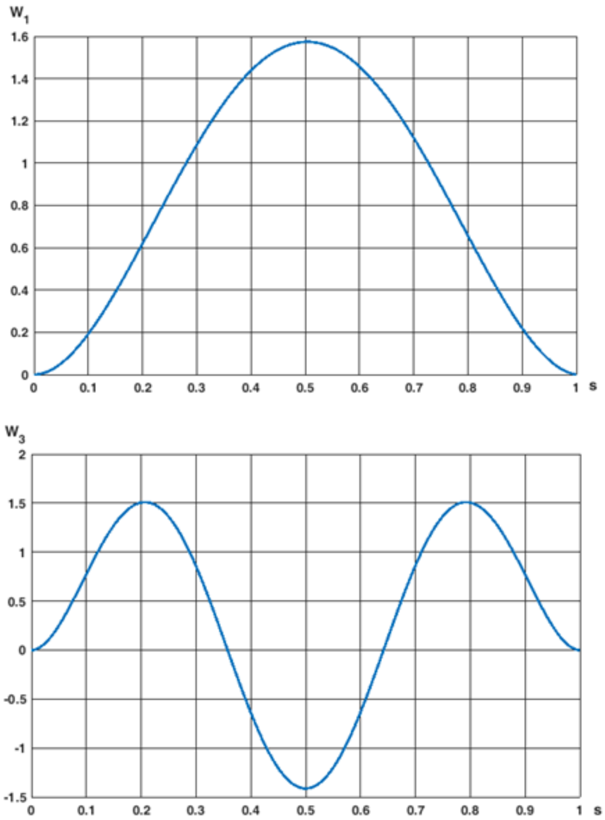


Figure 2. The first two symmetric eigen-modes

2. Electrostatic Field

The electrostatic forces are obtained from the expression of electrostatic potential. The distribution of this potential between electrodes $\Psi(x, z)$ is determined from Laplace's equation

$$\Delta\Psi = 0 \qquad (11)$$

Further we consider only symmetric case of conductive layer deformation. One (upper) layer has an electric potential V lower and other sidewalls have zero potential, believing that side walls are separated from conductive layer by thin insulating layer. The boundary conditions in this case have a form:

$$x = 0;\ \ \Psi = 0, \quad x = \frac{1}{2};\ \frac{\partial\Psi}{\partial x} = 0.$$
$$z = h;\quad \Psi = V, \quad z = 0;\ \ \Psi = 0 \qquad (12)$$

Let's introduce dimensionless coordinates $\xi = \frac{2x}{l}$, $\zeta = z/h$. The non-dimensional expression for potential $\psi = \Psi/V$ satisfies to the same equation (11).

The calculation of electric field is fulfilled taking in account the edge effect, assuming that the ratio of gap value h and length of layer l ($\varepsilon = \frac{h}{l}$) is small. Having separated the homogeneous solution, which corresponds to uniform distribution of potential $\psi_0 = \zeta$ and inputted the coordinate of left edge of layer $\eta = \xi/\varepsilon$, we obtain the boundary task relatively $\tilde{\psi}$, believing that $\psi = \tilde{\psi} + \zeta$

$$\varepsilon^2 \frac{\partial^2 \tilde{\psi}}{\partial \xi^2} + \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = 0 \quad \text{or} \quad \frac{\partial^2 \tilde{\psi}}{\partial \eta^2} + \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} = 0 \qquad (13)$$

with boundary conditions:

$$\xi = 0 (\eta = 0),\ \tilde{\psi} = -\zeta;$$
$$\xi = 1, \qquad \frac{\partial \tilde{\psi}}{\partial \tau} = 0, \qquad (14)$$
$$\zeta = 0, \quad \tilde{\psi} = 0; \quad \zeta = 1, \quad \tilde{\psi} = 0.$$

Using symmetry of electroelastic system, all subsequent relations will be written only for its left half ($\xi = 2s,\ 0 \le s \le 0.5,\ 0 \le \xi \le 1$). The solution of boundary task (13), (14) is searched by method of boundary layer $\tilde{\psi} = \tilde{\psi}_0(\eta, \xi) + \varepsilon \tilde{\psi}_1(\eta, \xi, \varepsilon)$. In a case of non-deformed plate-layer the expression of electrostatic potential has an approximated view:

$$\psi = \zeta + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp\left(-\frac{\pi n \xi}{\varepsilon}\right) \sin \pi n \zeta. \qquad (15)$$

Taking into account the beam deflection (upper plate of «capacitor») $w(s)$ the electric potential is determined by expression:

$$\psi = \frac{\zeta}{u(\xi)} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp\left(-\frac{\pi n \xi}{\varepsilon}\right) \sin \pi n \zeta,$$
$$u(\xi) = 1 - w(\xi) \qquad (16)$$

The distribution of potential ψ is shown in Fig.3.

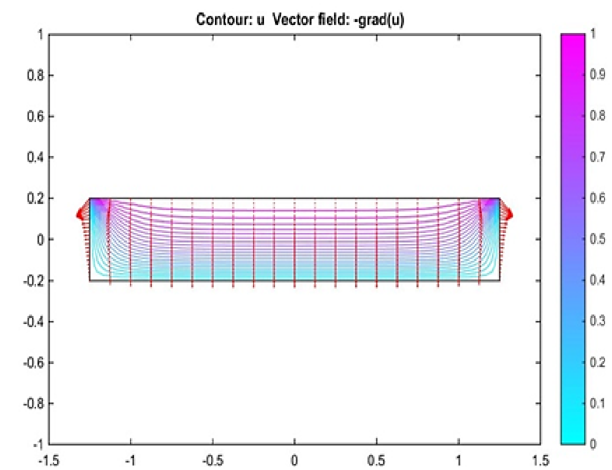


Figure 3. Distribution of electrostatic potential and electric intensity ($\varepsilon = 0.1$)

The distributed force of attraction acts in a normal of upper layer (Γ) can be found by expression:

$$f(s) = \left[\varepsilon^2 \left(\frac{\partial \psi}{\partial \xi}\right)^2 + \left(\frac{\partial \psi}{\partial \zeta}\right)^2\right]_{\Gamma}, \quad \zeta_{\Gamma} = u(s) = 1 - w(s). \qquad (17)$$

Differentiating electro potential on ξ we obtain

$$\frac{\partial \psi}{\partial \xi} = \frac{\zeta w'}{2(1-w)^2} - \frac{2}{\varepsilon} \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{2\pi n s}{\varepsilon}\right) \sin \pi n \zeta \ . \quad (18)$$

Here and future we will neglect the first item containing ε^2 in expression (17). Also in future we will neglect the changing of normal of the bounding upper elastic layer that leads to $\zeta = 1$. So electric force acting in a normal to elastic layer can be calculated using the expression

$$f(s) = \left(\frac{\partial \psi}{\partial \zeta}\right)^2, \qquad \frac{\partial \psi}{\partial \zeta} = \frac{1}{1-w} + S,$$
$$S = 2 \sum_{n=1}^{\infty} \exp\left(-\frac{2\pi n s}{\varepsilon}\right). \qquad (19)$$

The dependence $S(s)$ for left half of our system is shown in Fig.4

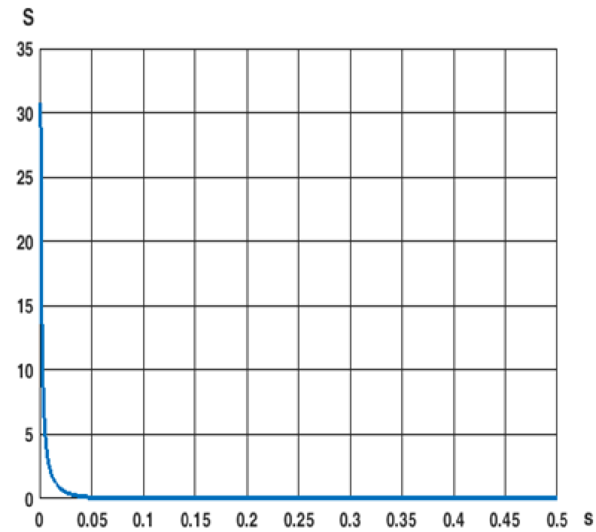


Figure 4. Description of edge effect

So we can see very fast decay of edge effect.

3. Electro-elasticity

The task of electro-elasticity consist from static equation of beam flexure under of action the electric force has a form

$$\varepsilon \alpha w^{IV} - w'' = \gamma^2 f(s, w), \qquad (20)$$

where $\gamma^2 = \frac{\epsilon_0 V^2 l^2}{2 P h^3}$ (ϵ_0 –dielectric permittivity) and homogeneous boundary conditions

$$w(0) = 0, w'(0) = 0, w(1) = 0, w'(1) = 0.$$

For solution of this boundary problem analytic-numerical method Newton – Kantorovich is used [10]. Corresponded operation equation may be written in form:

$$L(w) = 0, \qquad (21)$$

here

$$L(w) = \varepsilon \alpha w^{IV} - w'' - \gamma^2 f. \qquad (22)$$

Let's found $L'(w)$ – the Frechet derivative

$$L'(w) = \varepsilon \alpha \frac{d^4}{ds^4} - \frac{d^2}{ds^2} - \gamma^2 \frac{\partial f}{\partial w} \ . \qquad (23)$$

Taylor series $L(w)$ close to some w= w_0 in a linear approximation

$$L(w) = L(w_0) + L'(w_0)(w - w_0) = 0$$

Let's designate step of difference

$$\Delta_n = w_{n+1} - w_n, \qquad n = 0, 1, 2, \ldots$$

We can use iteration procedure:

$$L'(w_n)\Delta_n + L(w_n) = 0, \tag{24}$$

or

$$\varepsilon\alpha\Delta_n^{IV} - \Delta_n'' - \gamma^2 r_n \Delta_n + \varphi_n = 0,$$

$$\varphi_n = \varepsilon\alpha w_n^{IV} - w_n'' - \gamma^2 (\frac{\partial\psi}{\partial\zeta})^2|_{w=w_n}, \tag{25}$$

$$r_n = 2\frac{1}{(1-w_n)^2}(\frac{\partial\psi}{\partial\zeta})|_{w=w_n}.$$

If we know $w_n(s)$ – n approximation of unknown function (initial approximation is assigned), and after solution eq. (25) we obtain next approximation

$$w_{n+1} = w_n + \Delta_n \tag{26}$$

The solution of equation (25) will be look for as a sum of free oscillation modes (7)

$$\Delta_n(s) = \sum_{k=1}^{K} \beta_k W_k(s). \tag{27}$$

Summation is realized taking in account only symmetric modes. For determination coefficients β_k we can use Halerkin projections

$$2\int_0^{\frac{1}{2}} (\varepsilon\alpha\Delta_n^{IV} - \Delta_n'' - \gamma^2 r_n \Delta_n + \varphi_n). w_k(s)ds = 0 \tag{28}$$

Using transformation (28) yields nonlinear system for coefficients:

$$\Omega_k^2 \beta_k - 2\gamma^2 \sum_{r=1}^{K} R_{kr} \beta_r = 2\gamma^2 C_k - 2\Omega_k^2 B_k \tag{29}$$

where:

$$R_{kr} = \int_0^{1/2} r_n W_k W_r ds,$$

$$B_k = \int_0^{1/2} W_k w_n ds, \tag{30}$$

$$C_k = \int_0^{1/2} W_k (\frac{\partial\psi}{\partial\zeta})^2|_{w=w_n} ds.$$

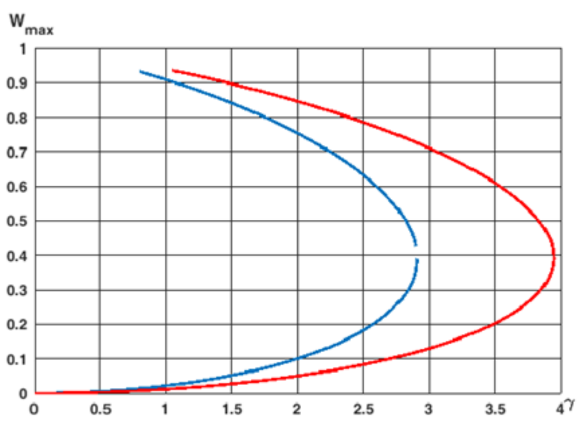


Figure 5. The results of calculation by iteration method ($\varepsilon = 0.1$)

Having solved system (29), i.e. determine unknown coefficients β_k from system (29) and consequently Δ_n we can find next approximation: $w_{n+1} = w_n + \Delta_n$.

The dependence w_{max} from value of parameter γ for case then $\varepsilon\alpha = 0.1\,; 0.2.$ (with account edge effect) is shown in Fig.5. The dependence $w = w(s)$ at $\varepsilon = 0.1$, $\gamma = 1.5$ on left part of beam is shown in Fig 6. Calculation fulfilled retaining twelve symmetric modes

The inner curve corresponds to the case then $\alpha = 1$, external for $\alpha = 2$. So the increasing of the beam stiffness leads to shift of branching curve to the right

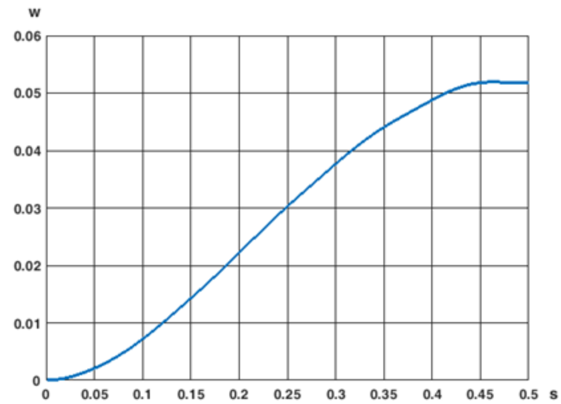


Figure 6. The form of the half of beam under the action of electric force

For solution task without edge effect it's enough in expression (19) for $\frac{\partial\psi}{\partial\zeta}$ take $S = 0$, i.e. to solve equation:

$$\varepsilon\alpha w^{IV} - w'' = \gamma^2 \frac{1}{(1-w)^2} \tag{31}$$

Comparing the solution of this equation with previous calculation are shown that taking in account the edge effect in our models practically not influence on the branching diagram (for example in Fig.5).

The task of flexure of tensile string (layer) under action of electrostatic force without taking in account first item in eq. (31) yields to analytic solution. In this case the boundary problem takes a form:

$$w'' + \gamma^2 \frac{1}{(1-w)^2} = 0,$$

$$w(0) = w(1) = 0. \tag{32}$$

The solution of this task taking in account the condition of symmetry relatively middle of string $w'(1/2) = 0$ leads to integral relation

$$\frac{1}{2}(w')^2 + \frac{\gamma^2}{1-w} = C, \tag{33}$$

where the constant C is determined from condition of symmetry as $C = \frac{\gamma^2}{1-w_m}$, w_m – flexure in a middle of the string. Using boundary conditions leads to equation relatively w_m

$$\int_0^{w_m} \sqrt{\frac{1-w}{w_m - w}} dw = \gamma \sqrt{\frac{1}{2(1-w_m)}}. \tag{34}$$

In result we obtain the algebraic equation

$$\sqrt{w_m} + (1 - w_m)\, ln\sqrt{\frac{1+\sqrt{w_m}}{1-\sqrt{w_m}}} = \gamma\sqrt{\frac{1}{2(1-w_m)}} \qquad (35)$$

This equation can be rewrite in form:

$$\gamma = \sqrt{2(1-w_m)}\,\sqrt{w_m} + \sqrt{2}\cdot(1- w_m)^{3/2}\, ln\sqrt{\frac{1+\sqrt{w_m}}{1-\sqrt{w_m}}} \qquad (36)$$

In result the dependence $w_m(\gamma)$ (36) is shown in Fig.7

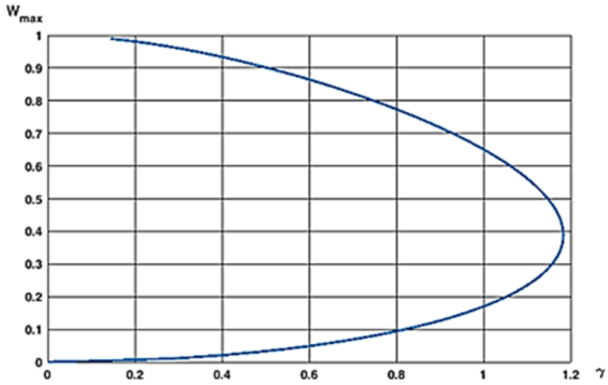


Figure 7. Analytic diagram of string branching

With increasing of parameter γ from zero until point of branching this equation has two solutions. The point of branching corresponds to tangency solution of (36) in point $w_m \sim 0.4$, that has place at $\gamma_* \sim 1.2$. After point of bifurcation this equation has not solution. It is evidently that at $\gamma < \gamma_*$ the solution with smaller amplitude of flexure is stable in contrast to unstable for of curve with bigger amplitude, both take place at the same value of electric potential.

4. Conclusions

The main achievement of this work is an investigation of the influence of the ratio between flexural and membrane stiffness of micro layer situated in electrostatic field on the point of pull-in effect. Moreover we suppose that the bending stiffness is much less than the stiffness of string. This assumption gives us the possibility to build the asymptotic approximation of pull-in branching curve.

Acknowledgements

The study was carried out by support of grant RFBR RAS 17-01-00414.

REFERENCES

[1] Grinberg Ya.S., Pashkin Yu.Ya. and Ilychev E.V. Nanomechanical resonators//Achievement of Physical science, 2912 – 4: V. 18, pp. 407 – 436.

[2] Wen-Ming Zhang, Han Yan, Zhi-Ke Peng, Guang Meng Electrostatic pull-in instability in MEMS/NEMS: A review // Sensors and Actuators A: Physical, 2014 Elsevier, pp.188-218

[3] John A. Pelesko and Tobin A. Driscoll The effect of the small-aspect-ratio approximation on canonical electrostatic MEMS models// Journal of Engineering Mathematics, 2006, V.53, pp. 239 - 252

[4] Slava Krylov [et al.] Symmetry breaking in an initially curved pre-stressed micro beam loaded by a distributed electrostatic force// International Journal of Solids and Structures, 51, 2014

[5] Bom K. [et al.] Nanomechanical resonators and their applications in biological/chemical detection: Nanomechanics principles // Physics Reports. 2011, V. 503, pp. 115 – 163

[6] Scott Bunch J. [et al.] Electromehanical Resonators from Graphene Sheets // 2007, V. 315, pp. 490 –493

[7] Lior Medina S. [et al.] Large-Scale Arrays of Single-Layer Graphene Resonators // Nano Letters. 2010, V.10, pp. 4869 – 4873.

[8] C. Lee, X. Wei, J. W. Kysar и J. Hone Measurement of the Elastic Properties and Intrinsic Strength of Monolayer Graphene// Science, V. 321, № 5887, pp. 385-388, 2010.

[9] Firsova N.E., Firsov Y.A. A new loss mechanism in graphene nanoresonators due to the synthetic electric fields caused by inherent out-of-plane membrane corrugations// J.Phys.D: Appl. Phys.,V.45, p., 2012.

[10] Fridman V.M. Theory of Elastic Oscillations. Equations and Methods. Springer, 257 p., 2018.